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A NOTE ON GRADED LIE H-PSEUDO-BIALGEBRAS

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We study the cohomology theory of graded Lie H-pseudoalgebras. Furthermore, using the differential of a 1-cochain in the reduced complex of graded Lie H-pseudoalgebras, we define the notion of graded Lie H-pseudobialgebras. The coboundary graded Lie H-pseudo-bialgebras and related CYBE theory are also considered.

Key words : Cohomology, graded Lie H-pseudo-bialgebra, CYBE.

1. Introduction

The notion of Lie H-pseudoalgebra over a cocommutative Hopf algebra [1] is a multivariable generalization of the concept of Lie conformal algebra [DK], introduced by Kac in connection with vertex algebras [10]. Algebraic properties of each element in a Lie H-pseudoalgebra may be recast in term of its Fourier coefficients, which are sometimes called creation and annihilation operators in the physical terminology. The space of all annihilation operators is a (typically infinite dimensional) Lie algebra, and the Lie bracket is continuous with respect to a linearly compact topology. Cartan and Guillemin's study of linearly compact infinite dimensional Lie algebras can then be usefully exploited in the study of such structures [5, 8, 9].

Classification problems, cohomology theory, representation theory and related other pseudoalgebras are discussed in [1-3, 6, 12, 15-17]. From the point of view of pseudo-duals, Boyallian and Liberati introduced the notions of Lie H-coalgebras and Lie H-pseudo-bialgebras in [4, 11]. Analogs of the CYBE and Drinfeld double in pseudoalgebras were also developed.

The purpose of this paper is to extend to the graded case some basic results of the fundamental articles [1, 4]. Here we generalize to the graded case some of these notions. In Section 2, we recall some materials on graded Lie H-pseudoalgebras. The cohomology theory of graded Lie H-pseudoalgebras with coefficients in an arbitrary module is considered in Section 3, more importantly, the connection between the cohomology of graded Lie H-pseudoalgebra and its corresponding annihilator algebra is discussed. After a brief account of the basic features in graded pseudoalgebra theory in Section 4, we define graded Lie H-pseudobialgebras, super Manin triples and relate them. Finally, we study the coboundary graded Lie H-pseudo-bialgebras and related CYBE theory.

Many of the results and proofs included here are straightforward extensions of those obtained in [4]. We hope however that this article could also be useful for a starter in the topic: we have, in particular, included detailed proofs of most of the facts announced here.

The reader may turn to [14] for the Hopf algebra terminology. The Sweedler notation is denoted by $\Delta(a) = a_{(1)} \otimes a_{(2)}$. Throughout the paper, all vector spaces, linear maps, and tensor products are considered over an algebraically closed field k, all Hopf algebras H are cocommutative, $X = H^* = \mathrm{Hom}_k(H, k)$ is the dual of H and $\sigma : H \otimes H \longrightarrow H \otimes H$ is the permutation $\sigma(f \otimes g) = g \otimes f$.

2. Graded Lie H-pseudo-bialgebras

In this section, we mainly recall some knowledge on graded Lie H-pseudoalgebras, the reader may refer to [17] for the more terminology on graded Lie H-pseudoalgebras and left symmetric H-pseudoalgebras.

By a grading group, we mean an abelian group $(G, +)$ together with a Z-bilinear symmetric function (bicharater) $(,) : G \times G \longrightarrow Z/2Z$. Elements of G will be called degrees. The identity of G is denoted by 0. For any grading group G, $e(G) = \{a \in G | (a, b) \equiv 0 (mod\ 2), b \in G\}$, and $G/e(G)$ is a grading group with $(x + e(G), y + e(G)) = (x, y)$ for $x + e(G), y + e(G) \in G/e(G)$. Throughout the paper, we denote (x, y) by $|x||y|$.

Definition 2.1 — [17]. A G-graded Lie H-pseudoalgebra over a graded H-module $L = \oplus_{g\in G} L_g$ is a triple $(L, \mu = [*])$, where $\mu \in \mathrm{Hom}_{H^{\otimes 2}}(L \otimes L, H^{\otimes 2} \otimes_H L)$, satisfying
graded-skew-commutativity,

$$[a * b] = -(-1)^{|a||b|}(\sigma \otimes_H I)[b * a].$$

graded-Jacobi identity,

$$(-1)^{|c||a|}[a * [b * c]] + \text{cyclic permutation} = 0$$

in $H^{\otimes 3} \otimes_H L$ for any $a \in L_\alpha,\ b \in L_\beta,\ c \in L_\gamma$.

In particular, for $G = \{0\}$, a graded Lie H-pseudoalgebra is just an ordinary Lie H-pseudoalgebra [1]. When $H = k$, a graded Lie H-pseudoalgebra is an ordinary graded Lie algebra.

For an arbitrary Hopf algebra H, we recall that the map $\mathcal{F} : H \otimes H \longrightarrow H \otimes H$ defined by the formula $\mathcal{F}(f \otimes g) = (f \otimes 1)(S \otimes I)\Delta(g) = fS(g_{(1)}) \otimes g_{(2)}$ is called the Fourier transform. $\mathcal{F}$ is a vector space isomorphism with an inverse given by $\mathcal{F}^{-1}(f \otimes g) = (f \otimes 1)\Delta(g) = fg_{(1)} \otimes g_{(2)}$. The Fourier transform is also called λ-bracket in [1].

Besides pseudobracket $[a * b]$, the authors of [1] introduced another bracket $[a, b] \in H \otimes L$ as the Fourier transform of $[a * b]$:

$$[a, b] = \sum_i \mathcal{F}(f_i \otimes g_i)(1 \otimes c_i) = \sum_i f_i S(g_{i(1)}) \otimes g_{i(2)} c_i \text{ if } [a * b] = \sum_i f_i \otimes g_i \otimes_H c_i.$$

In other words,

$$[a, b] = \sum_i h_i \otimes c_i \text{ if } [a * b] = \sum_i (h_i \otimes 1) \otimes_H c_i.$$

Then for $x \in H^*$, we define the x-bracket in L as follows

$$[a_x b] = \sum_i \langle S(x), f_i S(g_{i(1)})\rangle g_{i(2)} c_i = \sum_i \langle S(x), h_i\rangle c_i.$$

An equivalent definition of graded Lie H-pseudoalgebra is as follows. We call it graded Lie H-conformal algebra.

