Indian J. Pure Appl. Math., **48**(3): 391-410, September 2017
© Indian National Science Academy
DOI: 10.1007/s13226-017-0232-9

ALGORITHM FOR APPROXIMATING SOLUTIONS OF HAMMERSTEIN INTEGRAL EQUATIONS WITH MAXIMAL MONOTONE OPERATORS

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(*Received* 3 *October* 2016; *accepted* 15 *March* 2017)

Let X be a uniformly convex and uniformly smooth real Banach space with dual space X^*. Let $F : X \to X^*$ and $K : X^* \to X$ be bounded monotone mappings such that the Hammerstein equation $u + KFu = 0$ has a solution. An explicit iteration sequence is constructed and proved to converge strongly to a solution of this equation.

Key words : Bounded; maximal monotone mappings; Hammerstein equations; strong convergence.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^n$ be bounded. Let $k : \Omega \times \Omega \to \mathbb{R}$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be measurable real-valued functions. An integral equation (generally nonlinear) of *Hammerstein-type* has the form

$$u(x) + \int_{\Omega} k(x,y) f(y, u(y)) dy = w(x), \tag{1.1}$$

where the unknown function u and inhomogeneous function w lie in a Banach space E of measurable real-valued functions. If we define $F : \mathcal{F}(\Omega, \mathbb{R}) \to \mathcal{F}(\Omega, \mathbb{R})$ and $K : \mathcal{F}(\Omega, \mathbb{R}) \to \mathcal{F}(\Omega, \mathbb{R})$ by

$$Fu(y) = f(y, u(y)),\ x \in \Omega,$$

and

$$Kv(x) = \int_{\Omega} k(x,y) v(y) dy,\ x \in \Omega,$$

respectively, where $\mathcal{F}(\Omega, \mathbb{R})$ is a space of measurable real-valued functions defined from Ω to $\mathbb{R}$, then equation (1.1) can be put in the abstract form

$$u + KFu = 0. \tag{1.2}$$

where, without loss of generality, we have assumed that $w \equiv 0$.

Interest in (1.1) stems mainly from the fact that several problems that arise in differential equations, for instance, elliptic boundary value problems whose linear parts possess Green's function can, as a rule, be transformed into the form (1.1) (see e.g., Pascali and Sburian [27], chapter IV, p. 164). Equations of Hammerstein-type also play a crucial role in the theory of optimal control systems and in automation and network theory (see e.g., Dolezale [22]).

Several existence results have been proved for equations of Hammerstein-type (see e.g., Brézis and Browder [4, 5, 6], Browder [7], De Figueiredo and Gupta [9]).

1.1 *Approximation of solutions of Hammerstein equations*

In general, equations of Hammerstein-type are nonlinear and there is no known method to find close form solutions for them. Consequently, methods of approximating solutions of such equations, where solutions are known to exist, are of interest. Let H be a real Hilbert space. A *nonlinear* operator $A : H \to H$ is said to be *monotone* if for each $x, y \in H$, the following inequality holds

$$\langle Ax - Ay, x - y\rangle \geq 0. \tag{1.3}$$

The map A is said to be *angle-bounded* with angle $\beta > 0$ if

$$\langle Ax - Ay, z - y\rangle \leq \beta\langle Ax - Ay, x - y\rangle \tag{1.4}$$

for any triple elements $x, y, z \in H$. For $y = z$ inequality (1.4) implies the monotonicity of A. A monotone *linear* operator $A : H \to H$ is said to be *angle bounded* with angle $\alpha > 0$ if

$$|\langle Ax, y\rangle - \langle Ay, x\rangle| \leq 2\alpha\langle Ax, x\rangle^{\frac{1}{2}}\langle Ay, y\rangle^{\frac{1}{2}} \tag{1.5}$$

for all $x, y \in H$.

In the special case where *one of the operators is angle-bounded*, and the other is *bounded*, Brézis and Browder [4, 6] proved the strong convergence of a suitably defined *Galerkin approximation* to a solution of equation (1.2). In fact, they proved the following theorem.

***Theorem* 1.1** — *(Brézis and Browder [6]). Let H be a separable Hilbert space and C be a closed subspace of H. Let $K : H \to C$ be a bounded continuous monotone operator and $F : C \to H$ be angle-bounded and weakly compact mapping. For a given $f \in C$, consider the Hammerstein equation*

$$(I + KF)u = f \tag{1.6}$$

and its nth Galerkin approximation given by

$$(I + K_n F_n)u_n = P^* f, \tag{1.7}$$

where $K_n = P_n^* K P_n : H \to C_n$ and $F_n = P_n F P_n^* : C_n \to H$, the symbols have their usual meanings (see [6]). Then, for each $n \in \mathbb{N}$, the Galerkin approximation (1.7) admits a unique solution u_n in C_n and $\{u_n\}$ converges strongly in H to the unique solution $u \in C$ of the equation (1.6).

It is obvious that if an *iterative algorithm* can be developed for the approximation of solutions of equation of Hammerstein-type (1.2), this will certainly be a welcome complement to the Galerkin approximation method. Attempts had been made to approximate solutions of equations of Hammerstein-type using *Mann-type* (see e.g., Mann [25]) iteration scheme. However, the results obtained were not satisfactory (see [16]). The recurrence formulas used in these attempts, even in real Hilbert spaces, involved K^{-1} which is required to be strongly monotone when K is, and this, apart from limiting the class of mappings to which such iterative schemes are applicable, is also not convenient in any possible applications.

Let E be a real normed space with dual space E^*. A map $J : E \to 2^{E^*}$ defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = ||x|| ||x^*||, ||x|| = ||x^*||\}$$

is called the *normalized duality map* on E. A map $A : E \to E$ is called *accretive* if for each $x, y \in E$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y) \rangle \geq 0.$$

We note that in Hilbert spaces, $J = I$, the identity map on H. So, in Hilbert spaces, accretive operators are monotone.

A map $A : E \to E^*$ is called *monotone* if for each $x, y \in E$, the following inequality holds:

$$\langle Ax - Ay, x - y \rangle \geq 0.$$

Part of the difficulty in establishing *iterative algorithms* for approximating solutions of Hammerstein equations seems to be that the composition of two monotone maps need not be monotone.

The first satisfactory results on *iterative methods* for approximating solutions of Hammerstein equations *involving accretive-type mappings*, as far as we know, were obtained by Chidume and Zegeye [19, 20, 21].

Let X be a real Banach space and $F, K : X \to X$ be *accretive-type* mappings. Let $E := X \times X$. Then, Chidume and Zegeye [20, 21] defined $A : E \to E$ by

$$A[u,v] = [Fu - v, Kv + u] \text{ for } [u,v] \in E.$$

We note that $A[u,v]$=0 if and only if u solves (1.2) and $v = Fu$. The authors defined an iterative sequence and obtained strong convergence theorems in the Cartesian product space E, for solutions of Hammerstein equations under various continuity conditions on F and K, for special classes of real Banach spaces, X. It turns out that, in the case of a real Hilbert space, H, the operator A defined on $H \times H$ *is monotone whenever* F *and* K *are*. The method of proof used by Chidume and Zegeye provided the authors a clue for the establishment of the following coupled explicit iterative algorithm for computing a solution of the equation $u + KFu = 0$ *in the original space,* X. With initial vectors $u_0, v_0 \in X$, sequences $\{u_n\}$ and $\{v_n\}$ in X are defined iteratively as follows:

$$u_{n+1} = u_n - \alpha_n(Fu_n - v_n),\ n \geq 0, \tag{1.8}$$

$$v_{n+1} = v_n - \alpha_n(Kv_n + u_n),\ n \geq 0, \tag{1.9}$$

where α_n is a sequence in $(0,1)$ satisfying appropriate conditions. The recursion formulas (1.8) and (1.9) have been used successfully to approximate solutions of Hammerstein equations involving nonlinear *accretive-type mappings* (see e.g., Chidume and Djitte [12, 14], Chidume and Ofoedu [15], Chidume and Shehu [18], Chidume [10], and the references contained in them). The following theorem has been proved as a generalization of recent important results.

