Indian J. Pure Appl. Math., **48**(3): 423-427, September 2017

DOI: 10.1007/s13226-017-0240-9

ON ABSOLUTE CENTRAL AUTOMORPHISMS OF A GROUP FIXING THE CENTER ELEMENTWISE

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(*Received* 16 *August* 2016; *after final revision* 25 *January* 2017;
accepted 24 *March* 2017)

Let G be a finite p-group. The automorphism α of a group G is said to be an absolute central automorphism, if for all $x \in G$, $x^{-1}x^{\alpha} \in L(G)$, where $L(G)$ is the absolute center of G.

In this paper, we obtain a necessary and sufficient condition that each absolute central automorphism of G fixes the center element-wise.

Key words : Autonilpotent group; absolute central automorphism; absolute center of group; purely non-abelian group; autocommutator subgroup.

1. Introduction and Results

Throughout, p denotes a prime number. Let G be a finite group. We denote by G', $Z(G)$, $\phi(G)$, and $Aut(G)$, respectively, the commutator subgroup, the center, the Frattini subgroup, and the automorphism group of G. For a group H and abelian group K, $Hom(H, K)$ denotes the group of all homomorphisms from H to K. If $\alpha \in Aut(G)$ and $g \in G$ then, $[g, \alpha] = g^{-1}g^{\alpha} = g^{-1}\alpha(g)$ is the autocommutator of g and α. Clearly for $x \in G$, by taking $\alpha = \varphi_x$ (an inner automorphism) we have $[g, \varphi_x] = g^{-1}g^{\varphi_x} = g^{-1}x^{-1}gx$, which is the ordinary commutator for the element g and x of G. The subgroup $K(G) = [G, Aut(G)] = \langle [g, \alpha] | g \in G, \alpha \in Aut(G) \rangle$ is called the autocommutator subgroup of G. (see [3, 4]) We may define the autocommutator of higher weight inductively as follows:

$$[g, \alpha_1, ..., \alpha_n] = \big[[g, \alpha_1, ..., \alpha_{n-1}], \alpha_n\big], \text{ for all } \alpha_1, \alpha_2, ..., \alpha_n \in Aut(G), g \in G \text{ and } n \geq 1.$$

Assume $K_0(G) = G$ and $K_1(G) = K(G)$. Then for $n \geq 1$ we may define:

$K_n(G) = [K_{n-1}(G), Aut(G)] = \big\langle [g, \alpha_1, \alpha_2, ..., \alpha_n] | g \in G, \alpha_1, \alpha_2, ..., \alpha_n \in Aut(G) \big\rangle$. One can easily see that $\gamma_n(G) \leq K_n(G)$, $n \geq 1$ and $K_n(G)$ is characteristic subgroup of G. Hence we obtain the following descending series of G :

$$G \supseteq K_1(G) = K(G) \supseteq K_2(G) \supseteq ... \supseteq K_n(G) \supseteq ...$$

The absolute center of G is defined as follows:

$L(G) = \{x \in G \mid [x, \alpha] = 1$, for all $\alpha \in Aut(G)\}$, which is contained in $Z(G)$, the center of G. Now assume $L_1(G) = L(G)$. Then n^{th}-absolute center of G is defined in the following way $L_n(G)/L_{n-1}(G) = L\big(G/L_{n-1}(G)\big)$ for $n \geq 2$. Now we recall (from [4]) a group G is an autonilpotent group if $L_n(G) = G$ for some $n \geq 1$. Since (from [3]) $L_n(G) \leq Z_n(G)$ so every autonilpotent group is nilpotent. One observe (from [4]) that if $L_n(G) = G$ then, $K_n(G) =< 1 >$.

An automorphism α called absolute central if $[g, \alpha] \in L(G)$ for all $g \in G$ [3]. We define the subgroup $Var(G) = \{\alpha \in Aut(G) | [g, \alpha] \in L(G)$ for all $g \in G\}$ which is normal subgroup of $Aut(G)$. For a group G we define [3]:

$C_{Aut(G)}(Var(G)) = \{\alpha \in Aut(G); \alpha\beta = \beta\alpha$ for all $\beta \in Var(G)\}$ the centralizer of $Var(G)$ in $Aut(G)$. We denote by $C_{Var(G)}(Z(G))$ the group of all absolute central automorphisms of G fixing $Z(G)$ element-wise and $E(G) = [G, C_{Aut(G)}(Var(G))]$. One can easily see that $E(G)$ is subgroup of $K(G)$ which is contained in $K(G)$. If G be a group then, $E(G)$ is characteristic subgroup of G and containing G'. ($G' = [G, Inn(G)]$) [3].