Definition 2.2 — [17]. A graded Lie H-conformal algebra over a graded H-module L is a triple $(L, [,])$, where $[,] : L \otimes L \longrightarrow H \otimes L$ is defined as above, satisfying the following properties ($a \in L_\alpha,\ b \in L_\beta,\ c \in L_\gamma,\ h \in H$):

H-sesqui-linearity,

$$[ha, b] = (h \otimes 1)[a, b], \quad [a, hb] = (1 \otimes h_{(2)})[a, b](S(h_{(1)}) \otimes 1).$$

graded-skew-symmetry, if $[a, b] = \sum_i h_i \otimes c_i$, then

$$[b, a] = -(-1)^{|a||b|} \sum_i S(h_{i(1)}) \otimes h_{i(2)} c_i. \tag{2.1}$$

graded-Jacobi identity,

$$[a, [b, c]] - (-1)^{|a||b|}(\sigma \otimes I)[b, [a, c]] = (\mathcal{F}^{-1} \otimes I)[[a, b], c] \tag{2.2}$$

in $H \otimes H \otimes L$, where $\sigma : H \otimes H \longrightarrow H \otimes H$ is the permutation $\sigma(f \otimes g) = g \otimes f$.

One can also reformulate Definition 2.1 in terms of the x-bracket.

Definition 2.3 — [17]. A graded Lie H-conformal algebra over a graded H-module L is a triple $(L, [_x])$, where $x \in X,\ [_x] : L \otimes L \longrightarrow L$ is defined as above, satisfying the following properties ($a,\ b,\ c \in L,\ h \in H,\ x,\ y \in X$):

Locality, for any basis $\{x_i\}$ of X, $[a_{x_i} b] \neq 0$ for only a finite number of i.

H-sesqui-linearity,

$$[ha_x b] = [a_{xh} b], \ \ [a_x hb] = h_{(2)}[a_{S(h_{(1)})x} b].$$

Graded-skew-symmetry,

$$[a_x b] = -(-1)^{|a||b|} \sum_i < x, S(h_{i(1)}) > S(h_{i(2)})[b_{x_i} a],$$

where $\{x_i\}$ and $\{h_i\}$ are dual bases in X and H.

Graded-Jacobi identity,

$$[a_x[b_y c]] - (-1)^{|a||b|}[b_y[a_x c]] = [[a_{x_{(2)}} b]_{yx_{(1)}} c].$$

Definition 2.4 — [17]. A G-graded H-module $M = \oplus_{g\in G} M_g$ is said to be a module over a G-graded Lie H-pseudoalgebra L, if it is endowed with an operation $\rho \in \mathrm{Hom}_H(L \otimes M, H^{\otimes 2} \otimes_H M)$, written as $a * c = \rho(a \otimes c)$, which satisfies

$$a * (b * c) - (-1)^{|b||a|}(b * (a * c)) = [a * b] * c$$

for all $a, b \in L, c \in M$.

In the following, we recall the (graded) annihilation algebra of (graded) H-pseudoalgebra and its property.

Let Y be an H-bimodule which is a commutative associative H-differential algebra. For a (graded) left H-module L, recall that $\mathcal{A}_Y L = Y \otimes_H L$ becomes a (graded) left H-module via $h(x \otimes_H a) = (hx) \otimes_H a$ for any $h \in H$ and $x \otimes_H a \in \mathcal{A}_Y L$. If in addition, L is a (graded) Lie H-pseudoalgebra with pseudobracket $[a * b] = \sum_i f_i \otimes g_i \otimes_H e_i$, for $a,\ b \in L$, then $\mathcal{A}_Y(L)$ becomes a (graded) Lie H-differential algebra with bracket given by

$$[x \otimes_H a, y \otimes_H b] = \sum_i (xf_i)(yg_i) \otimes_H e_i.$$

In particular, when $Y = X$, $\mathcal{A}(L) = \mathcal{A}_X L = X \otimes_H L$ is a (graded) Lie H-differential algebra.

Proposition 2.5 — [17]. Any representation (M, ρ_M) of a graded Lie H-pseudoalgebra $(L, [*])$ is naturally a representation of $\mathcal{A}(L)$, with action

$$(x \otimes_H a)m = a_x m = \sum_i < S(x), f_i S(g_{i(1)}) > g_{i(2)} e_i, \text{ if } a * m = \sum_i (f_i \otimes g_i \otimes_H) e_i,$$

for $x \in X, a \in A$ and $m \in M$. This action is compatible with the action of H, that is, $h(\mathfrak{a}m) = (h_1\mathfrak{a})(h_2 m)$ for any $h \in H,\ \mathfrak{a} \in \mathcal{A}(L),\ m \in M$, and satisfies the locality condition: for any basis $\{x_i\}$ of X, $a_{x_i} m \neq 0$ for only a finite number of i. Conversely, any representation (M, ρ_M) of $\mathcal{A}(L)$ satisfying the above conditions is naturally a representation of L, with

$$a * m = \sum_i (S(h_i) \otimes 1) \otimes_H a_{x_i} m$$

where $\{h_i\},\ \{x_i\}$ are dual basis in H and X.

Definition 2.6 — [17]. Let $V = \oplus_{\alpha \in G} V_\alpha$ and $W = \oplus_{\alpha \in G} W_\alpha$ be two graded left H-modules. An H-pseudolinear map from V to W of degree s is a k-linear map

$$\varphi : V_\alpha \longrightarrow H \otimes H \otimes_H W_{\alpha+s}$$

for every $\alpha \in G$ such that $\varphi(hv) = (1 \otimes h \otimes_H 1)\varphi(v)$.

We denote the space of all such φ by $\mathrm{Chom}(V, W)_s$. Then $\mathrm{Chom}(V, W)_s$ is a left H-module with action $(h\varphi)(v) = (h \otimes 1 \otimes_H)\varphi(v)$ for $\varphi \in \mathrm{Chom}(V, W)_s$ and $h \in H$. Thus $\mathrm{Chom}(V, W) = \oplus_{\alpha \in G} \mathrm{Chom}(V, W)_\alpha$ is a G-graded left H-module. If $V = W$, then $\mathrm{Chom}(V, W)_s$ and $\mathrm{Chom}(V, W)$ are denoted by $\mathrm{cend}V_s$ and $\mathrm{cend}V$ respectively. Throughout the paper, unless otherwise specified, we always denote $V^* = \mathrm{Chom}(V, k)$.

3. Cohomology of Graded Lie H-pseudoalgebras

In this section, we focus on the cohomology of graded Lie H-pseudoalgebra.