***Theorem* 1.2** — *(Chidume and Djitte, [11]). Let H be a real Hilbert space and $F, K : H \to H$ be bounded and maximal monotone operators. Let $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ be sequences in H defined iteratively from arbitrary points $u_1,\ v_1 \in H$ as follows:*

$$u_{n+1} = u_n - \lambda_n(Fu_n - v_n) - \lambda_n\theta_n(u_n - u_1),\ n \geq 1, \tag{1.10}$$

$$v_{n+1} = v_n - \lambda_n(Kv_n + u_n) - \lambda_n\theta_n(v_n - v_1),\ n \geq 1, \tag{1.11}$$

where $\{\lambda_n\}_{n=1}^{\infty}$ and $\{\theta_n\}_{n=1}^{\infty}$ are sequences in $(0,1)$ satisfying the following conditions:

(i) $\lim_{n\to\infty} \theta_n = 0$,

(ii) $\sum_{n=1}^{\infty} \lambda_n\theta_n = \infty, \quad \lambda_n = o(\theta_n)$,

(iii) $\lim_{n\to\infty} \dfrac{\left(\frac{\theta_{n-1}}{\theta_n} - 1\right)}{\lambda_n\theta_n} = 0.$

Suppose that $u + KFu = 0$ *has a solution in* H*. Then, there exists a constant* $d_0 > 0$ *such that if* $\lambda_n \leq d_0\theta_n$ *for all* $n \geq n_0$ *for some* $n_0 \geq 1$*, then the sequence* $\{u_n\}_{n=1}^{\infty}$ *converges to* u^**, a solution of* $u + KFu = 0$*.*

Remark 1 : It is known that monotone mappings were studied in Hilbert spaces by Zarantonello [31], Minty [26], Kačurovskii [24] and a host of other authors as a result of their usefulness in numerous applications. Consider the following for example:

Let $f : H \to \mathbb{R} \cup \{\infty\}$ be a proper convex function. The *subdifferential* of f at $x \in H$ is defined by

$$\partial f(x) = \big\{x^* \in H : f(y) - f(x) \geq \big\langle y - x, x^*\big\rangle \;\; \forall\, y \in H\big\}.$$

It is easy to check that $\partial f : H \to 2^H$ is a *monotone operator* on H, and that $0 \in \partial f(x)$ *if and only if* x *is a minimizer of* f. Setting $\partial f \equiv A$, it follows that solving the inclusion $0 \in Au$, in this case, is solving for a minimizer of f.

In fact, Pascali and Sburian in [27] made the following remark.

> . . . The monotone maps constitute the most manageable class, because of the very simple structure of the monotonicity condition. The monotone mappings appear in a rather wide variety of contexts, since they can be found in many functional equations. Many of them appear also in calculus of variations, as subdifferential of convex funtions (Pascali and Sburian [27], p. 101).

Remark 2 : Even though the class of monotone-type operators have a wider variety of applications than the class of accretive-type operators in Banach spaces, virtually all the results on the approximation of solutions of Hammerstein equations are either proved in Hilbert spaces or in a Banach space in the case where the operators K and F are *accretive-type* mappings (see [13, 15, 17, 18]). To the best of our knowledge, there are very few results on the approximation of solutions of Hammerstein-type equations in Banach spaces (in the case where the operators K and F are *monotone-type operators*).

It may be that, part of the difficulty is that since the operator F maps E to E^* and K maps E^* to E the recursion formulas used for accretive-type mappings may no longer make sense under these settings. Moreover, most of the inequalities used in proving convergence when the operators are accretive-type involve the normalized duality mappings which also appears in the definition of accretive operators. However, the definition of monotone mappings does not involve the normalized duality mappings. This creates computational difficulties in attempting to use standard Banach space inequalities in proving convergence results for *monotone-type* mappings.

It is our purpose in this paper to construct a coupled iteration process and prove its strong convergence to a solution of $u + KFu = 0$ in uniformly convex and uniformly smooth real Banach spaces, where the operators K and F are maximal *monotone* and bounded. Furthermore, our result extends and generalizes Theorem 1.2. Our method of proof is also of independent interest.

2. PRELIMINARIES

Definition 2.1 — Let E be a normed space with $dimE \geq 2$. The *modulus of convexity* of E is the function $\delta_E : (0, 2] \rightarrow [0, 1]$, defined by

$$\delta_E(\epsilon) := inf\Big\{1 - \Big\|\frac{x+y}{2}\Big\| : \|x\| = \|y\| = 1; \epsilon = \|x - y\|\Big\}.$$

In the sequel, we shall need the following definitions and results. Let E be a smooth real Banach space with dual E^*. The function $\phi : E \times E \rightarrow \mathbb{R}$, is defined by,

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \text{ for } x, y \in E, \tag{2.1}$$

where J is the normalized duality mapping from E into 2^{E^*}. It was introduced by Alber and has been studied by Alber [1], Alber and Guerre-Delabriere [2], Kamimura and Takahashi [23], Reich [28] and a host of other authors. If $E = H$, a real Hilbert space, then equation (2.1) reduces to $\phi(x, y) = \|x - y\|^2$ for $x, y \in H$. It is obvious from the definition of the function ϕ that

$$(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2 \text{ for } x, y \in E. \tag{2.2}$$

Define a map $V : X \times X^* \rightarrow \mathbb{R}$ by

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^*\rangle + \|x^*\|^2. \tag{2.3}$$

Then, it is easy to see that

$$V(x, x^*) = \phi(x, J^{-1}(x^*)) \, \forall x \in X, \; x^* \in X^*. \tag{2.4}$$

Lemma 2.2 — (Alber, [1]). Let X be a reflexive strictly convex and smooth Banach space with X^* as its dual. Then,

$$V(x, x^*) + 2\langle J^{-1}x^* - x, y^*\rangle \leq V(x, x^* + y^*) \tag{2.5}$$

for all $x \in X$ and $x^*, y^* \in X^*$.

Lemma 2.3 — (Kamimura and Takahashi, [23]). Let X be a real smooth and uniformly convex Banach space, and let $\{x_n\}$ and $\{y_n\}$ be two sequences of X. If either $\{x_n\}$ or $\{y_n\}$ is bounded and $\phi(x_n, y_n) \to 0$ as $n \to \infty$, then $\|x_n - y_n\| \to 0$ as $n \to \infty$.

Lemma 2.4 — (Xu [30]). Let ρ_n be a sequence of non-negative real numbers satisfying the relation:

$$\rho_{n+1} \leq (1 - \beta_n)\rho_n + \beta_n \zeta_n + \gamma_n,\ n \geq 0, \tag{2.6}$$

where,
(i) $\beta_n \in [0, 1]$, $\sum \beta_n = \infty$; (ii) $\limsup \zeta_n \leq 0$; (iii) $\gamma_n \geq 0$; $(n \geq 0)$, $\sum \gamma_n < \infty$. Then, $\rho_n \to 0$ as $n \to \infty$.

Remark 3 : Let E^* be a strictly convex dual Banach space with a Fréchet differentiable norm and $A : E \to 2^{E^*}$, be a maximal monotone map. Let $z \in E^*$ be fixed. Then for every $\lambda > 0$, there exists a unique $x_\lambda \in E$ such that $z \in Jx_\lambda + \lambda A x_\lambda$ (see Reich [29], p. 342). Setting $R_\lambda z = x_\lambda$, we have the resolvent $R_\lambda := (J + \lambda A)^{-1} : E^* \to E$ of A, for every $\lambda > 0$. A celebrated result of Reich follows.

