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FINITE GROUPS WITH GIVEN σ-EMBEDDED AND σ-n-EMBEDDED SUBGROUPS[1]

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Let G be a finite group and $\sigma = \{\sigma_i | i \in I\}$ be a partition of the set of all primes $\mathbb{P}$. A set $\mathcal{H}$ of subgroups of G is said to be a complete Hall σ-set of G if every non-identity member of $\mathcal{H}$ is a Hall σ_i-subgroup of G and $\mathcal{H}$ contains exactly one Hall σ_i-subgroup of G for every $\sigma_i \in \sigma(G)$. A subgroup H is said to be σ-permutable if G possesses a complete Hall σ-set $\mathcal{H}$ such that $HA^x = A^xH$ for all $A \in \mathcal{H}$ and all $x \in G$. Let H be a subgroup of G. Then we say that: (1) H is σ-embedded in G if there exists a σ-permutable subgroup T of G such that $HT = H^{\sigma G}$ and $H \cap T \leq H_{\sigma G}$, where $H_{\sigma G}$ is the subgroup of H generated by all those subgroups of H which are σ-permutable in G, and $H^{\sigma G}$ is the σ-permutable closure of H, that is, the intersection of all σ-permutable subgroups of G containing H. (2) H is σ-n-embedded in G if there exists a normal subgroup T of G such that $HT = H^G$ and $H \cap T \leq H_{\sigma G}$. In this paper, we study the properties of the new embedding subgroups and use them to determine the structure of finite groups.

Key words : Finite group; σ-embedded subgroup; σ-n-embedded subgroup; σ-permutable subgroup; supersoluble.

1. INTRODUCTION

Throughout this paper, all groups are finite and G always denotes a group. Moreover, n is an integer, $\mathbb{P}$ is the set of all primes. The symbol $\pi(n)$ denotes the set of all primes dividing n and $\pi(G) = \pi(|G|)$, the set of all primes dividing the order of G.

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In what follows, $\sigma = \{\sigma_i | i \in I\}$ is some partition of $\mathbb{P}$, that is, $\mathbb{P} = \bigcup_{i\in I} \sigma_i$ and $\sigma_i \cap \sigma_j = \emptyset$ for all $i \neq j$. Π is always supposed to be a non-empty subset of the set σ and $\Pi^{'} = \sigma\backslash\Pi$. We write $\sigma(n) = \{\sigma_i | \sigma_i \cap \pi(n) \neq \emptyset\}$ and $\sigma(G) = \sigma(|G|)$.

Following [1] and [2], G is said to be σ-primary if $G = 1$ or $|\sigma(G)| = 1$; n is a Π-number if $\pi(n) \subseteq \bigcup_{\sigma_i\in\Pi} \sigma_i$; a subgroup H of G is called a Π-subgroup of G if $|H|$ is a Π-number; a subgroup H of G is called a Hall Π-subgroup of G if H is a Π-subgroup of G and $|G : H|$ is a $\Pi^{'}$-number. We use $|G|_{\sigma_i}$ to denote the order of the Hall σ_i-subgroup of G, $O^{\Pi}(G)$ denotes the subgroup of G generated by all its $\Pi^{'}$-subgroups and $O_{\Pi}(G)$ denotes the subgroup of G generated by all its normal Π-subgroups. Instead of $O^{\{\sigma_i\}}(G)$ we write $O^{\sigma_i}(G)$. A set $\mathcal{H}$ of subgroups of G is said to be a complete Hall Π-set of G if every non-identity member of $\mathcal{H}$ is a Hall σ_i-subgroup of G for some $\sigma_i \in \Pi$ and $\mathcal{H}$ contains exactly one Hall σ_i-subgroup of G for every $\sigma_i \in \Pi \cap \sigma(G)$. In particular, when $\Pi = \sigma$, we call $\mathcal{H}$ a complete Hall σ-set of G. If G has a complete Hall σ-set $\mathcal{H} = \{H_1, \cdots, H_t\}$ such that $H_i \neq 1$ and $H_iH_j = H_jH_i$ for all i, j, then $\mathcal{H}$ is said to be a σ-basis of G. Following [1] and [3], G is said to be Π-*full* (resp. σ-*full* or a σ-*group*) if G possesses a complete Hall Π-set (resp. Hall σ-set); *a* Π-*full (resp.* σ-*full) group of Sylow type* if every subgroup of G is a D_{σ_i}-group for all $\sigma_i \in \Pi \cap \sigma(G)$ (resp. $\sigma_i \in \sigma(G)$); a subgroup H of G is called σ-subnormal in G if there is a subgroup chain $H = H_0 \leq H_1 \leq \cdots \leq H_t = G$ such that either H_{i-1} is normal in H_i or $H_i/(H_{i-1})_{H_i}$ is σ-primary for all $i = 1, 2, \cdots, t$; a group G is called σ-soluble if every chief factor of G is σ-primary.

It is well known that embedded subgroups and supplemented subgroups play an important role in the theory of finite group. For example, a subgroup H of G is said to be c-normal [4] in G if G has a normal subgroup T of G such that $G = HT$ and $H \cap T \leq H_G$, where H_G is the normal core of H. A subgroup H of G is called s-embedded [5] in G if G has an s-permutable subgroup T such that $H^{sG} = HT$ and $H \cap T \leq H_{sG}$, where H_{sG} is the subgroup of H generated by all those subgroups of H which are s-permutable in G (note that a subgroup A of G is said to be s-permutable in G if $AP = PA$ for any Sylow subgroup P of G) and H^{sG} the intersection of all s-permutable subgroups of G containing H. A subgroup H of G is called n-embedded [5] in G if G has a normal subgroup T such that $H^G = HT$ and $H \cap T \leq H_{sG}$. A subgroup H of G is called σ-permutable [1] in G if G possesses a complete Hall σ-set $\mathcal{H}$ such that $HA^x = A^xH$ for all $A \in \mathcal{H}$ and all $x \in G$. Note that in the case when $\sigma = \{\{2\}, \{3\}, \cdots\}$, then a σ-permutable subgroup is just an s-permutable subgroup. By using the above embedded subgroups and supplemented subgroups, people have obtained a series of interesting results (see, for example [1-8]). Based on this fact, we consider the following new embedded subgroups:

Definition 1.1 — Let H be a subgroup of a group G. We say that:

(1) H is σ-embedded in G if there exists a σ-permutable subgroup T of G such that $HT = H^{\sigma G}$ and $H \cap T \leq H_{\sigma G}$.

(2) H is σ-n-embedded in G if there exists a normal subgroup T of G such that $HT = H^G$ and $H \cap T \leq H_{\sigma G}$.

Here $H_{\sigma G}$ is the σ-core of H, that is, the subgroup of H generated by all those subgroups of H which are σ-permutable in G and $H^{\sigma G}$ is the σ-permutable closure of H, that is, the intersection of all σ-permutable subgroups of G containing H.

It is clear that every c-normal subgroup, every n-embedded subgroup and every σ-permutable subgroup are σ-n-embedded in G. Moreover, if G is a σ-full group of Sylow type, then every s-embedded subgroup, every n-embedded subgroup, every σ-n-embedded subgroup and every σ-permutable subgroup of G are all σ-embedded subgroup of G. But, the following example shows that the converse is not true.

Example 1.2 : Let $G = (C_{29} \rtimes C_7) \times A_5$, where $C_{29} \rtimes C_7$ is a non-abelian group of order 203 and A_5 is the alternating group of degree 5. Let B be a subgroup of A_5 of order 12. Let $\sigma = \{\sigma_1, \sigma_2, \sigma_3\}$, where $\sigma_1 = \{2, 3, 5\}$, $\sigma_2 = \{29\}$ and $\sigma_3 = \{2, 3, 5, 29\}'$. It is clear that G is σ-soluble and so G is a σ-full group of Sylow type by [9, Theorem B]. We claim that B is σ-n-embedded in G and so σ-embedded in G, but B is neither s-embedded in G, nor n-embedded in G and nor c-normal in G. In fact, it is easy to see that $\mathcal{H} = \{A_5, C_{29}, C_7\}$ is a complete Hall σ-set of G. Let $H = C_{29} \rtimes C_7$. Then $H \trianglelefteq G$ and $C_7^x \leq H \leq C_G(A_5)$ for all $x \in G$. Hence $BC_7^x = C_7^x B$ for all $x \in G$, which implies that B is σ-permutable in G, and so B is σ-n-embedded in G and also σ-embedded in G (see Lemma 2.7(1) and Lemma 2.8(1) below). Clearly, B_{sG} is subnormal in G by [10, Lemmas 2.6 and 2.8], and so is subnormal in A_5 by [11, Chapter A, 14.1]. It follows that $B_{sG} = 1$ for A_5 is a simple group. If B is s-embedded in G, then there exists an s-permutable subgroup T of G such that $BT = B^{sG}$ and $B \cap T \leq B_{sG} = 1$. Since $1 \neq B^{sG}$ is s-permutable in G by [5, Lemma 2.5], and so $B^{sG} = A_5$ by the same discussion as above. Hence $BT = A_5$ and $B \cap T = 1$. But as T is s-permutable in G and $T \leq A_5$, we have that T is subnormal in G and so T is subnormal in A_5. This implies that $T = 1$ or $T = A_5$. If $T = 1$, then $B = A_5$, a contradiction. If $T = A_5$, then $B = B \cap T = 1$, also a contradiction. Hence B is not s-embedded in G. Consequently B is neither n-embedded in G and nor c-normal in G.