Let $(L, [*])$ be a graded Lie H-pseudoalgebra, (M, ρ_M) a graded L-module. Define $C^n(L, M)$ $(n \geq 1)$, consisting of all cochains $\gamma \in \mathrm{Hom}_{H^{\otimes n}}(L^{\otimes n}, H^{\otimes n} \otimes_H M)$ that are skew-symmetric. Explicitly, γ has the following properties:

H-polylinearity, for any $h_i \in H,\ a_i \in L$,

$$\begin{aligned} &\gamma(h_1 a_1 \otimes h_2 a_2 \otimes \cdots \otimes h_{n-1} a_{n-1} \otimes h_n a_n) \\ = \ &(h_1 \otimes h_2 \otimes \cdots \otimes h_{n-1} \otimes h_n \otimes_H 1)\gamma(a_1 \otimes a_2 \otimes \cdots \otimes a_{n-1} \otimes a_n). \end{aligned}$$

Graded-skew-symmetry,

$$\begin{aligned}&\gamma(a_1\otimes\cdots\otimes a_{i+1}\otimes a_i\otimes\cdots\otimes a_n)\\ =\ &-(-1)^{|a_i||a_{i+1}|}(\sigma_{i,i+1}\otimes_H I)\gamma(a_1\otimes\cdots\otimes a_i\otimes a_{i+1}\otimes\cdots\otimes a_n),\end{aligned}$$

where $\sigma_{i,i+1}: H^{\otimes n}\longrightarrow H^{\otimes n}$ is the transposition of the ith and $(i+1)$st factors.

For $n\geq 1$, the map $d: C^n(L,M)\longrightarrow C^{n+1}(L,M)$ is defined as follows:

$$\begin{aligned}&(d\gamma)(a_1\otimes a_2\otimes\cdots\otimes a_n\otimes a_{n+1})\\ =\ &\sum_{1\leq i\leq n+1}(-1)^{i+1+|a_i|(|\gamma|+|a_1|+\ldots+|a_{i-1}|)}(\sigma_{1\to i}\otimes_H I)(a_i)*\gamma(a_1\otimes\cdots\otimes\hat{a}_i\otimes\cdots\otimes a_{n+1})\\ &+\sum_{1\leq i<j\leq n+1}(-1)^{i+j+(|a_i|+|a_j|)(|a_1|+\ldots+|a_{i-1}|)+|a_j|(|a_{i+1}|+\ldots+|a_{j-1}|)}(\sigma_{1\to i,2\to j}\otimes_H I)\\ &\times\gamma([a_i*a_j]\otimes a_1\otimes\cdots\otimes\hat{a}_i\otimes\cdots\hat{a}_j\otimes\cdots\otimes a_{n+1}),\end{aligned}\tag{3.1}$$

where $\sigma_{1\to i}$ is the permutation $h_i\otimes h_1\otimes\cdots\otimes h_{i-1}\otimes h_{i+1}\otimes\cdots\otimes h_{n+1}\mapsto h_1\otimes\cdots\otimes h_{n+1}$, $\sigma_{1\to i,2\to j}$ is the permutation $h_i\otimes h_j\otimes h_1\otimes\cdots\otimes h_{i-1}\otimes h_{i+1}\otimes\cdots\otimes h_{j-1}\otimes h_{j+1}\otimes\cdots\otimes h_{n+1}\mapsto h_1\otimes\cdots\otimes h_{n+1}$, and $\hat{a}_i$ means omitted. We can verify that $d^2=0$ by direct computation.

The cohomology of the resulting complex $C^\bullet(L,M)$ is called the cohomology of $(L,[*])$ with coefficients in (M,ρ_M) and is denoted by $H^\bullet(L,M)$. $H^\bullet(L,M)$ is called the reduced cohomology.

We modify the above definition by replacing everywhere $\otimes_H$ by $\otimes$. Let $\tilde{C}^n(L,M)$ consist of all graded skew-symmetric cochains $\gamma\in\mathrm{Hom}_{H^{\otimes n}}(L^{\otimes n},H^{\otimes n}\otimes M)$. Then we can define a differential $\tilde{d}: C^n(L,M)\longrightarrow C^{n+1}(L,M)$ by (3.1) with $\otimes_H$ replaced everywhere by $\otimes$; then again $\tilde{d}^2=0$. The corresponding cohomology $\tilde{H}^\bullet(L,M)$ will be called the basic cohomology of $(L,[*])$ with coefficients in (M,ρ_M).

In the sequel, we will explain that the cohomology theory of graded Lie H-pseudoalgebra describes its extensions.

***Theorem* 3.1** — *Let $(L,[*])$ be a G-graded Lie H-pseudoalgebra, and (M,ρ_M) be a G-graded L-module, considered as a graded Lie H-pseudoalgebra with respect to the zero pseudobracket. Then the equivalence classes of H-split abelian extensions*

$$0\longrightarrow M\longrightarrow\hat{L}\longrightarrow L\longrightarrow 0$$

of the graded Lie H-pseudoalgebra $(L,[*])$ correspond bijectively to $H^2(L,M)$.

PROOF : It is similar to Theorem 15.1 [1]. We give a sketch proof. Denote $\hat{L} = L \oplus M$, whose pseudobracket is given by $[a\hat{*}b]$. For any $a,\ b \in L,\ \ m,\ m' \in M$, we have

$$[a\hat{*}m] = a * m,\ [m\hat{*}m'] = 0,\ [a\hat{*}b] - [a * b] =: \gamma(a \otimes b) \in H^{\otimes 2} \otimes_H M. \tag{3.2}$$

For $\gamma \in C^2(L, M)$, by identity (3.1) and the graded-Jacobi identity of L and $\hat{L}$, we can prove that $d\gamma = 0$.

Conversely, given an element of $H^2(L, M)$, we can choose a representative $\gamma \in C^2(L, M)$ and define an action $[\hat{*}]$ by (3.2). Then $[\hat{*}]$ depends only on the γ. □

Let $(\mathcal{A}(L), [,])$ be the graded annihilator algebra of graded Lie H-pseudoalgebra $(L, [*])$. By Proposition 2.5, any graded L-module (M, ρ_M) has a natural structure of a graded $\mathcal{A}(L)$-module.