Lemma 2.5 — (Reich, [29]). Let E^* be a strictly convex dual Banach space with a Fréchet differentiable norm and let $A : E \to E^*$ be maximal monotone such that $A^{-1}0 \neq \emptyset$. Let $z \in E^*$ be an arbitrary but fixed vector. For each $\lambda > 0$, there exists a unique $x_\lambda \in E$ such that $z \in Jx_\lambda + \lambda A x_\lambda$. Furthermore, x_λ converges strongly to a unique $v \in A^{-1}0$.

Lemma 2.6 — (Alber, [1]). Let X be a uniformly convex Banach space. Then for any $R > 0$ and any $x,\ y \in X$ such that $\|x\| \leq R$, $\|y\| \leq R$ the following inequality holds:

$$\langle Jx - Jy, x - y \rangle \geq (2L)^{-1} \delta_X (c_2^{-1} \|x - y\|), \tag{2.7}$$

where $c_2 = 2 \max\{1, R\}$, $1 < L < 1.7$.

Define

$$K := 4RLsup\{\|Jx - Jy\| : \|x\| \leq R, \|y\| \leq R\} + 1 \tag{2.8}$$

Lemma 2.7 — (Alber, [1]). Let X be a uniformly smooth and strictly convex Banach space. Then for any $R > 0$ and any $x,\ y \in X$ such that $\|x\| \leq R$, $\|y\| \leq R$ the following inequality holds:

$$\langle Jx - Jy, x - y \rangle \geq (2L)^{-1} \delta_{X^*} (c_2^{-1} \|Jx - Jy\|), \tag{2.9}$$

where $c_2 = 2 \max\{1, R\}$, $1 < L < 1.7$.

Lemma 2.8 — (Alber, [1]). Let X be a reflexive strictly convex and smooth Banach space with dual X^*. Let $W : X \times X \to \mathbb{R}$ be defined by $W(x,y) = \frac{1}{2}\phi(y,x)$. Then,

$$\phi(y,x) - \phi(y,z) \geq 2\langle Jx - Jz, z - y\rangle, \tag{2.10}$$

and

$$W(x,y) \leq \langle Jx - Jy, x - y\rangle, \tag{2.11}$$

for all $x, y, z \in X$

Lemma 2.9 — From Lemma 2.5, setting $\lambda_n := \frac{1}{\theta_n}$ where $\theta_n \to 0$ as $n \to \infty$, $\theta_n \leq \theta_{n-1}\ \forall\ n \geq 1$, $\left(\frac{\theta_{n-1}-\theta_n}{\theta_n}K\right) \leq 1$, $z = Jv$ for some $v \in E$, and $y_n := \left(J + \frac{1}{\theta_n}A\right)^{-1} z$, we obtain that:

$$Ay_n = \theta_n(Jv - Jy_n), \tag{2.12}$$

$$y_n \to y^* \in A^{-1}0,$$

where K is as in lemma 2.6 and $A : E \to E^*$ is maximal monotone. We observe that equation (2.12) yields

$$Jy_{n-1} - Jy_n + \frac{1}{\theta_n}\Big(Ay_{n-1} - Ay_n\Big) = \frac{\theta_{n-1} - \theta_n}{\theta_n}\Big(Ju - Jy_{n-1}\Big).$$

Taking the duality pairing of this with $y_{n-1} - y_n$ and using monotonicity of A, we obtain that

$$\langle Jy_{n-1} - Jy_n, y_{n-1} - y_n\rangle \leq \frac{\theta_{n-1} - \theta_n}{\theta_n}\Big\|Ju - Jy_{n-1}\Big\|\|y_{n-1} - y_n\|.$$

We observe that if E is uniformly convex and uniformly smooth, using Lemma 2.6 we obtain,

$$(2L)^{-1}\delta_E(c_2^{-1}\|y_{n-1} - y_n\|) \leq \frac{\theta_{n-1} - \theta_n}{\theta_n}\Big\|Ju - Jy_{n-1}\Big\|\|y_{n-1} - y_n\|,$$

which gives

$$\|y_{n-1} - y_n\| \leq c_2\delta_E^{-1}\left(\frac{\theta_{n-1} - \theta_n}{\theta_n}K\right), \text{ for some } K > 0. \tag{2.13}$$

Similarly, using equation 2.9 of Lemma 2.7, we obtain that,

$$\|Jy_{n-1} - Jy_n\| \leq c_2\delta_{E^*}^{-1}\left(\frac{\theta_{n-1} - \theta_n}{\theta_n}K\right), \text{for some } K > 0. \tag{2.14}$$

Lemma 2.10 — Let E be a smooth real Banach space with dual E^* and the function $\phi : E \times E \to \mathbb{R}$ defined by,

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \text{ for } x, y \in E,$$

where J is the normalized duality mapping from E into 2^{E^*}. Then,

$$\phi(y,x) = \phi(x,y) + 2\langle x, Jy\rangle - 2\langle y, Jx\rangle. \tag{2.15}$$

Lemma 2.11 — [10]. Let $p > 1$ and $r > 0$ be two fixed real number and X be a Banach space. Then the following are equivalent.

(i) X is uniformly convex.

(ii) There is a continuous, strictly increasing convex function

$$g : \mathbb{R}^+ \to \mathbb{R}^+,\ g(0) = 0$$

such that

$$||x+y||^p \geq ||x||^p + p\langle y, f_x\rangle + g(||y||) \tag{2.16}$$

for every $x, y \in B_r(0)$ and $f_x \in J_p(x)$.

(iii) There is a continuous, strictly increasing convex function

$$g : \mathbb{R}^+ \to \mathbb{R}^+,\ g(0) = 0$$

such that

$$\langle x - y, f_x - f_y\rangle \geq g(||x-y||) \tag{2.17}$$

for every $x, y \in B_r(0)$ and $f_x \in J_p(x),\ f_y \in J_p(y)$, where $B_r(0) := \{u \in X : ||u|| \leq r\}$.

Lemma 2.12 — [10]. Let $q > 1$ and $r > 0$ be two fixed real number and X be a smooth Banach space. Then the following are equivalent.

(i) X is uniformly smooth.

(ii) There is a continuous, strictly increasing convex function

$$g : \mathbb{R}^+ \to \mathbb{R}^+,\ g(0) = 0$$

such that for every $x, y \in B_r(0)$, we have

$$||x+y||^q \leq ||x||^q + q\langle y, J_q(x)\rangle + g(||y||) \tag{2.18}$$

(iii) There is a continuous, strictly increasing convex function

$$g : \mathbb{R}^+ \to \mathbb{R}^+,\ g(0) = 0$$

such that for all $x, y \in B_r(0)$, we have

$$\langle x - y, J_q(x) - J_q(y)\rangle \leq g(||x - y||), \tag{2.19}$$

where $B_r(0) := \{u \in X : ||u|| \leq r\}$.

Lemma 2.13 — Let $X,\ X^*$ be uniformly convex and uniformly smooth real Banach spaces. Let $E = X \times X^*$ with the norm $\|z\|_E = (\|u\|_X + \|v\|_{X^*})^{\frac{1}{2}}$, for any $z = [u, v] \in E$. Let $E^* = X^* \times X$ denote the dual space of E. For arbitrary $x = [x_1, x_2] \in E$, define the map $J_E : E \to E^*$ by

$$J_E(x) = J_E[x_1, x_2] := [J_X(x_1), J_{X^*}(x_2)],$$

so that for arbitrary $z_1 = [u_1, v_1],\ z_2 = [u_2, v_2]$ in E, the duality pairing $\langle \cdot, \cdot\rangle$ is given by

$$\langle z_1, J_E\rangle := \langle u_1, J_X(u_2)\rangle + \langle v_1, J_{X^*}(v_2)\rangle.$$

Then, E is uniformly smooth and uniformly convex.