Let $A = BC_7$. Then A is σ-n-embedded but is not σ-permutable in G. Indeed, if A is σ-permutable in G, then by Lemma 2.1(5) and [1, Theorem B], we have that $C_7 \leq A \leq O_\Pi(G) \leq$

$C_G(O_{\sigma_2}(G)) = C_G(C_{29})$, where $\Pi = \{\sigma_1, \sigma_3\}$, which contradicts the fact that $C_{29} \rtimes C_7$ is a non-abelian group. Thus A is not σ-permutable and $A_{\sigma G} = B$. It is clear that $T = C_{29}A_5 \trianglelefteq G$, $G = AT$ and $A \cap T = B$. Then $A^G = A(A^G \cap T)$ and $A \cap (A^G \cap T) = B = A_{\sigma G}$. This shows that A is σ-n-embedded in G.

Note that if σ is the smallest partition of $\mathbb{P}$, that is, σ_i is an one-element set for any $i \in I$, then it is clear that H is s-embedded (n-embedded) in G if and only if H is σ-embedded (σ-n-embedded) in G. It is also clear that if G is a σ-full group of Sylow type and H is σ-n-embedded in G, then H is σ-embedded in G. But the converse is not true (see [5, Example 1.2]).

Our main goal here is to prove the following theorem.

Main Theorem — *Let $\mathcal{F}$ be a saturated formation containing all supersoluble groups and G a group with a normal subgroup E such that $G/E \in \mathcal{F}$. Suppose that G is a σ-full group of Sylow type and $\mathcal{H} = \{W_1, W_2, \cdots, W_t\}$ is a complete Hall σ-set of G such that W_i is a nilpotent σ_i-subgroup for all $i = 1, \cdots, t$. If for every non-cyclic Hall σ_i-subgroup $W_i \cap E$ of E either every maximal subgroup of $W_i \cap E$ or every cyclic subgroup H of $W_i \cap E$ with prime order and order 4 (if the Sylow 2-subgroup P of E is non-abelian and $H \nleq Z_\infty(G)$) is σ-n-embedded in G for all $i = 1, 2, \cdots, t$, then $G \in \mathcal{F}$.*

In section 3, we give the proof of the main Theorem. In section 4, we will give some application of our results.

All unexplained terminologies and notations are standard, as in [11] and [12].

2. Preliminaries

Lemma 2.1 — (See [1, Lemma 2.6] and [3, Lemma 2.1]). Let A, K and N be subgroups of G. Suppose that A is σ-subnormal in G and N is normal in G. Then:

(1) $A \cap K$ is σ-subnormal in K.

(2) AN/N is σ-subnormal in G/N.

(3) If $N \leq K$ and K/N is σ-subnormal in G/N, then K is σ-subnormal in G.

(4) If $H \neq 1$ is a Hall Π-subgroup of G and A is not a Π'-group, then $A \cap H \neq 1$ is a Hall Π-subgroup of A.

(5) If G is Π-full and A is a Π-group, then $A \leq O_\Pi(G)$.

Recall that the σ-radical of G is the product of all normal σ-soluble subgroups of G and denote it by $R_\sigma(G)$ and $R_\sigma(G)$ is σ-soluble (see [9]). Moreover, any extension of the σ-soluble group by a σ-soluble group is a σ-soluble group (see also [9, Lemma 2.1]).

Lemma 2.2 — If G has a complete Hall σ-set $\mathcal{H} = \{W_1, \cdots, W_t\}$ such that W_i is nilpotent for all $i = 1, \cdots, t$ and H is a σ-subnormal σ-soluble subgroup of G, then $H \leq R_\sigma(G)$.

PROOF : Assume that this Lemma is false and let (G, H) be a counterexample with $|G| + |H|$ as small as possible.

By hypothesis, there exists a subgroup chain $H = H_0 \leq H_1 \leq \cdots \leq H_{l-1} \leq H_l = G$ such that either H_{i-1} is normal in H_i or $H_i/(H_{i-1})_{H_i}$ is σ-primary for all $i = 1, \cdots, l$. Let $M = H_{l-1}$. Then M is normal in G or G/M_G is σ-primary. Without loss of generality, We can assume that $M \neq G$. By Lemma 2.1(1), H is σ-subnormal in M. Hence $H \leq R_\sigma(M)$ by the choice of (G, H). If M is normal in G, then $H \leq R_\sigma(M) \leq R_\sigma(G)$ for $R_\sigma(M)$ is characteristic in M, a contradiction. Therefore G/M_G is σ-primary, which means that G/M_G is a σ_i-group and so $G/M_G = W_iM_G/M_G$ is nilpotent. It follows that HM_G is subnormal in G. Since $HM_G \leq M < G$ and H is σ-subnormal in HM_G by Lemma 2.1(1), $H \leq R_\sigma(HM_G)$ by the choice of (G, H). But as $R_\sigma(HM_G)$ is subnormal in G, we have that $R_\sigma(HM_G) \leq R_\sigma(G)$ by [9, Lemma 2.3]. Hence $H \leq R_\sigma(HM_G) \leq R_\sigma(G)$. This contradiction completes the proof. □

Let $\mathcal{L}$ be some non-empty set of subgroups of G and E a subgroup of G. Following [1], a subgroup A of G is called $\mathcal{L}$-permutable if $AH = HA$ for all $H \in \mathcal{L}$; $\mathcal{L}^E$-permutable if $AH^x = H^xA$ for all $H \in \mathcal{L}$ and all $x \in E$. In particular, a subgroup H of G is σ-permutable if G possesses a complete Hall σ-set $\mathcal{H}$ such that H is $\mathcal{H}^G$-permutable.

Lemma 2.3 — (See [1, Lemma 2.8] and [3, Lemma 2.2]). Let H, K and N be subgroups of a σ-group G. Let $\mathcal{H} = \{H_1, H_2, \cdots, H_t\}$ be a complete Hall σ-set of G and $\mathcal{L} = \mathcal{H}^K$. Suppose that H is $\mathcal{L}$-permutable and N is normal in G.

(1) If $H \leq E \leq G$, then H is $\mathcal{L}^*$-permutable, where $\mathcal{L}^* = \{H_1 \cap E, H_2 \cap E, \cdots, H_t \cap E\}^{K \cap E}$. In particular, if G is a σ-full group of Sylow type and H is σ-permutable in G, then H is σ-permutable in E.

(2) The subgroup HN/N is $\mathcal{L}^{**}$-permutable, where $\mathcal{L}^{**} = \{H_1N/N, \cdots, H_tN/N\}^{KN/N}$.

(3) If G is a σ-full group of Sylow type and E/N is a σ-permutable subgroup of G/N, then E is σ-permutable in G.

(4) If K is $\mathcal{L}$-permutable, then $\langle H, K \rangle$ is $\mathcal{L}$-permutable (see also [11, Chapter A, 1.6(a)]). In particular, for any subgroup A of G, $A_{\sigma G}$ is σ-permutable in G.

Lemma 2.4 — (See [1, Lemma 3.1]). Let H be a σ_1-subgroup of a σ-group G. Then H is σ-permutable in G if and only if $O^{\sigma_1}(G) \leq N_G(H)$.

Lemma 2.5 — (See [1, Lemma 3.2]). Let H and K be subgroups of G. If G is a σ-full group of Sylow type and H is σ-permutable in G, then $H \cap K$ is σ-permutable in K.

The following Lemma directly follows from Lemma 2.3, Lemma 2.5 and [1, Theorem C].