Lemma 1.1 — Let G be an autonilpotent group. Then for any nontrivial normal subgroup N of G, $L(G) \cap N \neq 1$.

PROOF : By induction we obtain

$L_i(G) = \{x \in G | [x, \alpha_1, ..., \alpha_i] = 1$ for all $\alpha_1, ..., \alpha_i \in Aut(G)\}$.

Since G is an autonilpotent group then, for some natural number n, $L_n(G) = G$. So there exist at least a positive integer i, such that $N \cap L_i(G) \neq< 1 >$. Now we have $[N \cap L_i(G), Aut(G)] \subseteq N \cap L_{i-1}(G) =< 1 >$ and $N \cap L_i(G) \subseteq N \cap L(G)$. Hence $N \cap L(G) = N \cap L_i(G) \neq< 1 >$. Specially by taken $N = L(G)$ we obtain $L(G) \neq 1$. □

The following lemma gives the important property of $E(G)$; while $K(G)$ does not carry over such a property.

Lemma 1.2 [3] — Let G be a group. Then $Var(G)$ acts trivially on the subgroup $E(G)$ of G.

A non-abelian group G is called purely non-abelian if it has no non-trivial abelian direct factor.

For a finite p-group G we define $\Omega_1(G) = \langle x \in G \mid x^p = 1 \rangle$.

We recall that an automorphism α is called central automorphism if $[g, \alpha] = g^{-1}g^{\alpha} \in Z(G)$ and define $Aut_c(G) = \{\alpha \in Aut(G) \big| [g, \alpha] \in Z(G)\}$ which is a normal subgroup of $Aut(G)$ [1]. Adney and Yen in [1] prove that if G is a purely non-abelian finite group then, there exist a bijection between $Aut_c(G)$ and $Hom\big(G/G', Z(G)\big)$. Also, Jamali and mousavi in [2] prove that if G is a finite group such that $Z(G) \leq G'$ then, $Aut_c(G) \cong Hom\big(G/G', Z(G)\big)$.

Similarly we have the following theorems about absolute central automorphisms [3] :

***Theorem* 1.3** — [3]. *Let G be a group such that $L(G)$ is contained in $E(G)$. Then:*

$$Var(G) \cong Hom\big(G/E(G), L(G)\big).$$

***Theorem* 1.4** — [3]. *Let G be a purely non-abelian finite group. Then:*
$Var(G) \cong Hom\big(G, L(G)\big)$.

The following result gives a description of the centralizer of the center of G in $Var(G)$.

Corollary 1.5 — Let G be a purely non-abelian finite group. Then:

$$Var(G) \cong Hom\big(G/E(G), L(G)\big).$$

PROOF : By Lemma 1.2 and Theorem 1.4. □

***Theorem* 1.6** — [3]. *Let G be a group. Then*
$C_{Var(G)}\big(Z(G)\big) \cong Hom\big(G/E(G)Z(G), L(G)\big)$.

Attar in [5] find necessary and sufficient condition that

$$Aut_c(G) = C_{Aut_c(G)}(Z(G)).$$

Similarly in this paper we obtain the necessary and sufficient condition that we have $Var(G) = C_{Var(G)}(Z(G))$.

Let G be a non-abelian finite p-group . Then by assumption:

$$G/E(G) = C_{p^{a_1}} \times \cdots \times C_{p^{a_k}},$$

where C_{pa_i} is a cyclic group of order p^{a_i}, and $a_1 \geq a_2 \geq \cdots \geq a_k \geq 1$. Let

$$G/E(G)Z(G) = C_{p^{b_1}} \times \cdots \times C_{p^{b_l}} \quad \text{and} \quad L(G) = C_{p^{c_1}} \times \cdots \times C_{p^{c_m}}$$

where $b_1 \geq b_2 \geq \cdots \geq b_l \geq 1$ and $c_1 \geq \cdots c_m \geq 1$. Since $G/E(G)Z(G)$ is a quotient of $G/E(G)$, we have $l \leq k$ and $b_i \leq a_i$ for all $1 \leq i \leq l$.