PROOF : Let $x = [x_1, x_2]$, $y = [y_1, y_2]$ be arbitrary elements of E. Then,

$$\begin{aligned}
&\langle x - y, J_E(x) - J_E(y)\rangle \\
&= \Big\langle [x_1 - y_1, x_2 - y_2], \Big[J_X(x_1) - J_X(y_1), J_{X^*}(x_2) - J_{X^*}(y_2)\Big]\Big\rangle \\
&= \Big\langle x_1 - y_1, J_X(x_1) - J_X(y_1)\Big\rangle + \Big\langle x_2 - y_2, J_{X^*}(x_2) - J_{X^*}(y_2)\Big\rangle \\
&\leq g_1^*(\|x_1 - y_1\|) + g_2^*(\|x_2 - y_2\|),
\end{aligned}$$

where $g_1^*,\ g_2^*$ are strictly increasing continuous and convex functions on $\mathbb{R}^+$ and $g_1^*(0) = g_2^*(0) = 0$. It follows that:

$$\Big\langle x - y, J_E(x) - J_E(y)\Big\rangle \leq g^*(\|x - y\|),$$

where $g^*(\|x - y\|) := g_1^*(\|x_1 - y_1\|) + g_2^*(\|x_2 - y_2\|)$. Hence the result follows from inequality (2.19) of Lemma 2.12 that E is uniformly smooth.

Also,

$$\begin{aligned}
&\langle x - y, J_E(x) - J_E(y)\rangle \\
&= \Big\langle [x_1 - y_1, x_2 - y_2], \Big[J_X(x_1) - J_X(y_1), J_{X^*}(x_2) - J_{X^*}(y_2)\Big]\Big\rangle \\
&= \Big\langle x_1 - y_1, J_X(x_1) - J_X(y_1)\Big\rangle + \Big\langle x_2 - y_2, J_{X^*}(x_2) - J_{X^*}(y_2)\Big\rangle \\
&\geq g_1(\|x_1 - y_1\|) + g_2(\|x_2 - y_2\|),
\end{aligned}$$

where g_1, g_2 are strictly increasing continuous and convex functions on $\mathbb{R}^+$ and $g_1(0) = g_2(0) = 0$. It follows that:

$$\Big\langle x - y, J_E(x) - J_E(y) \Big\rangle \geq g(\|x - y\|),$$

where $g(\|x-y\|) := g_1(\|x_1 - y_1\|) + g_2(\|x_2 - y_2\|)$. Hence the result follows from inequality (2.17) Lemma 2.11 that E is uniformly convex. □

In what follows, we shall need the following theorem.

***Theorem* 2.14** — (Browder [8]). *Let E be a strictly convex and reflexive Banach space with a strictly convex conjugate space E^*, T_1 a maximal monotone mapping from E to E^*, T_2 a hemicontinuous monotone mapping of all of E into E^* which carries bounded subsets of E into bounded subsets of E^*. Then, the mapping $T = T_1 + T_2$ is a maximal monotone map of E into E^*.*

Lemma 2.15 — Let E be a uniformly convex and uniformly smooth real Banach and $F : E \to E^*, K : E^* \to E$ be maximal monotone. Define $A : E \times E^* \to E^* \times E$ by

$$A[u, v] = [Fu - v, Kv + u] \;\forall\; [u, v] \in E \times E^*.$$

Then, A is maximal monotone.

PROOF : Let $S, T : E \times E^* \to E^* \times E$ be defined as

$$S[u, v] = [Fu, Kv], \quad T[u, v] = [-v, u].$$

Then, $A = S + T$. It suffices to show that S and T are maximal monotone. Observe that S is monotone. Let $h = [h_1, h_2] \in E^* \times E$. Since $F,\ K$ are maximal monotone, take $u = (J + \lambda F)^{-1} h_1$ and $v = (J_* + \lambda K)^{-1} h_2$, where J_* is the normalized duality map on E^*. Then, $(J + \lambda S)w = h$, where $w = [u, v]$. Hence, S is maximal monotone.

Clearly, T is bounded and monotone. Furthermore it is continuous. Hence, it is hemi-continuous. Therefore by theorem (2.14) above, $A = S + T$ is maximal monotone. □

Remark 4 : From Lemma 2.5, setting $\lambda_n := \frac{1}{\theta_n}$ where $\theta_n \to 0$ as $n \to \infty$, $\theta_n \leq \theta_{n-1} \;\forall\; n \geq 1$, $\left(\frac{\theta_{n-1} - \theta_n}{\theta_n} K\right) \leq 1$, $z = [z_1, z_2] = J_{E \times E^*}[u, v]$ for some $[u, v] \in E \times E^*$, and $x_\lambda = [y_n, y_n^*] := \left(J_{E \times E^*} + \frac{1}{\theta_n} A\right)^{-1} [z_1, z_2]$, we obtain that:

$$Jy_n + \frac{1}{\theta_n}(Fy_n - y_n^*) = z_1, \;\forall n \geq 0, \text{ and} \tag{2.20}$$

$$J_* y_n^* + \frac{1}{\theta_n}(Ky_n^* + y_n) = z_2 \;\forall n \geq 0; \tag{2.21}$$

Remark 5 : Let $y_n \to y$ and $y_n^* \to y^*$. From lemma 2.5 we have that $[y_n, y_n^*]$ converges to a point in $A^{-1}0$. This implies that $[y, y^*] \in A^{-1}0$. Consequently, $A[y, y^*] = 0$, that is, $Fy - y^* = 0$ and $Ky^* + y = 0$. Hence, $y^* = Fy$ and $y + KFy = 0$.

3. Main Result

In Theorem 3.1 below, the sequences $\{\lambda_n\}_{n=1}^{\infty}$ and $\{\theta_n\}_{n=1}^{\infty}$ are in $(0,1)$ and are assumed to satisfy the following conditions:

(i) $\lambda_n, \theta_n \to 0$ as $n \to \infty,\ (\frac{\theta_{n-1}-\theta_n}{\theta_n}K)\ \leq 1,\ \sum_{n=1}^{\infty} \lambda_n\theta_n = \infty;$

(ii) $\lambda_n \leq \gamma_0\theta_n,\ [\delta_E^{-1}(\lambda_n M_1^*) + \delta_{E^*}^{-1}(\lambda_n M_2^*)] \leq \gamma_0\theta_n;$

(iii) $\sum_{n=1}^{\infty} \delta_E^{-1}(\lambda_n M_1^*)M^* < \infty, \sum_{n=1}^{\infty} \delta_{E^*}^{-1}(\lambda_n M_2^*)M^* < \infty;$

(iv) $\dfrac{\delta_E^{-1}\left(\frac{\theta_{n-1}-\theta_n}{\theta_n}K\right)}{\lambda_n\theta_n} \to 0,\ \dfrac{\delta_{E^*}^{-1}\left(\frac{\theta_{n-1}-\theta_n}{\theta_n}K\right)}{\lambda_n\theta_n} \to 0$ as $n \to \infty,$

for some constants $M^* > 0,\ M_1^* > 0,\ M_2^* > 0,\ K > 0$ and $\gamma_0 > 0$; where δ_E is the modulus of convexity of E and δ_{E^*} is the modulus of convexity of E^*.