Lemma 2.6 — Let G be a σ-full group of Sylow type and $H \leq K \leq G$. Then:

(1) $H_{\sigma G} \leq H_{\sigma K}$.

(2) $H^{\sigma G}$ is σ-permutable in G and $H^{\sigma K} \leq H^{\sigma G} \leq H^G$.

(3) If H is normal in G, then $(K/H)^{\sigma(G/H)} = K^{\sigma G}/H$ and $(K/H)_{\sigma(G/H)} = K_{\sigma G}/H$.

(4) If H is σ-permutable in G, then H^x is σ-permutable in G for all $x \in G$.

Lemma 2.7 — Let G be a σ-full group of Sylow type and $H \leq K \leq G$.

(1) If H is σ-permutable in G, then H is σ-embedded in G.

(2) If H is σ-embedded in G, then H is σ-embedded in K.

(3) Suppose that H is normal in G. Then K/H is σ-embedded in G/H if and only if K is σ-embedded in G.

(4) Suppose that H is normal in G. Then for every σ-embedded subgroup E of G with $(|H|, |E|) = 1$, HE/H is σ-embedded in G/H.

(5) If H is σ-embedded in G, then H^x is σ-embedded in G for all $x \in G$.

PROOF : (1) This is evident.

(2) If H is σ-embedded in G, then there exists a σ-permutable subgroup T of G such that $HT = H^{\sigma G}$ and $H \cap T \leq H_{\sigma G}$. Let $T_0 = H^{\sigma K} \cap T$. Then $T_0 = K \cap T \cap H^{\sigma K}$. By Lemma 2.5, $K \cap T$ is σ-permutable in K and so by [1, Theorem C] T_0 is σ-permutable in K. By Lemma 2.6(2), we have that $H^{\sigma K} \leq H^{\sigma G}$ and so $H^{\sigma K} = H(H^{\sigma K} \cap T) = HT_0$. Moreover, $H \cap T_0 = H \cap T \leq H_{\sigma G} \leq H_{\sigma K}$ by Lemma 2.6(1). This shows that H is σ-embedded in K.

(3) First assume that K/H is σ-embedded in G/H and let T/H be a σ-permutable subgroup of G/H such that $(K/H)(T/H) = (K/H)^{\sigma(G/H)}$ and $(K/H) \cap (T/H) \leq (K/H)_{\sigma(G/H)}$. Then by Lemma 2.3(3), T is σ-permutable in G, and by Lemma 2.6(3), $KT = K^{\sigma G}$ and $K \cap T \leq K_{\sigma G}$. This shows that K is σ-embedded in G. Now assume that K is σ-embedded in G. Then G has a

σ-permutable subgroup T such that $KT = K^{\sigma G}$ and $K \cap T \leq K_{\sigma G}$. By Lemma 2.3(2), TH/H is σ-permutable in G/H. By Lemma 2.6(3), $(K/H)(TH/H) = K^{\sigma G}/H = (K/H)^{\sigma(G/H)}$ and $(K/H) \cap (TH/H) = (K \cap T)H/H \leq K_{\sigma G}/H = (K/H)_{\sigma(G/H)}$. Hence K/H is σ-embedded in G/H.

(4) Suppose that E is σ-embedded in G and let T be a σ-permutable subgroup of G such that $ET = E^{\sigma G}$ and $E \cap T \leq E_{\sigma G}$. Clearly, $HET = HE^{\sigma G} = (HE)^{\sigma G}$. Since $(|H|, |E|) = 1$, we have that $(|HE \cap T : H \cap T|, |HE \cap T : E \cap T|) = (|(HE \cap T)H : H|, |(HE \cap T)E : E|) = 1$. Hence $HE \cap T = (H \cap T)(E \cap T)$ by [11, Chapter A, 1.6(b)]. Then $HE \cap T = (H \cap T)(E \cap T) \leq HE_{\sigma G} \leq (HE)_{\sigma G}$ by Lemma 2.3(4). This shows that HE is σ-embedded in G. It follows from (3) that HE/H is σ-embedded in G/H.

(5) This follows from Lemma 2.6(4). □

Lemma 2.8 — Let G be a σ-full group of Sylow type and $H \leq K \leq G$.

(1) If H is σ-permutable in G, then H is σ-n-embedded in G.

(2) If H is σ-n-embedded in G, then H is σ-n-embedded in K.

(3) Suppose that H is normal in G. Then K/H is σ-n-embedded in G/H if and only if K is σ-n-embedded in G.

(4) Suppose that H is normal in G. Then for every σ-n-embedded subgroup E of G with $(|H|, |E|) = 1$, HE/H is σ-n-embedded in G/H.

(5) If H is σ-n-embedded in G, then H^x is σ-n-embedded in G for all $x \in G$.

PROOF See the proof of Lemma 2.7. □

Let P be a p-group. If P is not a non-abelian 2-group, then we use $\Omega(P)$ to denote the subgroup $\Omega_1(P)$. Otherwise, $\Omega(P) = \Omega_2(P)$.

Lemma 2.9 — (See [13, Lemma 4.3]). Let C be a Thompson critical subgroup (see [14, p. 185]) of a nontrivial p-group of P.

(1) If p is odd, then the exponent of $\Omega(C)$ is p.

(2) If P is a non-abelian 2-group, then the exponent of $\Omega(C)$ is 4.

Recall that a class of groups $\mathcal{F}$ is called a formation if it is closed under taking homomorphic images and subdirect products. A formation $\mathcal{F}$ is called saturated if $G \in \mathcal{F}$ whenever $G/\Phi(G) \in \mathcal{F}$ (see, for example, [12]). The $\mathcal{F}$-residual of G, denoted by $G^{\mathcal{F}}$, is the smallest normal subgroup of

G with quotient in $\mathcal{F}$. In this paper, we use $\mathcal{U}$ to denote the class of all supersoluble groups; $Z_{\mathcal{U}}(G)$ denotes the $\mathcal{U}$-hypercenter of a group G, that is, the product of all such normal subgroups H of G whose G-chief factors have prime order.

Lemma 2.10 — (See [13, Lemma 4.4] and [15, Lemma 2.12]). Let P be a normal p-subgroup of a group G and C a Thompson critical subgroup of P. If either $P/\Phi(P) \leq Z_{\mathcal{U}}(G/\Phi(P))$ or $\Omega(C) \leq Z_{\mathcal{U}}(G)$, then $P \leq Z_{\mathcal{U}}(G)$.

Lemma 2.11 — (See [10, Lemma 2.16]). Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and G be a group with a normal subgroup E such that $G/E \in \mathcal{F}$. If E is cyclic, then $G \in \mathcal{F}$.

Lemma 2.12 — (See [16, Theorem 1.8.17]). Let P be a nilpotent normal subgroup of a group G. If $P \cap \Phi(G) = 1$, then P is a direct product of some minimal normal subgroups of G.

Recall that a class $\mathcal{F}$ is called Fitting formation if $\mathcal{F}$ is both Fitting class and formation. A group G is said to be p-closed if G has a normal Sylow p-subgroup.

Lemma 2.13 — (See [17, p. 34]). Let p be a prime. Then the class of all p-closed groups is a saturated fitting formation.

3. Proof of Main Theorem

Lemma 3.1 — Let G be a σ-full group of Sylow type and $\mathcal{H} = \{W_1, W_2, \cdots, W_t\}$ be a complete Hall σ-set of G such that the Hall σ_i-subgroup W_i of G is nilpotent for all $i = 1, \cdots, t$. Let $p \in \sigma_1$, where p is the smallest prime dividing $|G|$. If every maximal subgroup of W_1 or every cyclic subgroup H of W_1 of prime order and order 4 (if a Sylow p-subgroup P of W_1 is a non-abelian 2-group and $H \nleq Z_{\infty}(G)$) is σ-embedded in G, then G is soluble.

Proof : First note that if G is σ-soluble, then G is soluble. In fact, if G is σ-soluble, then for every chief factor H/K of G, H/K is σ-primary, and so H/K is a σ_i-group, for some $\sigma_i \in \sigma(G)$. Hence $H/K \leq W_iK/K$ since W_iK/K is the Hall σ_i-subgroup of G/K. But since W_i is nilpotent, H/K is elementary abelian p-group. This shows that G is soluble. Therefore we only prove that G is σ-soluble.

Suppose that it is false and let G be a counterexample of minimal order. Then clearly, $t > 1$. By the well-known Feit-Thompson's theorem, we have $p = 2$. Without loss of generality, we may assume that P is a Sylow 2-subgroup of W_1. We now proceed by the following steps.