Let $\bar{C}^n(L, M) = \mathrm{Hom}(\mathcal{A}(L)^{\otimes n}, M)$ $(n \geq 1)$ be the cochain complex of $\mathcal{A}(L)$ in M, $\bar{d} : \bar{C}^n(L, M) \longrightarrow \bar{C}^{n+1}(L, M)$ its differential. Just as in [1] and [13], we define $\Theta : \tilde{C}^n(L, M) \longrightarrow \bar{C}^n(L, M)$ as follows: for any $\tilde{\gamma} \in \tilde{C}^n(L, M),\ (x_1 \otimes_H a_1) \otimes \cdots \otimes (x_n \otimes_H a_n) \in \mathcal{A}(L)\otimes^n$,

$$\Theta(\tilde{\gamma})(x_1 \otimes_H a_1 \otimes \cdots \otimes x_n \otimes_H a_n) = \tilde{\gamma}_{x_1,\, \cdots ,x_n}(a_1 \otimes \cdots \otimes a_n),$$

where

$$\tilde{\gamma}_{x_1,\, \cdots ,x_n}(a_1 \otimes \cdots \otimes a_n) = \sum_i < S(x_1), g_{i1} > \cdots < S(x_n), g_{in} > e_i,$$

if $\tilde{\gamma}(a_1 \otimes \cdots \otimes a_n) = \sum_i (g_{i1} \otimes \cdots \otimes g_{in}) \otimes e_i$.

Proposition 3.2 — $\bar{d}\Theta = \Theta\tilde{d}$.

PROOF : For any $(x_1 \otimes_H a_1) \otimes \cdots \otimes (x_{n+1} \otimes_H a_{n+1}) \in \mathcal{A}(L)\otimes^{n+1},\ \tilde{\gamma} \in \tilde{C}^n(L, M)$,

$$\begin{aligned} & \bar{d}\Theta(\tilde{\gamma})\Big((x_1 \otimes_H a_1) \otimes \cdots \otimes (x_{n+1} \otimes_H a_{n+1})\Big) \\ = & \sum_{1 \leq i \leq n+1} (-1)^{i+1+|a_i|(|\Theta(\tilde{\gamma})|+|a_1|+\ldots+|a_{i-1}|)}(x_i \otimes_H a_i)\Theta(\tilde{\gamma})\Big((x_1 \otimes_H a_1) \otimes \cdots \otimes \widehat{x_i \otimes_H a_i} \\ & \otimes \cdots \otimes (x_{n+1} \otimes_H a_{n+1})\Big) + \sum_{1 \leq i < j \leq n+1} (-1)^{i+j+(|a_i|+|a_j|)(|a_1|+\ldots+|a_{i-1}|)} \end{aligned}$$

$$
\begin{aligned}
& (-1)^{|a_j|(|a_{i+1}|+\ldots+|a_{j-1}|)}\Theta(\tilde{\gamma})\Big([x_i \otimes_H a_i, x_j \otimes_H a_j] \otimes (x_1 \otimes_H a_1) \otimes \cdots \otimes (\widehat{x_i \otimes_H a_i}) \\
& \otimes \cdots \otimes (\widehat{x_j \otimes_H a_j}) \otimes \cdots \otimes (x_{n+1} \otimes_H a_{n+1})\Big) \\
= & \sum_{1 \le i \le n+1} (-1)^{i+1+|a_i|(|\Theta(\tilde{\gamma})|+|a_1|+\ldots+|a_{i-1}|)}\Big(x_i \otimes_H a_i\Big)\tilde{\gamma}_{x_1,\,\cdots,\hat{x_i},\cdots,x_{n+1}}(a_1 \otimes \cdots \otimes \hat{a_i} \otimes a_{n+1}) \\
& + \sum_{1 \le i < j \le n+1} (-1)^{i+j+(|a_i|+|a_j|)(|a_1|+\ldots+|a_{i-1}|)+|a_j|(|a_{i+1}|+\ldots+|a_{j-1}|)} \\
& \times \tilde{\gamma}_{\sum (x_i h_m)(x_j),\, x_1,\cdots\hat{x_i},\cdots\hat{x_j},\cdots,x_{n+1}}\Big(e_m \otimes a_1 \otimes \cdots \otimes \hat{a_i} \otimes \cdots \otimes \hat{a_j} \otimes a_{n+1}\Big),
\end{aligned}
$$

where we put $[a_i * a_j] = \sum_m h_m \otimes 1 \otimes_H e_m$, hence

$$
[x_i \otimes_H a_i, x_j \otimes_H a_j] = \sum_m (x_i h_m)(x_j) \otimes_H e_m,\ [a_i, a_j] = \sum_m h_m \otimes e_m.
$$

We suppose that

$$
\tilde{\gamma}\Big(e_m \otimes a_1 \otimes \cdots \otimes \hat{a_i} \otimes \cdots \otimes \hat{a_j} \otimes a_{n+1}\Big) = \sum_q l_{q1} \otimes l_{q2} \otimes \cdots l_{qn} \otimes w_q \in H^{\otimes n} \otimes M,
$$

$$
\tilde{\gamma}(a_1 \otimes \cdots \otimes \hat{a_i} \otimes a_{n+1}) = \sum_t g_{t1} \otimes g_{t2} \otimes \cdots \otimes g_{tn} \otimes v_t \in H^{\otimes n} \otimes M,
$$

and

$$
a_i * v_t = \sum_p u_p \otimes 1 \otimes_H \lambda_p.
$$

It follows that

$$
\begin{aligned}
& \Big(x_i \otimes_H a_i\Big)\tilde{\gamma}_{x_1,\,\cdots\hat{x_i},\cdots,x_{n+1}}(a_1 \otimes \cdots \otimes \hat{a_i} \otimes a_{n+1}) \\
= & \Big(x_i \otimes_H a_i\Big)\sum_t < S(x_1), g_{t1} > \cdots < S(x_n), g_{tn} > v_t \\
= & \sum_t < S(x_1), g_{t1} > \cdots < S(x_n), g_{tn} > a_{i_{x_i}} v_t \\
= & \sum_{t,\,p} < S(x_1), g_{t1} > \cdots < S(x_n), g_{tn} >< S(x_i), u_p > \lambda_p,
\end{aligned}
$$

and

$$
\begin{aligned}
& \tilde{\gamma}_{\sum (x_i h_m)(x_j),\, x_1,\cdots\hat{x_i},\cdots\hat{x_j},\cdots,x_{n+1}}\Big(e_m \otimes a_1 \otimes \cdots \tilde{a_i} \otimes \cdots \tilde{a_j} \otimes a_{n+1}\Big) \\
= & \sum_m < S((x_i h_m)(x_j)), l_{q1} >< S(x_1), l_{q2} > \cdots < S(x_{n+1}), l_{qn} > w_q \\
= & \sum_m < (x_i h_m)(x_j), S(l_{q1}) >< S(x_1), l_{q2} > \cdots < S(x_{n+1}), l_{qn} > w_q
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_m < (x_i h_m)\otimes x_j, \Delta(S(l_{q1})) >< S(x_1), l_{q2} > \cdots < S(x_{n+1}), l_{qn} > w_q \\
&= \sum_m < x_i h_m, S(l_{q1})_{(1)} >< x_j, S(l_{q1})_{(2)} >< S(x_1), l_{q2} > \cdots < S(x_{n+1}), l_{qn} > w_q \\
&= \sum_m < x_i, S(l_{q1(1)})S(h_m) >< x_j, S(l_{q1(2)}) >< S(x_1), l_{q2} > \cdots < S(x_{n+1}), l_{qn} > w_q \\
&= \sum_m < S(x_i), h_m l_{q1(1)} >< S(x_j), l_{q1(2)} >< S(x_1), l_{q2} > \cdots < S(x_{n+1}), l_{qn} > w_q.
\end{aligned}
$$