***Theorem* 3.1** — *Let E be a uniformly convex and uniformly smooth real Banach space and $F : E \to E^*$, $K : E^* \to E$ be maximal monotone and bounded maps. For $u_1 \in E,\ \ v_1 \in E^*$, define the sequences $\{u_n\}$ and $\{v_n\}$ in E and E^*, respectively by*

$$u_{n+1} = J^{-1}(Ju_n - \lambda_n(Fu_n - v_n) - \lambda_n\theta_n(Ju_n - Ju_1)),\ n \geq 1, \tag{3.1}$$

$$v_{n+1} = J_*^{-1}(J_*v_n - \lambda_n(Kv_n + u_n) - \lambda_n\theta_n(J_*v_n - J_*v_1)),\ n \geq 1, \tag{3.2}$$

Assume that the equation $u + KFu = 0$ has a solution. Then, the sequences $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ converge strongly to u^* and v^*, respectively, where u^* is the solution of $u+KFu = 0$ with $v^* = Fu^*$.

Proof : We first prove that the sequences $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ are bounded.
For (u_n, v_n), $(u^*, v^*) \in E \times E^*$ where u^* is a solution of (1.2) with $v^* = Fu^*$, set $w_n = (u_n, v_n)$ and $w^* = (u^*, v^*)$. Define $\Lambda : (E \times E^*) \times (E \times E^*) \to \mathbb{R}$ by

$$\Lambda(w_1, w_2) = \phi(u_1, u_2) + \phi(v_1, v_2), \tag{3.3}$$

where $w_1 = (u_1, v_1)$ and $w_2 = (u_2, v_2)$. Let $E \times E^*$ be endowed with the norm $\|(u, v)\| = (\|u\|_E^2 + \|v\|_{E^*}^2)^{\frac{1}{2}}$. We show that $\Lambda(w^*, w_n) \leq r$, for all $n \geq 1$ and for some $r > 0$.

Using the fact that F and K are bounded, define

$$M_1 := \sup\{||(Fu - v) + \theta(Ju - Ju_1)|| : (u, v) \in B_{E\times E^*}, \theta \in (0,1)\} + 1;$$

$$M_2 := \sup\{||(Kv+u)+\theta(J_*v - J_*v_1)|| : (u,v) \in B_{E\times E^*}\, \theta \in (0,1)\} + 1;$$

$$M_3 := \sup\{||Ju - \lambda(Fu-v) - \lambda\theta(Ju - Ju_1) - Ju_1|| : (u,v) \in B_{E\times E^*}, \lambda, \theta \in (0,1)\} + 1;$$

$$M_4 := \sup\{||J^{-1}(Ju - \lambda(Fu-v) - \lambda\theta(Ju - Ju_1)) - u^*|| : (u,v) \in B_{E\times E^*}, \lambda, \theta \in (0,1)\} + 1;$$

$$M_5 := \sup\{||J_*v - \lambda(Kv+u) - \lambda\theta(J_*v - J_*v_1) - J_*v_1|| : (u,v) \in B_{E\times E^*}, \lambda, \theta \in (0,1)\} + 1;$$

$$M_6 := \sup\{||J_*^{-1}(J_*v - \lambda(Kv+u) - \lambda\theta(J_*v - J_*v_1)) - v^*|| : (u,v) \in B_{E\times E^*}, \lambda, \theta \in (0,1)\} + 1;$$

$$M_1^* = 2LM_1M_4$$

$$M_2^* = 2LM_2M_6$$

$$M^* =: \max\{2c_2M_1, 2c_2M_2, 2c_2M_3, 2c_2M_5, 2M_1M_4 + 2M_2M_6\};$$

where c_2 and L are the constants appearing in Lemma 2.6 and $B_{E\times E^*} = \{w \in E \times E^* : \Lambda(w^*, w) \leq r\}$. Let $r > 0$ be such that

$$\frac{r}{5} \geq \Lambda(w^*, w_1).$$

Define

$$\gamma_0 := \min\Big\{1, \frac{r}{5M^*}, \frac{1}{M_1^*}, \frac{1}{M_2^*}\Big\}.$$

Claim:

$\Lambda(w^*, w_n) \leq r, \ \forall\ n \geq 1$. The proof of this claim is by induction. By construction, we have $\Lambda(w^*, w_1) \leq r$. Assume that $\Lambda(w^*, w_n) \leq r$ for some $n \geq 1$. This implies that

$$\phi(u^*, u_n) + \phi(v^*, v_n) \leq r, \text{ for some } n \geq 1.$$

We prove that $\Lambda(w^*, w_{n+1}) \leq r$. Suppose, for contradiction, that this is not the case, then $\Lambda(w^*, w_{n+1}) > r$. From lemma (2.6), we have that

$$\begin{aligned}(2L)^{-1}\delta_E(c_2^{-1}||u_{n+1} - u_n||) \quad &\leq ||Ju_{n+1} - Ju_n||||u_{n+1} - u_n|| \\ &\leq \lambda_n M_1 M_4.\end{aligned}$$

This yields

$$||u_{n+1} - u_n|| \leq c_2\delta_E^{-1}(\lambda_n M_1^*), \quad M_1^* = 2LM_1M_4. \tag{3.4}$$

Also, using lemma 2.7, we obtain

$$||v_{n+1} - v_n|| \leq c_2\delta_{E^*}^{-1}(\lambda_n M_2^*), \quad M_2^* = 2LM_2M_6. \tag{3.5}$$

Using the definition of u_{n+1}, equation (2.4) and inequality (2.5) with

$$y^* = \lambda_n(Fu_n - v_n) + \lambda_n\theta_n(Ju_n - Ju_1),$$

we obtain

$$\begin{aligned}
\phi(u^*, u_{n+1}) &= V(u^*, Ju_n - \lambda_n(Fu_n - v_n) - \lambda_n\theta_n(Ju_n - Ju_1)) \\
&\leq V(u^*, Ju_n) - 2\Big\langle J^{-1}(Ju_n - \lambda_n(Fu_n - v_n) \\
&\quad -\lambda_n\theta_n(Ju_n - Ju_1)) - u^*, \lambda_n(Fu_n - v_n) + \lambda_n\theta_n(Ju_n - Ju_1)\Big\rangle \\
&= \phi(u^*, u_n) - 2\Big\langle u_{n+1} - u^*, \lambda_n\Big((Fu_n - v_n) + \theta_n(Ju_n - Ju_1\Big)\Big\rangle \\
&= \phi(u^*, u_n) - 2\Big\langle u_{n+1} - u_n, \lambda_n\Big((Fu_n - v_n) + \theta_n(Ju_n - Ju_1)\Big)\Big\rangle \\
&\quad -2\Big\langle u_n - u^*, \lambda_n\Big((Fu_n - v_n) + \theta_n(Ju_n - Ju_1)\Big)\Big\rangle.
\end{aligned}$$

Which implies that

$$\begin{aligned}
\phi(u^*, u_{n+1}) &\leq \phi(u^*, u_n) + 2\|u_{n+1} - u_n \\
&\quad \|\Big|\Big|\lambda_n\Big((Fu_n - v_n) + \theta_n(Ju_n - Ju_1)\Big)\Big|\Big| \\
&\quad -2\lambda_n\Big\langle u_n - u^*, \Big((Fu_n - v_n) + \theta_n(Ju_n - Ju_1)\Big)\Big\rangle.
\end{aligned} \tag{3.6}$$