(1) G is not 2-nilpotent.

If G is 2-nilpotent, then, clearly, G is soluble, a contradiction.

(2) W_1 is not cyclic (This follows from (1) and [18, IV, Theorem 2.8]).

(3) Every maximal subgroup of W_1 is σ-embedded in G.

Suppose that some maximal subgroup of W_1 is not σ-embedded in G. Then by hypothesis every cyclic subgroup H of W_1 of prime order and order 4 (if P is a non-abelian 2-group and $H \nleq Z_\infty(G)$) is σ-embedded in G. Since G is not 2-nilpotent by (1), by [18, IV, Theorem 5.4] and [16, Theorem 3.4.11], G has a 2-closed Schmidt subgroup $A = A_2 \rtimes A_q$, where A_2 is a Sylow 2-subgroup of A of exponent 2 or 4 (if A_2 is non-abelian), A_q is a Sylow q-subgroup of A, $A_2/\Phi(A_2)$ is a A-chief factor, $Z_\infty(A) = \Phi(A)$ and $\Phi(A) \cap A_2 = \Phi(A_2)$.

We claim that $|A_2/\Phi(A_2)| = 2$. Without loss of generality, we can assume that A_2 is contained in W_1. By Lemma 2.7(2), every cyclic subgroup H of A_2 with prime order and order 4 (if A_2 is non-abelian and $H \nleq Z_\infty(G)$) is σ-embedded in A. If $q \in \pi(W_1)$, then A is a σ_1-group. So $A \le W_1^x$ for G is a σ-full group of Sylow type, which means that A is nilpotent, a contradiction. Hence $q \notin \sigma_1$. Assume that there exists a minimal subgroup $X/\Phi(A_2)$ of $A_2/\Phi(A_2)$ such that $X/\Phi(A_2)$ is not σ-permutable in $A/\Phi(A_2)$. Let $x \in X \backslash \Phi(A_2)$ and $L = \langle x \rangle$. Then $X = L\Phi(A_2)$ and $|L| = 2$ or 4. If $L = L_{\sigma A}$, then L is σ-permutable in A by Lemma 2.3(4). So $X/\Phi(A_2) = L\Phi(A_2)/\Phi(A_2)$ is σ-permutable in $A/\Phi(A_2)$ by Lemma 2.3(2), a contradiction. Hence we may assume that $L_{\sigma A} < L$. If $L \le Z_\infty(G)$, then $L \le Z_\infty(A) \cap A_2 = \Phi(A) \cap A_2 = \Phi(A_2)$, a contradiction. Hence by the hypothesis, there exists a σ-permutable subgroup T of A such that $LT = L^{\sigma A} \le A_2$ and $L \cap T \le L_{\sigma A} < L$. Since $T\Phi(A_2)/\Phi(A_2)$ is σ-permutable in $A/\Phi(A_2)$ by Lemma 2.3(2), it follows that $A_q\Phi(A_2)/\Phi(A_2) \le N_{A/\Phi(A_2)}(T\Phi(A_2)/\Phi(A_2))$ by Lemma 2.4. Then $T\Phi(A_2)/\Phi(A_2) \trianglelefteq A/\Phi(A_2)$. Hence $T\Phi(A_2)/\Phi(A_2) = 1$ or $A_2/\Phi(A_2)$. If $T\Phi(A_2)/\Phi(A_2) = A_2/\Phi(A_2)$, then $T = A_2$, and so $L = L \cap T \le L_{\sigma A} < L$, a contradiction. Hence $T \le \Phi(A_2)$. Since $LT = L^{\sigma A}$ is σ-permutable in A by Lemma 2.6(2), we have that $X/\Phi(A_2) = L\Phi(A_2)/\Phi(A_2) = LT\Phi(A_2)/\Phi(A_2)$ is σ-permutable in $A/\Phi(A_2)$, a contradiction. The contradiction shows that every minimal subgroup of $A_2/\Phi(A_2)$ is σ-permutable in $A/\Phi(A_2)$. It follows that every minimal subgroup of $A_2/\Phi(A_2)$ is s-permutable in $A/\Phi(A_2)$. Hence $|A_2/\Phi(A_2)| = 2$ by [10, Lemma 2.11]. This implies that A is nilpotent, a contradiction. Thus we have (3).

(4) $O_{\sigma_1}(G) = 1$.

If $O_{\sigma_1}(G) \neq 1$, then $\overline{\mathcal{H}} = \{W_1/O_{\sigma_1}(G), W_2O_{\sigma_1}(G)/O_{\sigma_1}(G), \cdots, W_tO_{\sigma_1}(G)/O_{\sigma_1}(G)\}$ is a complete Hall σ-set of $G/O_{\sigma_1}(G)$ and $W_iO_{\sigma_1}(G)/O_{\sigma_1}(G) \simeq W_i/W_i \cap O_{\sigma_1}(G)$ is nilpotent. If $O_{\sigma_1}(G) = W_1$, then $|G/W_1|$ is an odd number and so G/W_1 is soluble. Hence G is soluble, a contradiction. Let $M/O_{\sigma_1}(G)$ be a maximal subgroup of $W_1/O_{\sigma_1}(G)$. Then M is a maximal subgroup of W_1. By (3) and Lemma 2.7(3), $M/O_{\sigma_1}(G)$ is σ-embedded in $G/O_{\sigma_1}(G)$. Therefore the hypothesis holds for $G/O_{\sigma_1}(G)$. The choice of G implies that $G/O_{\sigma_1}(G)$ is σ-soluble and so G is σ-soluble, a contradiction. Hence $O_{\sigma_1}(G) = 1$.

(5) Final contradiction.

For any maximal subgroup V of W_1 we have $V_{\sigma G} = 1$. Indeed, by Lemma 2.3(4) $V_{\sigma G}$ is σ-permutable in G, so it is σ-subnormal in G by [1, Theorem B]. Hence $V_{\sigma G} \leq O_{\sigma_1}(G) = 1$ by (4) and Lemma 2.1(5). By (2), $W_1 = V_1V_2$ for some maximal subgroups V_1 and V_2 of W_1. By (3) V_i is σ-embedded in G. So G has a σ-permutable subgroup T_i such that $V_i^{\sigma G} = V_iT_i$ and $V_i \cap T_i \leq (V_i)_{\sigma G} = 1$ for $i = 1, 2$. By [1, Theorem B], T_i is σ-subnormal in G. Hence by Lemma 2.1(4), $W_1 \cap T_i$ is a Hall σ_1-subgroup of T_i. But since $V_i \cap T_i = 1$, we have that $|T_i|_{\sigma_1} = |W_1 \cap T_i| = |W_1 \cap T_i : V_i \cap T_i| \leq |W_1 : V_i| = q$, where $q \in \sigma_1$. This implies that T_i is soluble and so $T_i \leq R_\sigma(G)$ by Lemma 2.2. It follows that $V_i^{\sigma G} = V_iT_i = V_i(R_\sigma(G) \cap V_i^{\sigma G})$, and so $V_i^{\sigma G}$ is σ-soluble. By [1, Theorem B and Theorem C] $V_i^{\sigma G}$ is σ-subnormal in G. So $V_i^{\sigma G} \leq R_\sigma(G)$ by Lemma 2.2. Since $W_1 = V_1V_2 \leq \langle V_1^{\sigma G}, V_2^{\sigma G} \rangle \leq R_\sigma(G)$. $|G/R_\sigma(G)|$ is a σ_1'-number and so it is an odd number. This implies that G is σ-soluble. The final contradiction completes the proof. □

***Theorem* 3.2** — *Let G be a σ-full group of Sylow type and $\mathcal{H} = \{W_1, \cdots, W_t\}$ be a complete Hall σ-set of G such that W_i is a nilpotent σ_i-subgroup for all $i = 1, \cdots, t$. If every maximal subgroup of every non-cyclic subgroup W_i is σ-n-embedded in G for $i = 1, \cdots, t$, then G is supersoluble.*

PROOF : Suppose that this assertion is false and let G be a counterexample of minimal order. Then:

(1) G is soluble and so G is not a simple group.

Let $p \in \sigma_1$ and P be a Sylow p-subgroup of G, where p is the smallest prime dividing $|G|$. If W_1 is cyclic, then P is cyclic. So G is p-nilpotent by [18, IV, Theorem 2.8] and so G is soluble. Now assume that W_1 is non-cyclic. Since a σ-n-embedded subgroup of G is σ-embedded in G, by hypothesis and Lemma 3.1 we have that G is soluble. The choice of G implies that G is not a simple group.