On the other hand,

$$
\Theta\tilde{d}(\tilde{\gamma})\Big((x_1\otimes_H a_1)\otimes\cdots\otimes(x_{n+1}\otimes_H a_{n+1})\Big) = \tilde{d}(\tilde{\gamma})_{x_1,\cdots,x_{n+1}}(a_1\otimes\cdots\otimes a_{n+1}),
$$

$$
\begin{aligned}
&\tilde{d}(\tilde{\gamma})(a_1\otimes\cdots\otimes a_{n+1}) \\
= &\sum_{1\le i\le n+1} (-1)^{i+1+|a_i|(|\gamma|+|a_1|+\ldots+|a_{i-1}|)}(\sigma_{1\to i}\otimes I)a_i\tilde{\gamma}(a_1\otimes\cdots\otimes\hat{a}_i\otimes\cdots\otimes a_{n+1}) \\
&+\sum_{1\le i<j\le n+1} (-1)^{i+j+(|a_i|+|a_j|)(|a_1|+\ldots+|a_{i-1}|)+|a_j|(|a_{i+1}|+\ldots+|a_{j-1}|)}(\sigma_{1\to i,2\to j}\otimes I) \\
&\times\tilde{\gamma}\Big([a_i,a_j]\otimes a_1\otimes\cdots\otimes\hat{a}_i\otimes\cdots\hat{a}_j\otimes\cdots\otimes a_{n+1}\Big). \qquad (3.3)
\end{aligned}
$$

Meanwhile,

$$
\begin{aligned}
a_i\tilde{\gamma}(a_1\otimes\cdots\otimes\hat{a}_i\otimes\cdots\otimes a_{n+1}) &= a_i\sum_t g_{t1}\otimes g_{t2}\otimes\cdots\otimes g_{tn}\otimes v_t \\
&= \sum_{t,\,p} u_p\otimes g_{t1}\otimes g_{t2}\otimes\cdots\otimes g_{tn}\otimes\lambda_p, \qquad (3.4)
\end{aligned}
$$

and

$$
\begin{aligned}
&\tilde{\gamma}\Big([a_i,a_j]\otimes a_1\otimes\cdots\otimes\hat{a}_i\otimes\cdots\hat{a}_j\otimes\cdots\otimes a_{n+1}\Big) \\
= &\sum_m \tilde{\gamma}\Big(h_m\otimes e_m\otimes a_1\otimes\cdots\otimes\hat{a}_i\otimes\cdots\hat{a}_j\otimes\cdots\otimes a_{n+1}\Big) \\
= &\sum_m (h_m\otimes 1^{\otimes n-1})(\Delta\otimes I^{\otimes n-2})(l_{q1}\otimes l_{q2}\otimes\cdots\otimes l_{qn}) \\
= &\sum_m h_m l_{q1(1)}\otimes l_{q1(2)}\otimes l_{q2}\otimes\cdots\otimes l_{qn}. \qquad (3.5)
\end{aligned}
$$

Hence

$$
\begin{aligned}
(3.3) = &\sum_{1\le i\le n+1} (-1)^{i+1+|a_i|(|\Theta(\tilde{\gamma})|+|a_1|+\ldots+|a_{i-1}|)}(\sigma_{1\to i}\otimes I)\times(3.4) \\
+ &\sum_{1\le i<j\le n+1} (-1)^{i+j+(|a_i|+|a_j|)(|a_1|+\ldots+|a_{i-1}|)+|a_j|(|a_{i+1}|+\ldots+|a_{j-1}|)}(\sigma_{1\to i,2\to j}\otimes I)\times(3.5).
\end{aligned}
$$

It follows that $\bar{d}\Theta = \Theta\tilde{d}$. $\square$

Recall that H has a decreasing filtration $F_{-1}H \supset F_0H \supset \cdots$, where

$$F^nH = 0(n<0), F^0H = k[G(H)]$$

and for $n \geq 1$,

$$F^nH = span_k\{h \in H \mid \Delta(h) \in F^0H \otimes h + h \otimes F^0H + \sum_{i=1}^{n-1} F^iH \otimes F^{n-i}H\}.$$

It has the following properties:

$$(F^nH)(F^mH) \subset F^{n+m}H,$$

$$\Delta(F^nH) \subset \sum_{i=0}^{n} F^iH \otimes F^{n-i}H,$$

$$S(F^nH) \subset F^nH,$$

and

$$\bigcup_n F^nH = H.$$

And X has a decreasing filtration $X = F_{-1}X \supset F_0X \supset \cdots$, where $F_nX = (F^nH)^{\perp}$.

Let L_0 be a finite dimensional k-subspace of L which generates L over H, and set

$$F_i\mathcal{A}(L) = \{x \otimes_H l | x \in F_iX, l \in L\}.$$

Since H is cocommutative, $\bigcup_n F^nH = H$, hence $\bigcap F_iX = 0$. This leads to

$$\bigcap F_i\mathcal{A}(L) = 0.$$

For more details about these filtrations one can turn to [1].

Proposition 3.3 — The above map $\Theta : \tilde{C}^{\bullet}(L,M) \longrightarrow \bar{C}^{\bullet}(L,M)$ is an isomorphism from the complex $\tilde{C}^{\bullet}(L,M)$ to the subcomplex $\bar{C}^{\bullet}_{\mathrm{GF}}(L,M)$ of $\bar{C}^{\bullet}(L,M)$ consisting of local cochains, i.e., cochains $\Theta(\tilde{\gamma})$ satisfying

$$\Theta(\tilde{\gamma})\Big((x_1 \otimes_H a_1) \otimes \cdots \otimes (x_n \otimes_H a_n)\Big) = 0 \tag{3.6}$$

for any fixed $x_2, \cdots, x_n$ and $a_1, \cdots, a_n$, and $x_1 \in F_kX$ for $k \gg 0$.