Observe that using the monotonicity of F and J, we have:

$$\begin{aligned}
&\Big\langle u_n - u^*, \Big((Fu_n - v_n) + \theta_n(Ju_n - Ju_1)\Big)\Big\rangle \\
\geq\ &\langle u_n - u^*, (Fu^* - v_n)\rangle + \theta_n\langle u_n - u_{n+1}, Ju_n - Ju_{n+1}\rangle \\
&+\theta_n\langle u_n - u_{n+1}, Ju_{n+1} - Ju_1\rangle + \theta_n\langle u_{n+1} - u^*, Ju_n - Ju_{n+1}\rangle \\
&+\theta_n\langle u_{n+1} - u^*, Ju_{n+1} - Ju_1\rangle \\
\geq\ &\langle u_n - u^*, (Fu^* - v_n)\rangle - \theta_n||u_n - u_{n+1}||||Ju_{n+1} - Ju_1|| \\
&-\theta_n||u_{n+1} - u^*||||Ju_n - Ju_{n+1}|| + \theta_n\langle u_{n+1} - u^*, Ju_{n+1} - Ju_1\rangle.
\end{aligned}$$

Substituting into inequality (3.6), we obtain

$$\begin{aligned}
\phi(u^*, u_{n+1}) &\leq \phi(u^*, u_n) + 2\|u_{n+1} - u_n\|\Big|\Big|\lambda_n\Big((Fu_n - v_n) + \theta_n(Ju_n - Ju_1)\Big)\Big|\Big| \\
&\quad -2\lambda_n\langle u_n - u^*, (Fu^* - v_n)\rangle + 2\lambda_n\theta_n||u_n - u_{n+1}||||Ju_{n+1} - Ju_1|| \\
&\quad +2\lambda_n\theta_n||u_{n+1} - u^*||||Ju_n - Ju_{n+1}|| - 2\lambda_n\theta_n\langle u_{n+1} - u^*, Ju_{n+1} - Ju_1\rangle.
\end{aligned}$$

Now, using inequality (2.10) of lemma 2.8 and inequality (3.4), we have that

$$\begin{aligned}\phi(u^*,u_{n+1}) &\leq \phi(u^*,u_n)-\lambda_n\theta_n\phi(u^*,u_{n+1})+\lambda_n\theta_n\phi(u^*,u_1) \\ &+\lambda_n\delta_E^{-1}(\lambda_n M_1^*)(2c_2M_1)+2\lambda_n\theta_n(\lambda_n M_1)M_4 \\ &+\lambda_n\theta_n[\delta_E^{-1}(\lambda_n M_1^*)(2c_2M_3)]-2\lambda_n\langle u_n-u^*,(Fu^*-v_n)\rangle.\end{aligned} \tag{3.7}$$

Similarly, using the fact that K and J_* are monotone, inequality (2.10) of lemma 2.8 and inequality (3.5), we have

$$\begin{aligned}\phi(v^*,v_{n+1}) &\leq \phi(v^*,v_n)-\lambda_n\theta_n\phi(v^*,v_{n+1})+\lambda_n\theta_n\phi(v^*,v_1) \\ &+\lambda_n\delta_{E^*}^{-1}(\lambda_n M_2^*)(2c_2M_2)+2\lambda_n\theta_n(\lambda_n M_2)M_6 \\ &+\lambda_n\theta_n[\delta_{E^*}^{-1}(\lambda_n M_2^*)(2c_2M_5)]-2\lambda_n\langle v_n-v^*,(Kv^*+u_n)\rangle.\end{aligned} \tag{3.8}$$

Observe that since $u^*+KFu^*=0$, setting $Fu^*=v^*$, we obtain that $Kv^*=-u^*$, and these equations yield

$$2\lambda_n\langle u_n-u^*,(v_n-Fu^*)\rangle+2\lambda_n\langle v_n-v^*,-(Kv^*+u_n)\rangle=0,$$

so that adding (3.7) and (3.8), we obtain

$$\begin{aligned}r &< \Lambda(w^*,w_{n+1}) \\ &\leq \Lambda(w^*,w_n)-\lambda_n\theta_n\Lambda(w^*,w_{n+1})+\lambda_n\theta_n\Lambda(w^*,w_1)+\lambda_n[\delta_E^{-1}(\lambda_n M_1^*)+\delta_{E^*}^{-1}(\lambda_n M_2^*)]M^* \\ &\quad +\lambda_n\theta_n[\delta_E^{-1}(\lambda_n M_1^*)+\delta_{E^*}^{-1}(\lambda_n M_2^*)]M^*+\lambda_n\theta_n(\lambda_n M^*).\end{aligned}$$

So we have

$$\begin{aligned}r<\Lambda(w^*,w_{n+1}) &\leq \Lambda(w^*,w_n)-\lambda_n\theta_n\Lambda(w^*,w_{n+1})+\lambda_n\theta_n\Lambda(w^*,w_1) \\ &\quad +\lambda_n\theta_n(\gamma_0\theta_n)M^*+\lambda_n\theta_n\gamma_0M^*+\lambda_n\theta_n\gamma_0M^* \\ &\leq r-\lambda_n\theta_n r+\lambda_n\theta_n\frac{r}{5}+\lambda_n\theta_n\frac{r}{5}+\lambda_n\theta_n\frac{r}{5}+\lambda_n\theta_n\frac{r}{5} \\ &< r.\end{aligned}$$

This is a contradiction, hence, $\Lambda(w^*,w_{n+1})\leq r$ and so $\Lambda(w^*,w_n)\leq r$ for all $n\geq 1$. As a result, we have $\phi(u^*,u_n)\leq r$ and $\phi(v^*,v_n)\leq r$ for all $n\geq 1$. Thus from inequality (2.2), we have that $\{u_n\}_{n\geq 1}$ and $\{v_n\}_{n\geq 1}$ are bounded.

We now prove that $\{u_n\}$ converges strongly to a solution of the Hammerstein equation.

Using equation (2.4), lemmas 2.10 and 2.2, with $y^* = \lambda_n(Fu_n - v_n) + \lambda_n\theta_n(Ju_n - Ju_1)$, we have

$$
\begin{aligned}
\phi(y_n, u_{n+1}) &= \phi(y_n, J^{-1}(Ju_n - \lambda_n(Fu_n - v_n) - \lambda_n\theta_n(Ju_n - Ju_1))) \\
&\leq V(y_n, Ju_n) - 2\langle u_{n+1} - y_n, \lambda_n(Fu_n - v_n) + \lambda_n\theta_n(Ju_n - Ju_1)\rangle \\
&= \phi(u_n, y_n) + 2\langle u_n, Jy_n\rangle - 2\langle y_n, Ju_n\rangle - 2\lambda_n\langle u_{n+1} - y_n, (Fu_n - v_n) \\
&\quad +\theta_n(Ju_n - Ju_1)\rangle \\
&= V(u_n, Jy_n) + 2\langle u_n, Jy_n\rangle - 2\langle y_n, Ju_n\rangle - 2\lambda_n\langle u_{n+1} - y_n, (Fu_n - v_n) \\
&\quad +\theta_n(Ju_n - Ju_1)\rangle \\
&\leq V(u_n, Jy_{n-1}) - 2\langle y_n - u_n, Jy_{n-1} - Jy_n\rangle + 2\langle u_n, Jy_n)\rangle - 2\langle y_n, Ju_n)\rangle \\
&\quad -2\lambda_n\langle u_{n+1} - y_n, (Fu_n - v_n) + \theta_n(Ju_n - Ju_1)\rangle \\
&= \phi(y_{n-1}, u_n) + 2\langle y_{n-1}, Ju_n\rangle - 2\langle u_n, Jy_{n-1}\rangle - 2\langle y_n - u_n, Jy_{n-1} - Jy_n\rangle \\
&\quad +2\langle u_n, Jy_n\rangle - 2\langle y_n, Ju_n\rangle - 2\lambda_n\langle u_{n+1} - y_n, (Fu_n - v_n) + \theta_n(Ju_n - Ju_1)\rangle \\
&= \phi(y_{n-1}, u_n) + 2\langle y_{n-1} - y_n, Ju_n\rangle + 2\langle y_n, Jy_n - Jy_{n-1}\rangle \\
&\quad -2\lambda_n\langle u_{n+1} - y_n, (Fu_n - v_n) + \theta_n(Ju_n - Ju_1)\rangle.
\end{aligned}
$$