(2) G/R is supersoluble for every non-identity minimal normal subgroup of G.

Let R be a minimal normal subgroup of G. Then R is a p-group by (1). It is clear that the hypothesis holds on G/R. Hence G/R is supersoluble by the choice of G.

(3) R is the unique minimal normal subgroup of G, $|R| > p$ and $R \nleq \Phi(G)$ (This directly follows from (2)).

(4) Final contradiction.

Without loss of generality, we may assume that $R \leq W_1$. Then by (3) and [18, III, Lemma 3.3(a)], $R \nleq \Phi(W_1)$. Hence there exists a maximal subgroup V of W_1 such that $W_1 = RV$, $V_G = 1$ and $|W_1 : V| = p = |R : R \cap V|$. By (3), W_1 is not cyclic. Then by hypothesis, G has a normal subgroup T of G such that $VT = V^G$ and $V \cap T \leq V_{\sigma G}$. Then by [1, Theorem C], $V \cap T = V_{\sigma G} \cap T$ is σ-permutable in G. It follows from Lemma 2.4 that $O^{\sigma_1}(G) \leq N_G(V \cap T)$. On the other hand, $V \cap T$ is a normal subgroup of W_1. Hence $V \cap T$ is normal in G, and so $V \cap T = 1$ for $V_G = 1$. By (3), $T = 1$ or $R \leq T$. If $T = 1$, then $V = V^G \trianglelefteq G$, and so $V = 1$. It follows that $|R| = p$, a contradiction. Hence $R \leq T$ and $R \cap V \leq T \cap V = 1$, which also means that $|R| = p$, contrary to (3). The final contradiction completes the proof. □

Lemma 3.3 — Let G be a σ-full group of Sylow type and $\mathcal{H} = \{W_1, \cdots, W_t\}$ be a complete Hall σ-set of G such that W_i is a nilpotent σ_i-subgroup for all $i = 1, \cdots, t$. Let P be a normal p-subgroup of G and $P \leq W_i$ for some i. If every cyclic subgroup H of P with prime order and order 4 (if P is a non-abelian 2-group and $H \nleq Z_\infty(G)$) is σ-n-embedded in G, then $P \leq Z_{\mathcal{U}}(G)$.

PROOF : Assume that this assertion is false and let (G, P) be a counterexample with $|G| + |P|$ minimal. Then:

(1) G has a unique normal subgroup R such that P/R is a chief factor of G, $R \leq Z_{\mathcal{U}}(G)$ and $|P/R| > p$.

Let P/R be a chief factor of G. Then clearly, (G, R) satisfies the hypothesis of the Lemma. The choice of (G, P) implies that $R \leq Z_{\mathcal{U}}(G)$. If $|P/R| = p$, then $P/R \leq Z_{\mathcal{U}}(G/R)$ and so $P \leq Z_{\mathcal{U}}(G)$, a contradiction. Hence $|P/R| > p$. Assume that P/N is a chief factor of G with $P/N \neq P/R$. A same discussion as above, we have that $N \leq Z_{\mathcal{U}}(G)$. Then $P/R = NR/R \leq RZ_{\mathcal{U}}(G)/R \leq Z_{\mathcal{U}}(G/R)$. It follows from $R \leq Z_{\mathcal{U}}(G)$ that $P \leq Z_{\mathcal{U}}(G)$, a contradiction. Hence we have (1).

(2) The exponent of P is p or 4 (when P is a non-abelian 2-group).

Let C be a Thompson critical subgroup of P (see [14, p. 185]). If $\Omega(C) < P$, then $\Omega(C) \leq R \leq Z_{\mathcal{U}}(G)$ by (1). So $P \leq Z_{\mathcal{U}}(G)$ by Lemma 2.10, a contradiction. Hence $P = C = \Omega(C)$. Then by Lemma 2.9, the exponent of P is p or 4 (when P is a non-abelian 2-group).

(3) Final contradiction.

Since W_i/R is nilpotent, $P/R \cap Z(W_i/R) > 1$. Suppose that $L/R \leq P/R \cap Z(W_i/R)$ and $|L/R| = p$. Let $x \in L \backslash R$ and $H = \langle x \rangle$. Then $L = HR$ and $|H| = p$ or 4 (when P is a non-abelian 2-group) by (2). If $H = H_{\sigma G}$, then by Lemma 2.3(2)(4), HR/R is σ-permutable in G/R. Obviously, $HR/R \trianglelefteq W_i/R$ and by Lemma 2.4 $O^{\sigma_i}(G/R) \leq N_{G/R}(HR/R)$. This implies that $HR/R \trianglelefteq G/R$. But as P/R is a chief factor of G and $H \nleq R$, we have that $P = HR$. It follows that $P/R = HR/R = L/R$ is cyclic, which contradicts (1). Hence $H_{\sigma G} < H$. If $H \leq Z_\infty(G)$, then $1 \neq L/R = HR/R \leq Z_\infty(G)R/R \leq Z_\infty(G/R)$. So $P/R \cap Z_\infty(G/R) \neq 1$. Hence $P/R \leq Z_\infty(G/R)$. But as $R \leq Z_{\mathcal{U}}(G)$, it follows that $P \leq Z_{\mathcal{U}}(G)$, a contradiction. So H is σ-n-embedded in G by hypothesis, there exists a normal subgroup T of G such that $HT = H^G \leq P$ and $H \cap T \leq H_{\sigma G} < H$. By (1) we have that $T \leq R$ or $T = P$. If $T = P$, then $H = H \cap T \leq H_{\sigma G} < H$, a contradiction. So $T \leq R$. Similarly, we have that $H^G \leq R$ or $H^G = P$. If $H^G \leq R$, then $H \leq H^G \leq R$, a contradiction. So $H^G = P$. Hence $P = H^G = HT = HR$, and so $P/R = HR/R = L/R$ is cyclic, which contradicts (1). This final contradiction completes the Lemma. □

PROOF OF MAIN THEOREM : Suppose that this theorem is false and let (G, E) be a counterexample with $|G| + |E|$ minimal. Let P be a Sylow p-subgroup of E where p is the smallest prime dividing $|E|$. Without loss of generality we may assume that $P \leq W_1 \cap E$. We now proceed by the following steps.

(1) $W_1 \cap E$ is non-cyclic.

If $W_1 \cap E$ is cyclic, then P is cyclic. It follows from [18, IV, Theorem 2.8] that E is p-nilpotent. Let $E_{p'}$ be the normal Hall p'-subgroup of E. Then $E_{p'} \neq 1$ by Lemma 2.11. Clearly $E_{p'} \trianglelefteq G$ and $E/E_{p'}$ is a cyclic p-group. Since $(G/E_{p'})/(E/E_{p'}) \simeq G/E \in \mathcal{F}$, we have that $G/E_{p'} \in \mathcal{F}$ by Lemma 2.11. Clearly, $W_i \cap E_{p'} = W_i \cap E$ for $i \neq 1$, and $W_1 \cap E_{p'}$ is cyclic. Hence the hypothesis holds for $(G, E_{p'})$ and so $G \in \mathcal{F}$ by the choice of (G, E), a contradiction. So $W_1 \cap E$ is non-cyclic.

(2) If $E = P$ and every cyclic subgroup H of P with prime order and order 4 (if P is a non-abelian 2-group and $H \nleq Z_\infty(G)$) is σ-n-embedded in G, then $E \leq Z_{\mathcal{U}}(G)$. (This directly follows from Lemma 3.3).

(3) If every cyclic subgroup H of $W_1 \cap E$ with prime order and order 4 (if P is a non-abelian 2-group and $H \nleq Z_\infty(G)$) is σ-n-embedded in G, then E is not p-nilpotent.