PROOF : Similarly to proposition 2.5. $\square$

Note that the locality condition (3.6) means that $\Theta(\tilde{\gamma})$ is continuous when M is endowed with the discrete topology and $\mathcal{A}(L)$ with the topology as above. Accordingly, we have

Corollary 3.4 — The basic cohomology $\tilde{H}^{\bullet}(L, M)$ of a graded Lie H-pseudoalgebra L is isomorphic to the Gelfand-Fuchs cohomology $H^{\bullet}_{\mathrm{GF}}(\mathcal{A}(L), M)$ of its graded annihilation Lie algebra $\mathcal{A}(L)$.

For more details on Gelfand-Fuchs cohomology of graded Lie algebras, one can turn to the series works of Prof. Fuchs, for example [7].

4. Graded Lie H-pseudo-bialgebras

As for the non graded case [4], we used the cohomology theory of graded Lie H-pseudoalgebras, in order to get to the right notion of cocycle that will be the compatibility condition between pseudo-bracket and coproduct. Consequently, we introduce the notions of graded Lie H-coalgebras and Lie H-pseudo-bialgebras.

Proposition 4.1 — The G-graded H-module $M \otimes N$ is an L-module with the following action ($a \in L, m \in M$ and $n \in N$):

$$a * (m \otimes n) \sum_k (h_k \otimes 1) \otimes_H (m_k \otimes n) + (-1)^{|a||m|} \sum_l (h_l^{'} \otimes 1) \otimes_H (m \otimes n_l),$$

if

$$a * m = \sum_k (h_k \otimes 1) \otimes_H m_k \in (H \otimes H) \otimes_H M$$

and

$$a * n = \sum_l (h_l^{'} \otimes 1) \otimes_H n_l \in (H \otimes H) \otimes_H N.$$

Proof : Similarly to Lemma 4.1 [4]. □

Suppose that L be a graded H-module, denote by τ the super-twist map of $L \otimes L$, namely,

$$\tau(x \otimes y) = (-1)^{|x||y|} y \otimes x$$

for all $x, y \in L$. Denote by ξ the super-cyclic map, that is,

$$\xi = (\tau \otimes 1)(1 \otimes \tau) : x_1 \otimes x_2 \otimes x_3 \longrightarrow (-1)^{|x_1|(|x_2|+|x_3|)} x_2 \otimes x_3 \otimes x_1$$

for $x_1, x_2, x_3 \in L$.

Now we give the following definition:

Definition 4.2 — A graded Lie H-coalgebra is a pair (L, δ) consisting of a graded H-module and $\delta : L \longrightarrow L \otimes L$ satisfying the following conditions:

$$\delta(L_\alpha) \subset \sum_{\alpha=\beta+\gamma} L_\beta \otimes L_\gamma, \tag{4.1}$$

$$Im\delta \subset Im(1 \otimes 1 - \tau), \tag{4.2}$$

$$(1 \otimes 1 \otimes 1 + \xi + \xi^2)(\delta \otimes I)\delta = 0, L \longrightarrow L \otimes L \otimes L. \tag{4.3}$$

This is just the definition of graded Lie coalgebra, compatible with the H-module structure of L.

Let L be a finite free graded H-module with basis $\{a_g\}_{g\in G}$. The dual basis of $\{a_g\}_{g\in G}$ in $L^* = \mathrm{Chom}(L, k)$ is defined by the set $\{a^{g'}\}_{g'\in G}$, where each $a^{g'} \in L^*$ is given by

$$a^g * (a_{g'}) = (1 \otimes 1) \otimes_H \delta_{gg'}.$$

Obviously, $\{a^g\}_{g\in G}$ is a linearly independent set such that H-generates L^*.

***Theorem* 4.3** — *(i) Let $(L = \oplus_{g\in G} H a_g, [*])$ be a finite free graded Lie H-pseudoalgebra, with pseudobracket given by*

$$[a_g * a_{g'}] = \sum_{k=1}^{N} (h_{g''}^{gg'} \otimes l_{g''}^{gg'}) \otimes_H a_{g''}.$$

Let $L^* = \mathrm{Chom}(L, k) = \oplus_{g\in G} H a^g$ be the dual of L, where $\{a^g\}$ is the dual basis corresponding to $\{a_g\}$. Define $\delta : L^* \longrightarrow L^* \otimes L^*$ as

$$\delta(a^{g''}) = \sum_{g,g'} S(h_{g''}^{gg'}) a^g \otimes S(l_{g''}^{gg'}) a^{g'}$$

and $\delta(ha^{g''}) = \Delta(h)\delta(a^{g''})$. Then (L^*, δ) is a graded Lie H-coalgebra.

(ii) Conversely, let (L, δ) be a finite graded Lie H-coalgebra. Then the left H-module $(L^*, [_x])$ is a graded Lie H-conformal algebra with the x-bracket defined by

$$[f\, {}_x\, g]_y(a) = \sum f_{x_{(2)}}(a_{(1)}) g_{yS(x_{(1)})}(a_{(2)})$$

with $f,\, g \in L^*$, $a \in L$, $x,\, y \in X = H^*$ and $\delta(a) = a_{(1)} \otimes a_{(2)}$.

PROOF : Similarly to the proof of Theorem 4.5 in [4]. □

Motivated by the definition of the differential of a 1-cochain in the reduced complex of a graded Lie H-pseudoalgebra, we introduce the following notion.

Definition 4.4 — A graded Lie H-pseudo-bialgebra is a triple $(L, [*], \delta)$ such that $(L, [*])$ is a graded Lie pseudoalgebra, (L, δ) is a graded H-coalgebra and they satisfy the cocycle condition:

$$a * \delta(b) - (-1)^{|a||b|}(\sigma \otimes_H I)(b * \delta(a)) = \delta([a * b]) \tag{4.4}$$

for all $a \in L_\alpha,\ b \in L_\beta$.

A morphism $f : L \longrightarrow L'$ of graded Lie H-pseudo-bialgebras is an H-morphism such that $f[*] = [*](f \otimes f)$ and $\delta f = (f \otimes f)\delta$.

Remark 4.5 : The compatibility condition (4.4) for the cocommutator $\delta : L \longrightarrow L \otimes L$ and pseudobracket $[*]$ in the definition of a graded Lie H-pseudo-bialgebra is indeed the condition that δ is a 1-cocycle of graded Lie H-pseudoalgebra $(L, [*])$ with coefficients in $L \otimes L$ in the reduced complex in Section 3.