Applying monotonicity of F and using equations (2.20), (2.11), (3.4), (2.13) and (2.14), we have

$$
\begin{aligned}
\phi(y_n, u_{n+1}) &\leq \phi(y_{n-1}, u_n) + ||y_n - y_{n-1}||C_1 + ||Jy_n - Jy_{n-1}||C_2 + 2\lambda_n||u_{n+1} - u_n||M_1 \\
&\quad -2\lambda_n\langle u_n - y_n, (Fu_n - v_n) + \theta_n(Ju_n - Ju_1)\rangle \\
&= \phi(y_{n-1}, u_n) + ||y_n - y_{n-1}||C_1 + ||Jy_n - Jy_{n-1}||C_2 + 2\lambda_n||u_{n+1} - u_n||M_1 \\
&\quad -2\lambda_n\langle u_n - y_n, (Fu_n - v_n) + \theta_n(Ju_n - Jy_n - \frac{1}{\theta_n}(Fy_n - y_n^*))\rangle \\
&\leq \phi(y_{n-1}, u_n) + ||y_n - y_{n-1}||C_1 + ||Jy_n - Jy_{n-1}||C_2 + 2\lambda_n||u_{n+1} - u_n||M_1 \\
&\quad -2\lambda_n\langle u_n - y_n, y_n^* - v_n\rangle - 2\lambda_n\theta_n\langle u_n - y_{n-1}, Ju_n - Jy_{n-1}\rangle \\
&\quad -2\lambda_n\theta_n\langle u_n - y_{n-1}, Jy_{n-1} - Jy_n\rangle - 2\lambda_n\theta_n\langle y_{n-1} - y_n, Ju_n - Jy_n\rangle \\
&\leq \phi(y_{n-1}, u_n) + ||y_n - y_{n-1}||C_1 + ||Jy_n - Jy_{n-1}||C_2 + 2\lambda_n||u_{n+1} - u_n||M_1 \\
&\quad -\lambda_n\theta_n\phi(y_{n-1}, u_n) + ||Jy_n - Jy_{n-1}||C_3 + ||y_n - y_{n-1}||C_4 - 2\lambda_n\langle u_n - y_n, y_n^* - v_n\rangle \\
&\leq \phi(y_{n-1}, u_n) - \lambda_n\theta_n\phi(y_{n-1}, u_n) + \delta_E^{-1}\Big(\frac{\theta_{n-1} - \theta_n}{\theta_n}K\Big)C_5 \qquad (3.9) \\
&\quad +\delta_{E^*}^{-1}\Big(\frac{\theta_{n-1} - \theta_n}{\theta_n}K\Big)C_6 + 2c_2\lambda_n\delta_E^{-1}(\lambda_n M_1^*)M_1 - 2\lambda_n\langle u_n - y_n, y_n^* - v_n\rangle,
\end{aligned}
$$

where $C_1,\ C_2,\ C_3,\ C_4$ are positive constants and $C_5 = c_2C_1 + c_2C_4,\ C_6 = c_2C_2 + c_2C_3$.

Similarly, applying monotonicity of K and using equations (2.21), (2.11), (3.5), (2.13) and (2.14), we have

$$\begin{aligned}\phi(y_n^*, v_{n+1}) &\leq \phi(y_{n-1}^*, v_n) - \lambda_n\theta_n\phi(y_{n-1}^*, v_n) + \delta_E^{-1}\Big(\frac{\theta_{n-1}-\theta_n}{\theta_n}K\Big)C_5^* \qquad (3.10)\\ &+\delta_{E^*}^{-1}\Big(\frac{\theta_{n-1}-\theta_n}{\theta_n}K\Big)C_6^* + 2c_2\lambda_n\delta_{E^*}^{-1}(\lambda_nM_2^*)M_2 - 2\lambda_n\langle v_n - y_n^*, u_n - y_n\rangle.\end{aligned}$$

where C_5^* and C_6^* are positive constants.

Hence, adding equations (3.9) and (3.10) we have

$$\begin{aligned}\Lambda(p_n, w_{n+1}) &\leq \Lambda(p_{n-1}, w_n) - \lambda_n\theta_n\Big(\phi(y_{n-1}, u_n) + \phi(y_{n-1}^*, v_n)\Big) + 2c_2\lambda_n\delta_E^{-1}(\lambda_nM_1^*)M_1\\ &+2c_2\lambda_n\delta_{E^*}^{-1}(\lambda_nM_2^*)M_2 + \delta_E^{-1}\Big(\frac{\theta_{n-1}-\theta_n}{\theta_n}K\Big)C_5 + \delta_E^{-1}\Big(\frac{\theta_{n-1}-\theta_n}{\theta_n}K\Big)C_5^*\\ &+\delta_{E^*}^{-1}\Big(\frac{\theta_{n-1}-\theta_n}{\theta_n}K\Big)C_6 + \delta_{E^*}^{-1}\Big(\frac{\theta_{n-1}-\theta_n}{\theta_n}K\Big)C_6^*.\end{aligned}$$

where $p_n = [y_n, y_n^*]$ is as in Remark 5. Letting $M^* = \max\{C_5 + C_5^*, C_6 + C_6^*, 2c_2M_1, 2c_2M_2\}$, we have

$$\begin{aligned}\Lambda(p_n, w_{n+1}) &\leq \Lambda(p_{n-1}, w_n) - \lambda_n\theta_n\Lambda(p_{n-1}, w_n) + \lambda_n\delta_E^{-1}(\lambda_nM_1^*)M^* + \lambda_n\delta_{E^*}^{-1}(\lambda_nM_2^*)M^*\\ &+\delta_E^{-1}\Big(\frac{\theta_{n-1}-\theta_n}{\theta_n}K\Big)M^* + \delta_{E^*}^{-1}\Big(\frac{\theta_{n-1}-\theta_n}{\theta_n}K\Big)M^*.\end{aligned}$$

Setting

$\rho_n := \Lambda(p_{n-1}, w_n)$; $\beta_n := \lambda_n\theta_n$; $\zeta_n := \Big(\frac{\delta_E^{-1}\big(\frac{\theta_{n-1}-\theta_n}{\theta_n}K\big)M^*}{\lambda_n\theta_n} + \frac{\delta_{E^*}^{-1}\big(\frac{\theta_{n-1}-\theta_n}{\theta_n}K\big)M^*}{\lambda_n\theta_n}\Big)$;
$\gamma_n := \lambda_n\delta_E^{-1}(\lambda_nM_1^*)M^* + \lambda_n\delta_{E^*}^{-1}(\lambda_nM_2^*)M^*$;
we have

$$\rho_{n+1} \leq (1-\beta_n)\rho_n + \beta_n\zeta_n + \gamma_n,\ n \geq 1.$$

It now follows from Lemma (2.4) that $\rho_n \to 0$ as $n \to \infty$, i.e., $\Lambda(p_{n-1}, w_n) \to 0$ as $n \to \infty$. Consequently, by Lemma 2.3, we obtain that $\lim||u_n - y_{n-1}|| = 0$. Hence using Remark 6, we have that the sequence $\{u_n\}_{n=1}^{\infty}$ converges strongly to a solution of (1.2). □

Remark 7 : We have (see e.g., Alber [3]) for $p > 1, q > 1, X = L^p, X^* = L^q,\ that$

$$\delta_{X^*}(\epsilon) = 1 - \Big(1 - \Big(\frac{\epsilon}{2}\Big)^q\Big)^{\frac{1}{q}},$$

and thus obtain also that

$$\delta_{X^*}^{-1}(\epsilon) = 2[1-(1-\epsilon)^q]^{\frac{1}{q}} \leq 2q^{\frac{1}{q}}\epsilon^{\frac{1}{q}}.$$

(The last inequality follows since $(1-\epsilon)^q > 1-q\epsilon$, for $q>1$).