2. Main Result

***Theorem* 2.1** — *Let G be a non-abelian finite p-group which is autonilpotent. Then $Var(G) = C_{Var(G)}(Z(G))$ if and only if $Z(G) \leq E(G)$ or $Z(G) \leq \Phi(G)$, $k = l$ and $c_1 \leq b_t$ where t is the largest integer between 1 and k such that $a_t > b_t$.*

PROOF : Suppose that $Var(G) = C_{Var(G)}(Z(G))$ and $Z(G) \not\leq E(G)$. We claim that $Z(G) \leq \Phi(G)$.

Assume that $Z(G)$ is not contained in $\Phi(G)$. Choose an element g in $Z(G)$ such that $g \notin M$ for some maximal subgroup M of G. Therefore $G = M\langle g\rangle$.

Let $1 \neq z \in \Omega_1(L(G)) \cap M$ (by Lemma 1.1). Then the map α define on G by $\alpha(mg^k) = mg^k z^k$ for every $m \in M$ and $k \in \{0, 1, \ldots, p-1\}$, is an absolute central automorphism. By the given hypothesis $g = \alpha(g) = gz$, whence $z = 1$, which is a contradiction. Hence $Z(G) \leq \Phi(G)$. Since $Z(G) \leq \Phi(G)$, it follows that $l = rank\big(G/E(G)Z(G)\big) = rank\big(G/E(G)\big) = k$ and G is purely non-abelian. Thus (by Corollary 1.5) we have $\big|Var(G)\big| = \big|Hom\big(G/E(G), L(G)\big)\big|$. On the other hand (by Theorem 1.6) we have

$$\big|Var(G)\big| = \big|C_{Var(G)}(Z(G))\big| = \big|Hom\big(G/E(G)Z(G), L(G)\big)\big|,$$

since $Var(G) = C_{Var(G)}(Z(G))$, therefore

$$\big|Hom\big(G/E(G)Z(G), L(G)\big)\big| = \big|Hom\big(G/E(G), L(G)\big)\big|.$$

Hence

$$\prod_{\substack{1\leq i\leq k\\ 1\leq j\leq m}} p^{\min\{a_i,c_j\}} = \prod_{\substack{1\leq i\leq l\\ 1\leq j\leq m}} p^{\min\{b_i,c_j\}}.$$

Since $a_i \geq b_i$, for all $1 \leq i < k$, we have $\min\{a_i, c_j\} \geq \min\{b_i, c_j\}$ for all $1 \leq i \leq k$, $1 \leq j \leq m$. Hence $\min\{a_i, c_j\} = \min\{b_i, c_j\}$ for all $1 \leq i \leq k$, $1 \leq j \leq m$. Since $Z(G) \not\leq E(G)$, there exists some $1 \leq i \leq k$ such that $a_i > b_i$. Let t be the largest integer between 1 and k such that $a_t > b_t$. We claim that $c_1 \leq b_t$. Suppose that $c_1 > b_t$. Thus $b_t = \min\{c_1, b_t\} = \min\{c_1, a_t\}$, which is impossible.

Conversely, if $Z(G) \leq E(G)$ then, every absolute central automorphism fixes $Z(G)$ (by Lemma 1.2) and so $Var(G) = C_{Var(G)}(Z(G))$. Now assume that $Z(G) \leq \Phi(G)$, $k = l$ and $c_1 \leq b_t$, since

$Z(G) \leq \Phi(G)$, G is purely non-abelian and so

$$\left|Var(G)\right| = \left|Hom\big(G/E(G), L(G)\big)\right| = \prod_{\substack{1\leq i\leq l\\ 1\leq j\leq m}} p^{\min\{b_i,c_j\}}.$$

Since $b_t \geq c_1$, we have

$$b_1 \geq b_2 \geq \cdots \geq b_{t-1} \geq b_t \geq c_1 \geq c_2 \geq \cdots \geq c_m \geq 1,$$

therefore $c_j \leq b_i \leq a_i$ for all $1 \leq j \leq m$ and $1 \leq i \leq t$, whence $\min\{a_i, c_j\} = c_j = \min\{b_i, c_j\}$ for all $1 \leq j \leq m$ and $1 \leq i \leq t$. Since $a_i = b_i$, for all $i > t$, we have $\min\{a_i, c_j\} = \min\{b_i, c_j\}$ for all $1 \leq j \leq m$ and $t+1 \leq i \leq t$. Thus $\min\{a_i, c_j\} = \min\{b_i, c_j\}$, for all $1 \leq i \leq k$, $1 \leq j \leq m$. Therefore, $Var(G) = C_{Var(G)}(Z(G))$(by Theorem 1.6). □

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