The following proposition provides a way to construct a graded Lie H-pseudo-bialgebra from a given graded Lie H-pseudo-bialgebra:

Proposition 4.6 — Let H' be a Hopf subalgebra of H, and $(L, [*], \bar{\delta})$ be a graded Lie H'-pseudo-bialgebra. Then $(Cur(L) = H \otimes_{H'} L, \gamma, \delta)$ is a graded Lie H-pseudo-bialgebra, where $\delta(f \otimes_{H'} a) = (f_{(1)} \otimes_{H'} a_{(1)}) \otimes (f_{(2)} \otimes_{H'} a_{(2)})$,

$$\begin{aligned}
\gamma((f \otimes_{H'} a) \otimes (g \otimes_{H'} b)) &= [(f \otimes_{H'} a) * (g \otimes_{H'} b)] \\
&= \Big((f \otimes g) \otimes_H 1\Big)[a * b] \\
&= \sum_i (ff_i \otimes gg_i) \otimes_H (1 \otimes_{H'} e_i)
\end{aligned}$$

if $\Delta(f) = f_{(1)} \otimes f_{(2)},\ \bar{\delta}(a) = a_{(1)} \otimes a_{(2)}$ and $[a * b] = \sum_i (f_i \otimes g_i) \otimes_{H'} e_i$.

PROOF : Obviously, $(Cur(L) = H \otimes_{H'} L, \gamma)$ is a graded Lie H-pseudoalgebra. In the sequel, we show that $(Cur(L) = H \otimes_{H'} L, \delta)$ is a graded Lie H-coalgebra, we only check that (4.3) holds. For any $f \otimes_{H'} a \in H \otimes_{H'} L$, by the definition of δ, we have

$$\begin{aligned}
(\delta \otimes I)\delta(f \otimes_{H'} a) &= (\delta \otimes I)(f_{(1)} \otimes_{H'} a_{(1)}) \otimes (f_{(2)} \otimes_{H'} a_{(2)}) \\
&= (f_{(11)} \otimes_{H'} a_{(11)}) \otimes (f_{(12)} \otimes_{H'} a_{(12)}) \otimes (f_{(2)} \otimes_{H'} a_{(2)}).
\end{aligned}$$

Since $(L, \bar{\delta})$ is a graded Lie H'-coalgebra,

$$(1 \otimes 1 \otimes 1 + \xi + \xi^2)(\bar{\delta} \otimes I)\bar{\delta} = 0,$$

that is,

$$\sum a_{(11)} \otimes a_{(12)} \otimes a_{(2)} + (-1)^{|a_{(11)}|(|a_{(12)}|+|a_{(2)}|)} a_{(12)} \otimes a_{(2)} \otimes a_{(11)}$$
$$+(-1)^{|a_{(2)}|(|a_{(11)}|+|a_{(12)}|)} a_{(2)} \otimes a_{(11)} \otimes a_{(12)} = 0.$$

Hence

$$(1 \otimes 1 \otimes 1 + \xi + \xi^2)(\delta \otimes I)\delta = 0.$$

It follows that $(Cur(L) = H \otimes_{H'} L, \delta)$ is a graded Lie H-coalgebra.

Finally, we check the compatibility condition (4.4):

$$\delta([(f \otimes_{H'} a) * (g \otimes_{H'} b)]) = (f \otimes_{H'} a) * \delta(g \otimes_{H'} b) - (-1)^{|a||b|}(\sigma \otimes_H I)(g \otimes_{H'} b) * \delta(f \otimes_{H'} a).$$

Suppose

$$[a * b] = \sum_i (f_i \otimes g_i) \otimes_{H'} e_i,\ [a * b_{(1)}] = \sum_i l_i \otimes k_i \otimes_{H'} c_i,$$
$$[a * b_{(2)}] = \sum_i p_i \otimes q_i \otimes_{H'} w_i,\ [b * a_{(1)}] = \sum_i m_i \otimes n_i \otimes_{H'} r_i,$$

and

$$[b * a_{(2)}] = \sum_i s_i \otimes t_i \otimes_{H'} z_i.$$

Then

$$\bar{\delta}([a * b]) = \bar{\delta}\Big(\sum_i f_i \otimes g_i \otimes_{H'} e_i\Big) = \sum_i \big(f_i \otimes g_i\big) \otimes_{H'} \big(e_{i(1)} \otimes e_{i(2)}\big),$$

$$a * \bar{\delta}(b) = \sum_i \big(l_i \otimes k_i\big) \otimes_{H'} \big(c_i \otimes b_{(2)}\big) + \sum_j (-1)^{|a||b_{(1)}|}\big(p_j \otimes q_j\big) \otimes_{H'} \big(b_{(1)} \otimes w_j\big),$$

$$b * \bar{\delta}(a) = \sum_i \big(m_i \otimes n_i\big) \otimes_{H'} \big(r_i \otimes a_{(2)}\big) + \sum_j (-1)^{|a_{(1)}||b|}\big(s_j \otimes t_j\big) \otimes_{H'} \big(a_{(1)} \otimes z_j\big),$$

$$\begin{aligned} \delta([(f \otimes_{H'} a) * (g \otimes_{H'} b)]) &= \delta\Big(\sum_i (ff_i \otimes gg_i) \otimes_H (1 \otimes_{H'} e_i)\Big) \\ &= \sum_i \big(ff_i \otimes gg_i\big) \otimes_H \big((1 \otimes_{H'} e_{i(1)}) \otimes (1 \otimes_{H'} e_{i(2)})\big), \end{aligned} \quad (4.5)$$

$$\begin{aligned}[(f \otimes_{H'} a) * (g_{(1)} \otimes_{H'} b_{(1)})] &= [(f \otimes_{H'} a) * (g_{(1)} \otimes_{H'} b_{(1)})] \\ &= \sum_i (fl_i \otimes g_{(1)} k_i) \otimes_H (1 \otimes_{H'} c_i), \end{aligned} \tag{4.6}$$

and

$$\begin{aligned}[(f \otimes_{H'} a) * (g_{(2)} \otimes_{H'} b_{(2)})] &= [(f \otimes_{H'} a) * (g_{(2)} \otimes_{H'} b_{(2)})] \\ &= \sum_i (fp_i \otimes g_{(2)} q_i) \otimes_H (1 \otimes_{H'} w_i). \end{aligned} \tag{4.7}$$

By (4.6) and (4.7), we have

$$\begin{aligned}(f \otimes_{H'} a) * \delta(g \otimes_{H'} b) &= \sum_i (fl_i \otimes g_{(1)} k_i) \otimes_H \big((1 \otimes_{H'} c_i) \otimes (g_{(2)} \otimes_{H'} b_{(2)})\big) \\ &+ \sum_j (-1)^{|a||b_{(1)}|} (fp_j \otimes g_{(2)} q_j) \otimes_H \big((g_{(1)} \otimes_{H'} b_{(1)} \\ &\otimes (1 \otimes_{H'} w_j)\big). \end{aligned} \tag{4.8}$$