Assume that E is p-nilpotent. Then E has a normal Hall p'-subgroup $E_{p'}$ and $E_{p'} \trianglelefteq G$. If $E_{p'} = 1$, then by (2) we have that $E \leq Z_{\mathcal{U}}(G)$, and so $G \in \mathcal{F}$. This contradiction shows that $E_{p'} \neq 1$. Then $\overline{\mathcal{H}} = \{W_1E_{p'}/E_{p'}, \cdots, W_tE_{p'}/E_{p'}\}$ is a complete Hall σ-set of $G/E_{p'}$ and $W_iE_{p'}/E_{p'} \simeq W_i/W_i \cap E_{p'}$ is nilpotent. Clearly $W_iE_{p'}/E_{p'} \cap E/E_{p'} = 1$ for $i = 2, \cdots, t$, and $W_1E_{p'}/E_{p'} \cap E/E_{p'} = E/E_{p'}$. If $W_1E_{p'}/E_{p'} \cap E/E_{p'} = E/E_{p'}$ is cyclic, then $G/E_{p'} \in \mathcal{F}$ by Lemma 2.11 and $(G/E_{p'})/(E/E_{p'}) \simeq G/E \in \mathcal{F}$. If $W_1E_{p'}/E_{p'} \cap E/E_{p'} = E/E_{p'}$ is non-cyclic, then for every cyclic subgroup $H/E_{p'}$ of $E/E_{p'}$ with prime order p or order 4, we have that $H = E_{p'} \rtimes L$, where $L = \langle x \rangle$ is of order p or 4 by Schur-Zassenhaus Theorem. So by the hypothesis, Lemma 2.8(4) and [12, Chapter 1, Theorem 2.6(d)], we have that $H/E_{p'} = E_{p'}L/E_{p'}$ is σ-n-embedded in $G/E_{p'}$ or $H/E_{p'} \leq Z_{\infty}(G/E_{p'})$. This shows that the hypothesis holds for $(G/E_{p'}, E/E_{p'})$. Therefore $G/E_{p'} \in \mathcal{F}$ by the choice of (G, E) and $E_{p'} \neq 1$. It is clear that $W_i \cap E = W_i \cap E_{p'}$ for $i = 2, \cdots, t$ and $W_1 \cap E_{p'} \leq W_1 \cap E$. So the hypothesis holds for $(G, E_{p'})$, and so $G \in \mathcal{F}$ by the choice of (G, E), a contradiction. Hence E is not p-nilpotent.

(4) Every maximal subgroup of $W_1 \cap E$ is σ-n-embedded in G.

By hypothesis and (1) either every maximal subgroup of $W_1 \cap E$ is σ-n-embedded in G or every cyclic subgroup H of $W_1 \cap E$ with prime order and order 4 (if P is a non-abelian 2-group and $H \nleq Z_{\infty}(G)$) is σ-n-embedded in G. Suppose that we have the second case. Then by (3), E is not p-nilpotent. So by [18, IV, Theorem 5.4] and [16, Theorem 3.4.11], E has a p-closed Schmidt subgroup $A = A_p \rtimes A_r$, where A_p is a Sylow p-subgroup of A of exponent p or 4 (when A_p is a non-abelian 2-group) and $A_p/\Phi(A_p)$ is an eccentric chief factor of A. By the same discussion as Lemma 3.1(3), we have that $|A_p/\Phi(A_p)| = p$. Hence $A_p/\Phi(A_p)$ is central in A for p is the smallest prime dividing $|E|$. This contradiction shows that every maximal subgroup of $W_1 \cap E$ is σ-n-embedded in G.

(5) E is q-closed, where q is the largest prime dividing $|E|$.

Let q be the largest prime dividing $|E|$. Without loss of generality, we may assume that $q \in \sigma_i$, Q is a Sylow q-subgroup of E and $Q \leq W_i \cap E$. We use $\mathcal{M}$ to denote the class of all q-closed groups. Suppose that E is not in $\mathcal{M}$. Then:

(a) $E = G$.

Indeed, if $E \neq G$, then the hypothesis holds for (E, E) by Lemma 2.8(2). The choice of (G, E) implies that E is supersoluble and so it is q-closed, which contradicts to our assumption on E.

(b) G is soluble (This follows from (4) and Lemma 3.1).

(c) $t = 2$, that is, $G = W_1W_2$.

Assume that $t > 2$. By (b) and [9, Theorem A], G has a σ-basis $\{H_1, H_2, \cdots, H_t\}$ and $W_j = H_j^{x_j}$ by [9, Theorem B]. So $W_i^{x_i^{-1}} W_j^{x_j^{-1}} = H_iH_j$ is a subgroup of G and so $W_iW_j^{x_j^{-1}x_i}$ is a subgroup of G for all $j \neq i$. By Lemma 2.8(2)(5) and (a), the hypothesis holds for $(W_iW_j^{x_j^{-1}x_i}, W_iW_j^{x_j^{-1}x_i})$. Since $t > 2$, $W_iW_j^{x_j^{-1}x_i} < G$. Hence $W_iW_j^{x_j^{-1}x_i}$ is q-closed. So $W_j^{x_j^{-1}x_i} \leq N_G(Q)$ for $j \neq i$. It follows that Q is normal in G for W_i is nilpotent, a contradiction. Hence (c) holds.

(d) Every minimal subgroup of W_2 is σ-n-embedded in G.

If W_2 is cyclic or every maximal subgroup of W_2 is σ-n-embedded in G, then by (4), (c) and Theorem 3.2, G is supersoluble, and so $G \in \mathcal{F}$, a contradiction. Hence we have (d).

(e) W_2 is a r-group for some prime r and so $G = W_1R$ where R is a Sylow r-subgroup of G.

If not, then by (b) and [9, Theorem B] W_1 is G-permutable with every Sylow subgroup of G. This means for any Sylow r-subgroup R of W_2, we have that $W_1R^x = R^xW_1$ for some $x \in G$. Then $W_1R^x < G$. By (4) and (d), W_1R^x is q-closed. If $Q \leq W_1$, then $R^x \leq N_G(Q)$. Therefore Q is normal in G for W_1 is nilpotent, a contradiction. So $Q \leq W_2$. Then W_1Q^x is q-closed. This means that $W_1^{x^{-1}} \leq N_G(Q)$. Hence Q is normal in G for W_2 is nilpotent. This contradiction shows that W_2 is a r-group.

(f) $r = q$, that is, $W_2 = Q$ is a q-group.

If not, then by (e) $Q \leq W_1$. Now we claim that $O_{\sigma_1}(G) = 1$. In fact, if $O_{\sigma_1}(G) \neq 1$, then by (4) and Lemma 2.8(3)(4), the hypothesis holds on $G/O_{\sigma_1}(G)$, and so $G/O_{\sigma_1}(G)$ is q-closed by induction on $G = E$. Then $QO_{\sigma_1}(G) \trianglelefteq G$. But since $QO_{\sigma_1}(G) \leq W_1$ and W_1 is nilpotent, we have that Q char $QO_{\sigma_1}(G)$, and so $Q \trianglelefteq G$, a contradiction. Hence $O_{\sigma_1}(G) = 1$. Since $p, q \in \sigma_1$. Let V be a maximal subgroup of W_1 such that $P \nleq V$. Then by (4) and (a), G has a normal subgroup T such that $V^G = VT$ and $V \cap T \leq V_{\sigma G}$. By Lemma 2.3(4) and [1, Theorem B] $V_{\sigma G}$ is σ-subnormal in G. So $V \cap T \leq V_{\sigma G} \leq O_{\sigma_1}(G) = 1$ by Lemma 2.1(5). Clearly, $|T|_{\sigma_1} \leq p$. If $|T|_{\sigma_1} = p$, then $W_1 \leq V^G$ by (b). Hence $V^G = W_1(V^G \cap R)$ by (e). If $V^G < G$, then V^G satisfies the hypothesis, so V^G is q-closed and so $Q \trianglelefteq G$, a contradiction. Hence $G = V^G = T \rtimes V$, so $R \leq T$. Since $|T|_{\sigma_1} = p$, so T is p-nilpotent by [18, IV, Theorem 2.8]. Hence $R \trianglelefteq T$ and so $R \trianglelefteq G$. Therefore $G/R \simeq W_1$ is nilpotent. By Lemma 3.3 and (d), $R \leq Z_{\mathcal{U}}(G)$. This implies that G is supersoluble and so $G \in \mathcal{F}$, a contradiction. Hence $|T|_{\sigma_1} = 1$, so $T \leq R$. By (d) and Lemma 2.8(2), every minimal subgroup of T is σ-n-embedded in V^G. Hence by Lemma 3.3 we have that $T \leq Z_{\mathcal{U}}(V^G)$. But $V^G/T \simeq V \leq W_1$ is nilpotent, it follows that V^G is supersoluble. Then Q char $V^G \trianglelefteq G$, so $Q \trianglelefteq G$. This contradiction shows that $r = q$ and $W_2 = Q$ is a q-group.

(g) $O^q(G) = G$ or $|Q| = q$.