Prototypes for our result are the following:

$$\theta_n = (n+1)^{-b}, \text{and } \lambda_n = (n+1)^{-a}\ n \geq 1,$$

where

$$0 < b < \frac{a}{r},\ a+b < \frac{1}{r},\ b < \frac{1}{K};\ \text{where } K > 0 \text{ is as defined in Lemma 2.6},\ r = max\{p,q\}.$$

For example, without loss of generality, if we set $r=p$, then taking

$$a := \frac{1}{(p+1)};\ \ b := min\Big\{\frac{1}{2K}, \frac{1}{2p(p+1)}\Big\},$$

conditions (i) to (iv) are satisfied.

Remark 8 : Theorem (3.1) is an extension of Theorem (1.2) to uniformly convex and uniformly smooth spaces.

Remark 9 : (see e.g., Alber [3], p.36). The analytical representations of duality mappings are known in a number of Banach spaces. For instance, in the spaces l^p, $L^p(G)$ and $W_m^p(G), p \in (1,\infty)$, $p^{-1}+q^{-1}=1$, respectively,

$$Jx = ||x||_{l^p}^{2-p} y \in l^q,\ \ y = (|x_1|^{p-2}x_1, |x_2|^{p-2}x_2, ...),\ \ x = (x_1, x_2, ...),$$

$$Jx = ||x||_{L^p}^{2-p}|x(s)|^{p-2}x(s) \in L^q(G),\ \ s \in G,$$

and

$$Jx = ||x||_{W_m^p}^{2-p} \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^{\alpha}(|D^{\alpha}x(s)|^{p-2}D^{\alpha}x(s)) \in W_{-m}^q(G), m > 0, s \in G$$

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REFERENCES

1. Y. Alber, *Metric and generalized projection operators in Banach spaces: Properties and applications*. In: Theory and Applications of Nonlinear Operators of Accretive and Monotone Type, (A. G. Kartsatos, Ed.), Marcel Dekker, New York (1996), pp. 15-50.
2. Y. Alber and S. Guerre-Delabriere, On the projection methods for fixed point problems, *Analysis (Munich)*, **21**(1) (2001), 17-39.
3. Y. Alber and I. Ryazantseva, *Nonlinear ill posed problems of monotone type*, Springer, London, UK, 2006, *J. Nonlinear Convex Anal.*, **15**(5) (2014), 851-865.
4. H. Brézis and F. E. Browder, Some new results about Hammerstein equations, *Bull. Amer. Math. Soc.*, **80** (1974), 567-572.
5. H. Brézis and F. E. Browder, Existence theorems for nonlinear integral equations of Hammerstein type, *Bull. Amer. Math. Soc.*, **81** (1975), 73-78.
6. H. Brézis and F. E. Browder, Nonlinear integral equations and systems of Hammerstein type, *Bull. Amer. Math. Soc.*, **82** (1976), 115-147.
7. F. E. Browder, *Nonlinera functional analysis and nonlinear integral equations of Hammerstein and Uryshon type*, Contributions to Nonlinear Functional Analysis, Academic Press, (1971), 425-500.
8. F. E. Browder, Existence and perturbation theorems for nonlinear maximal monotone operators in Banach spaces, *Bull. Amer. Math. Soc.*, **73**(3) (1967), 322-327.
9. F. E. Browder, D. G. De Figueiredo and C. P. Gupta, Maximal mono-tone operators and nonlinear integral equations of Hammerstein type, *Bull. Amer. Math. Soc.*, **76** (1970), 700-705.
10. C. E. Chidume. *Geometric properties of banach spaces and nonlinear iterations*, vol. 1965 of Lectures Notes in Mathematics, Springer, London, UK, 2009.
11. C. E. Chidume and N. Djitte, Approximation of solutions of nonlinear integral equations of Hammerstein type, *ISRN Mathematical Analysis*, **2012**, Article ID 169751, 12 pages, (2012).
12. C. E. Chidume and N. Djitte, Iterative approximation of solutions of nonlinear equations of Hammerstein-type, *Nonlinear Anal.*, **70** (2009), 4086-4092.
13. C. E. Chidume and N. Djitte, Convergence theorems for solutions of Hammerstein equations with accretive-type nonlinear operators, *PanAm. Math. J.*, **22**(2) (2012), 19-29.
14. C. E. Chidume and N. Djitte, Approximation of solutions of Hammerstein equations with bounded strongly accretive nonlinear operator, *Nonlinear Anal.*, **70** (2009), 4071-4078.
15. C. E. Chidume and E. U. Ofoedu, Solution of nonlinear integral equations of Hammerstein-type, *Nonlinear Anal.*, **74** (2011), 4293-4299.

16. C. E. Chidume and M. O. Osilike, Iterative solutions of nonlinear integral equations of Hammerstein-type, *J. Nigerian Math. Soc.*, **11** (1992), 9-18 (MR96c:65207).

17. C. E. Chidume and M. O. Osilike, Iterative solutions of nonlinear accretive operator equations in arbitrary Banach spaces, *Nonlinear Analysis-Theory Methods & Applications*, **36** (1999), 863-872.

18. C. E. Chidume and Y. Shehu, Approximation of solutions of generalised equations of Hammerstein-type, *Comput. Math. Appl.*, **63** (2012), 966-974.

19. C. E. Chidume and H. Zegeye, Approximation of solutions of nonlinear equations of Hammerstein-type in Hilbert space, *Proc. Amer. Math. Soc.*, **133**(3) (2005), 851-858.

20. C. E. Chidume and H. Zegeye, Iterative approximation of solutions of nonlinear equation of Hammerstein-type, *Abstr. Appl. Anal.*, **6** (2003), 353-367.

21. C. E. Chidume and H. Zegeye, Approximation os solutions of nonlinear equations of monotone and Hammerstein-type, *Appl. Anal.*, **82**(8) (2003), 747-758.

22. V. Dolezale, *Monotone operators and its applications in automation and network theory*, Studies in Automation and Control, **3**, Elsevier Science publ., New York, 1979.

23. S. Kamimura and W. Takahashi, Strong convergence of a proximal-type algorithm in a Banach space, *SIAM J. Optim.*, **13**(3) (2002), 938-945.

24. R. I. Kačurovskii, On monotone operators and convex functionals, *Uspekhi Mathematicheskikh Nauk*, **15**(4) (1960), 213-215.

25. W. R. Mann, Mean value methods in iteration, *Proc. Amer. Math. Soc.*, **4** (1953), 506-510.

26. G. J. Minty, Monotone (nonlinear) operators in Hilbert space, *Duke Math. J.*, **29**(4) (1962), 341-346.

27. D. Pascali and S. Sburian, *Nonlinear mappings of monotone type*, Editura Academia Bucuresti, Romania, 1978.

28. S. Reich, *A weak convergence theorem for the alternating methods with Bergman distance*, In: A. G. Kartsatos (Ed.), Theory and Applications of Nonlinear Operators of Accretive and Monotone Type, in Lecture notes in pure and Appl. Math., **178** (1996), Dekker, New York, 313-318.

29. S. Reich, *Constructive techniques for accretive and monotone operators*, Applied non-linear analysis, Academic Press, New York (1979), 335-345.

30. H. K. Xu, Iterative algorithms for nonlinear operators, *J. London Math. Soc.*, **66**(2) (2002), 240-256.

31. E. H. Zarantonello, Solving functional equations by contractive averaging, *Tech. Rep.*, **160**, U. S. Army Math. Research Center, Madison, Wisconsin, 1960.