Assume that $O^q(G) \neq G$ and let M be a maximal subgroup of G containing $O^q(G)$. Then M is normal in G and by (e) and (f) $M = W_1(M \cap Q)$. By Lemma 2.8(2), the hypothesis holds on (M, M). Hence $M \cap Q$ is normal in M. Suppose that $|Q| > q$. Then $M \cap Q \neq 1$. It is clear that the hypothesis still holds for $G/M \cap Q$ since $Q/M \cap Q$ is cyclic. Hence $G/M \cap Q$ is q-closed, and so

$Q/M \cap Q$ is normal in $G/M \cap Q$. But then G is q-closed, a contradiction. Hence $|Q| = q$. So we have (g).

(h) If V is a maximal subgroup of W_1 and $V_G = 1$, then V^G is q-closed.

Indeed, by (1), (4), $V \neq 1$ and G has a normal subgroup T such that $V^G = VT$ and $V \cap T \leq V_{\sigma G}$. Then $V \cap T = V_{\sigma G} \cap T$ is normal in G by the same discussion as Theorem 3.2(4). It follow from $V_G = 1$ that $V \cap T = 1$. So $V^G = T \rtimes V$. Hence $|T|_{\sigma_1} \leq t$ for some prime t since W_1 is nilpotent. Therefore $W_1 \cap T = 1$ or $W_1 \cap T$ is cyclic. Then T satisfies the hypothesis by Lemma 2.8(2) and (d), so T is q-closed. Let Q_0 be the Sylow q-subgroup of T. Then Q_0 is normal in V^G. This means that V^G is q-closed.

(i) If W_1 has two maximal subgroups V_1 and V_2 such that $W_1 = V_1V_2$ and $(V_1)_G = (V_2)_G = 1$, then W_1^G is q-closed.

Since $(V_1)_G = (V_2)_G = 1$, V_1^G and V_2^G are q-closed by (h). Hence by Lemma 2.13, $D = V_1^G V_2^G$ is q-closed. But $W_1 \leq D$ and $W_1^G \leq D$. Hence we have (i).

(j) $O_{\sigma_1}(G) \neq 1$.

Suppose that $O_{\sigma_1}(G) = 1$. Since $G = E = W_1Q$ by (c) and (f) and G is soluble by (b), we have that $1 \neq O_q(G) < Q$. Then by (g) $O^q(G) = G$. Since W_1 is not cyclic by (1) and (a), for some maximal subgroups V_1 and V_2 of W_1 we have that $W_1 = V_1V_2$ and $(V_1)_G = (V_2)_G = 1$. By (i) W_1^G is q-closed. Hence $G \neq W_1^G$ and so $O^q(G) \leq W_1^G < G$ by (f), which contradicts (g). Hence $O_{\sigma_1}(G) \neq 1$.

(k) $N = O_{\sigma_1}(G)$ is the only minimal normal subgroup of G contained in W_1, N is a r-group for some prime $r \in \sigma_1$ and $N = G^{\mathcal{M}} \nleq \Phi(G)$.

By (4), (d) and Lemma 2.8(3)(4), the hypothesis is still holds on G/N for every minimal normal subgroup N of G contained in W_1. By (j) $N \neq 1$. Hence G/N is q-closed. But as the class of all q-closed groups is a saturated formation by Lemma 2.13, we have that $N \nleq \Phi(G)$ and so $N = G^{\mathcal{M}}$ is the only minimal normal subgroup of G contained in W_1. Since G is soluble by (b), N is a r-group for some prime r. Hence $N \leq O_r(G) \leq O_{\sigma_1}(G)$. Since $O_{\sigma_1}(G) \leq W_1$ is nilpotent, $O_r(G) = O_{\sigma_1}(G)$. Let M be a maximal subgroup of G such that $G = NM$. Then $N \cap M \trianglelefteq G$, so $N \cap M = 1$. Hence $O_r(G) = N(O_r(G) \cap M)$. But $O_r(G) \cap M$ is normal in G by [11, Chapter A, Lemma 8.4], so $O_r(G) \cap M = 1$. Hence $N = O_r(G) = O_{\sigma_1}(G)$.

(l) $O_q(G) \neq 1$.

Assume that $O_q(G) = 1$. Then by (k), $F(G) = O_{\sigma_1}(G) = N$. Since G is soluble by (b), we have that $C_G(N) = C_G(F(G)) \leq F(G) = N$ by [16, Theorem 1.8.18]. Since $N \nleq \Phi(G)$ by (k), there exists a maximal subgroup V of W_1 such that $W_1 = NV$, and $V_G = 1$ by (k). Clearly, $|W_1 : V| = r = |N : N \cap V|$. By (4), there exists a normal subgroup T of G such that $VT = V^G$ and $V \cap T \leq V_{\sigma G}$. By the same discussion in Theorem 3.2(4), we have that $V \cap T = V_{\sigma G} \cap T$ is normal in G, so $V \cap T = 1$. Since N is a minimal normal subgroup of G, $N \cap T = 1$ or N. If $N \cap T = 1$, then $T \leq C_G(N) \leq N$, so $T = T \cap N = 1$. Hence $V^G = VT = V$ is normal in G, and so $V = 1$. It follows that $W_1 = N$ and $|W_1| = r$, which contradicts (1). Hence $N \cap T = N$, and so $N \leq T$. Consequently, $N \cap V \leq T \cap V = 1$. So $W_1 = NV = N \times V$. This implies that $V \leq C_G(N) \leq N$. Hence $V = 1$. This shows that $W_1 = N$ and $|W_1| = r$, a contradiction also. Hence $O_q(G) \neq 1$.

Final contradiction for (5).

By (k) there exists a maximal subgroup V of W_1 such that $W_1 = NV$, $V_G = 1$ and $|W_1 : V| = r = |N : N \cap V|$. By (4), there exists a normal subgroup T of G such that $VT = V^G$ and $V \cap T \leq V_{\sigma G}$. The same discussion as Theorem 3.2(4) we have that $V \cap T = V_{\sigma G} \cap T$ is normal in G, so $V \cap T = 1$. Clearly, $|T|_{\sigma_1} \leq r$. If $|T|_{\sigma_1} = 1$, then T is a q-group and so $T \leq Q$. Hence $W_1 \cap V^G = W_1 \cap VT = V(W_1 \cap T) = V$. As N is a minimal normal subgroup of G, we have that $N \cap V^G = 1$ or N. If $N \cap V^G = 1$, then $N \cap V = 1$ and so $|N| = r$. Since G/N is q-closed by (k), QN is normal in G. From $QN/N \simeq Q$, it follows that QN/N is supersoluble. Hence QN is supersoluble. Then $Q \trianglelefteq QN$. This implies that $Q \trianglelefteq G$, a contradiction. Hence $N \cap V^G = N$ and so $N \leq W_1 \cap V^G = V$. This contradiction shows that $|T|_{\sigma_1} = r$. Thus $|V^G|_{\sigma_1} = |V||T|_{\sigma_1} = |W_1|$ and $W_1 \leq V^G$. Obviously, $W_1^G \leq V^G$. Since V^G is q-closed by (h), W_1^G is q-closed. Hence $W_1^G \neq G$. By (e) and (f), G/W_1^G is a q-group. It follows that $O^q(G) \leq W_1^G < G$. Now by (g) $|Q| = q$. But as $O_q(G) \neq 1$ by (l), we have that $Q = O_q(G) \trianglelefteq G$. This contradiction completes the proof of (5).

(6) Let q be the largest prime dividing $|E|$ and Q be a Sylow q-subgroup of E. Then $Q \trianglelefteq G$ and $G/Q \in \mathcal{F}$. Consequently Q is not cyclic.

Indeed, by (5), E is q-closed, so $Q \trianglelefteq G$. By Lemma 2.8(3)(4), the hypothesis is still true for $(G/Q, E/Q)$. Hence $G/Q \in \mathcal{F}$ by the choice of (G, E). Thus we have (6).

(7) Suppose that $Q < E$ and without loss of generality we may assume that $q \in \sigma_i$. Then $W_i \cap E$ is non-cyclic and every maximal subgroup of $W_i \cap E$ is σ-n-embedded in G.

If $W_i \cap E$ is cyclic, then Q is cyclic, which contradicts (6). If there is a maximal subgroup of $W_i \cap E$ is not σ-n-embedded in G, then by hypothesis, every minimal subgroup of $W_i \cap E$ is σ-n-embedded in G. Clearly, $Q \cap W_i = Q \leq W_i \cap E$ and $W_j \cap Q = 1$ for all $j \neq i$. Hence (G, Q)

satisfies the hypothesis. The choice of (G, E) implies that $G \in \mathcal{F}$, a contradiction. Hence (7) holds.

(8) If $Q < E$, then $Q = O_{\sigma_i}(E)$, and so $Q = O_{\sigma_i}(G) \cap E = O_{\sigma_i}(E)$.

Assume that $Q < O_{\sigma_i}(E)$. Then since W_i is nilpotent, there exists a minimal normal subgroup R of G such that $R \leq O_r(E) \leq O_{\sigma_i}(E)$, where $r \in \sigma_i$ and $r \neq q$. By (7) and Lemma 2.8(3)(4), $(G/R, E/R)$ satisfies the hypothesis, so $G/R \in \mathcal{F}$. Then $G \in \mathcal{F}$ by (6) and $R \cap Q = 1$. This contradiction shows that (8) holds.

(9) If $Q < E$, then Q is a minimal normal subgroup of G.

We first claim that $Q \cap \Phi(G) = 1$. In fact, if $Q \cap \Phi(G) \neq 1$ and let N be a minimal normal subgroup of G contained in $Q \cap \Phi(G)$, then by (7), $(G/N, E/N)$ satisfies the hypothesis, so $G/N \in \mathcal{F}$. Consequently $G \in \mathcal{F}$, a contradiction. Therefore $Q \cap \Phi(G) = 1$. By Lemma 2.12, $Q = N_1 \times \cdots \times N_s$, where N_i is a minimal normal subgroup of G. It is also clear that $(G/N_i, E/N_i)$ satisfies the hypothesis. Hence $G/N_i \in \mathcal{F}$. This implies that $s = 1$ and Q is a minimal normal subgroup of G.

(10) $Q = E = P$.

If $Q < E$, then $Q \leq W_i \cap E$. Assume that $Q = W_i \cap E$. Let U be a maximal subgroup of Q such that $U \trianglelefteq W_i$. Then $U \neq 1$ by (6) and $U_G = 1$ by (9). Now by (7) there exists a normal subgroup T of G such that $UT = U^G$ and $U \cap T \leq U_{\sigma G}$. By the same discussion as Theorem 3.2(4), we have that $U \cap T = U_{\sigma G} \cap T$ is normal in G, so $U \cap T = 1$. By (9) $U^G = Q$. Since Q is a minimal normal subgroup of G, $T = 1$ or Q. If $T = 1$, then $U = U^G = Q$, a contradiction. Hence $Q = T$ and so $U = U \cap T = 1$. This contradiction shows that $Q < W_i \cap E$. Since $W_i \cap E$ is nilpotent and Q is a Sylow q-subgroup of $W_i \cap E$, we have that $W_i \cap E = Q \times S$, where S is a non-identity Hall subgroup of $W_i \cap E$. Hence $S \trianglelefteq W_i$. Let W be a maximal subgroup of Q such that $W \trianglelefteq W_i$. Let $V = WS$. Then $V \trianglelefteq W_i$, V is a maximal subgroup of $W_i \cap E$, and $V_G = 1$ by (8) and (9). Then $W_i \cap E = QV$, $|W_i \cap E : V| = q = |Q : Q \cap V|$ and $Q \cap V = W \neq 1$. By (7) there exists a normal subgroup T of G such that $VT = V^G$ and $V \cap T \leq V_{\sigma G}$. By a same discussion as above, we have that $V \cap T = 1$. Since Q is a minimal normal subgroup of G, $Q \cap V^G = 1$ or Q. If $Q \cap V^G = 1$, then $Q \cap V = 1$. This contradiction shows that $Q \leq V^G$. Then $W_i \cap E = QV \leq V^G \leq E$, so $W_i \cap E = W_i \cap V^G = V(W_i \cap T)$. Hence $q = |W_i \cap E : V| = |V(W_i \cap T) : V| = |W_i \cap T|$, and so $W_i \cap T$ is a q-group. But as $T \leq V^G \leq E$, $W_i \cap T \leq E$. Hence $W_i \cap T \leq Q$, so $W_i \cap T \leq Q \cap T$. Since Q is a minimal normal subgroup of G and $W_i \cap T \neq 1$, we have that $Q \leq T$. It follows that $Q \leq W_i \cap T \leq Q \cap T = Q$. Thus $Q = W_i \cap T$ is cyclic, which contradicts (6). Hence $Q = E = P$.

(11) $P = G^{\mathcal{F}}$ is a minimal normal subgroup of G and P is not cyclic.

Let N be a minimal normal subgroup of G contained in P. By (10) $W_1 \cap E = E = P$. Then by (4) and Lemma 2.8(3), $(G/N, E/N)$ satisfies the hypothesis. Hence $G/N \in \mathcal{F}$. This implies that N is the only minimal normal subgroup of G contained in P and $N \nleq \Phi(G)$. Clearly P is not cyclic. Let M be a maximal subgroup of G not containing N. Then $G = NM = PM$. By [11, Chapter A, Lemma 8.4] $P \cap M \trianglelefteq G$. Hence $P \cap M = 1$, so $P = N = G^{\mathcal{F}}$ is a minimal normal subgroup of G.

(12) Final contradiction.

Let V be a maximal subgroup of P such that $V \trianglelefteq W_1$. Then by (4) G has a normal subgroup T such that $VT = V^G$ and $V \cap T \leq V_{\sigma G}$. Similarly as above, we have that $V \cap T = V_{\sigma G} \cap T$ is normal in G and so $V \cap T = 1$ by (11). Since P is non-cyclic, $V \neq 1$. Hence $P = V^G = VT$. But since P is a minimal normal subgroup of G by (11), $T = 1$ or $T = P$. If $T = 1$, then $P = V$, a contradiction. Hence $T = P$, so $V = V \cap T = 1$, a contradiction also. The final contradiction completes the theorem.

4. Some Applications of The Results

The main theorem has many corollaries, we here cite some of them.

Corollary 4.1 — (Buckley [7]). Let G be a group of odd order. If all subgroups of G of prime order are normal in G, then G is supersoluble.

Corollary 4.2 — (Srinivasan [19]). If every maximal subgroups of every Sylow subgroups of G is normal or s-permutable in G, then G is supersoluble.

Corollary 4.3 — (Asaad [20]). If every subgroup of prime order and every cyclic subgroup of order 4 is permutable in G, then G is supersoluble.

Corollary 4.4 — (Wang [4]). If all cyclic subgroups of G with prime order and order 4 are c-normal in G, then G is supersoluble.

Corollary 4.5 — (Wang [4]). If all maximal subgroups of all Sylow subgroups of G are c-normal in G, then G is supersoluble.

Corollary 4.6 — (Wei [21]). Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and G a group with a normal subgroup E such that $G/E \in \mathcal{F}$. If all maximal subgroups of every non-cyclic Sylow subgroup of E are c-normal in G, then $G \in \mathcal{F}$.

Corollary 4.7 — (Ballester-Bolinches and Wang [6]). Let $\mathcal{F}$ be a saturated formation containing

$\mathcal{U}$. If all minimal subgroups and all cyclic subgroups with order 4 of $G^{\mathcal{F}}$ are c-normal in G, then $G \in \mathcal{F}$.

Corollary 4.8 — (Ballester-Bolinches and Pedraza-Aguilera [22]). Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and G a group with a normal subgroup E such that $G/E \in \mathcal{F}$. Assume that a Sylow 2-subgroup of G is abelian. If all minimal subgroups of E are permutable in G, then $G \in \mathcal{F}$.

Corollary 4.9 — (Ballester-Bolinches and Pedraza-Aguilera [22]). Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and G a group with a normal subgroup E such that $G/E \in \mathcal{F}$. If all minimal subgroups and all cyclic subgroups with order 4 of E are permutable in G, then $G \in \mathcal{F}$.

Corollary 4.10 — (Asaad [23]). Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and G a group with a normal subgroup E such that $G/E \in \mathcal{F}$. If every maximal subgroup of every Sylow subgroup of E is s-permutable in G, then $G \in \mathcal{F}$.

Corollary 4.11 — (Guo and Skiba [5]). Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and G a group with a normal subgroup E such that $G/E \in \mathcal{F}$. If for every non-cyclic Sylow subgroup P of E every maximal subgroup of P or every every cyclic subgroups H of P with prime order and order 4 (if P is a non-abelian 2-group and $H \nsubseteq Z_\infty(G)$) is n-embedded in G, then $G \in \mathcal{F}$.

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