Similarly,

$$\begin{aligned}(g \otimes_{H'} b) * \delta(f \otimes_{H'} a) &= \sum_i \big(gm_i \otimes f_{(1)} n_i\big) \otimes_H \big((1 \otimes_{H'} r_i) \otimes (f_{(2)} \otimes_{H'} a_{(2)})\big) \\ &+ \sum_j (-1)^{|a_{(1)}||b|} (gs_j \otimes f_{(2)} t_j) \otimes_H \big((f_{(1)} \otimes_{H'} a_{(1)} \\ &\otimes (1 \otimes_{H'} z_j)\big). \end{aligned} \tag{4.9}$$

In view of $(L, [*], \bar{\delta})$ being a graded Lie H'-pseudo-bialgebra,

$$\bar{\delta}([a * b]) = a * \bar{\delta}(b) - (-1)^{|a||b|} (\sigma \otimes_{H'} I)(b * \bar{\delta}(a)),$$

that is,

$$\begin{aligned} & \sum_i \big(f_i \otimes g_i\big) \otimes_{H'} \big(e_{i(1)} \otimes e_{i(2)}\big) \\ = & \sum_i \big(l_i \otimes k_i\big) \otimes_{H'} \big(c_i \otimes b_{(2)}\big) + \sum_j (-1)^{|a||b_{(1)}|} \big(p_j \otimes q_j\big) \otimes_{H'} \big(b_{(1)} \otimes w_j\big) \\ & - \sum_i (-1)^{|a||b|} \big(n_i \otimes m_i\big) \otimes_{H'} \big(r_i \otimes a_{(2)}\big) - \sum_j (-1)^{(|a|+|a_{(1)}|)|b|} \big(t_j \otimes s_j\big) \otimes_{H'} \big(a_{(1)} \otimes z_j\big). \end{aligned}$$

Hence $(4.5) = (4.8) - (\sigma \otimes_H I) \times (4.9)$, i.e.,

$$\delta([(f \otimes_{H'} a) * (g \otimes_{H'} b)]) = (f \otimes_{H'} a) * \delta(g \otimes_{H'} b) - (-1)^{|a||b|} (\sigma \otimes_H I)\big(g \otimes_{H'} b) * \delta(f \otimes_{H'} a)\big).$$

Therefore, $(Cur(L) = H \otimes_{H'} L, \gamma, \delta)$ is a graded Lie H-pseudo-bialgebra. $\square$

Remark 4.8 : More generally, let $\varphi : H' \longrightarrow H$ be a homomorphism of Hopf algebra, $(L, [*], \bar{\delta})$ a graded Lie H'-pseudobialgebra, then $(H \otimes_{H'} L, \gamma, \delta)$ is a graded Lie H-pseudobialgebra with

$$\delta(f \otimes_{H'} a) = (f_{(1)} \otimes_{H'} a_{(1)}) \otimes (f_{(2)} \otimes_{H'} a_{(2)}),$$

and

$$\gamma\Big((f \otimes_{H'} a) \otimes (g \otimes_{H'} b)\Big) = \sum_i \Big(f\varphi(f_i) \otimes g\varphi(g_i)\Big) \otimes_H (1 \otimes_{H'} e_i)$$

if $\Delta(f) = f_{(1)} \otimes f_{(2)}$, $\bar{\delta}(a) = a_{(1)} \otimes a_{(2)}$ and $[a * b] = \sum_i (f_i \otimes g_i) \otimes_{H'} e_i$.

In the sequel, we consider the theory of pseudo super Manin triple of graded Lie H-pseudoalgebra.

Let V be a graded H-module. A bilinear pseudo-form on V is a k-bilinear map $\langle\rangle : V \times V \longrightarrow (H \otimes H) \otimes_H k$ such that

$$\langle hv, w\rangle = ((h \otimes 1) \otimes_H 1)\langle v, w\rangle, \ \ \langle v, hw\rangle = ((1 \otimes h) \otimes_H 1)\langle v, w\rangle$$

for all $v, w \in V, h \in H$.

We call a bilinear pseudo-form symmetric if for all $v, w \in V$,

$$\langle v, w\rangle = (-1)^{|v||w|}(\sigma \otimes_H 1)\langle w, v\rangle.$$

A bilinear pseudo-form in a graded Lie pseudoalgebra L is called invariant if

$$\langle [a * b], c\rangle = \langle a, [b * c]\rangle$$

for all $a, b, c \in L$, where the usual composition rules of polylinear maps are applied in.

Given a bilinear pseudo-form on a H-module V, we have a homomorphism of H-modules, $\phi : V \longrightarrow V^* = \mathrm{Chom}(V, k)$, $v \mapsto \phi_v$, given as usual by

$$(\phi_v) * w = \langle v, w\rangle, \ \ v \in V.$$

Now, suppose that a bilinear pseudo-form satisfies that $\langle v, w\rangle = 0$ for all $w \in V$, implies $v = 0$. Then ϕ gives an injective map between V and V^*, but not necessarily surjective.

Following [4], a bilinear pseudo-form is called non-degenerate if ϕ gives an isomorphism between V and V^*.

Definition 4.9 — A (finite rank) pseudo super Manin triple is a triple of finite rank graded Lie pseudoalgebras (L, L_1, L_0), where L is equipped with a non-degenerate invariant symmetric bilinear pseudo-form $\langle , \rangle$ such that

(i) L_1, L_0 are graded Lie pseudosubalgebras of L and $L = L_0 \oplus L_1$ as H-module.

(ii) L_0 and L_1 are isotropic with respect to $\langle , \rangle$, that is $\langle L_i, L_i \rangle = 0$ for $i = 0, 1$.

***Theorem* 4.10** — *Let L be a graded Lie pseudoalgebra free of finite rank. Then there is a one-to-one correspondence between graded Lie pseudo-bialgebra structures on L and pseudo super Manin triples (R, R_1, R_0) such that $R_1 = L$.*

PROOF : Similarly to the proof of Theorem [6.2] in [4]. □

5. COBOUNDARY GRADED LIE H-PSEUDO-BIALGEBRAS

Let $(L, [*])$ be a graded Lie H-pseudoalgebra, $r = \sum_i a_i \otimes b_i \in L \otimes L$. We define the classical Yang-Baxter equation (CYBE) as follows:

$$[[r, r]] = \sum_{i,j} \mu^3_{-1}\Big((-1)^{|a_i||b_j|}[a_j, a_i] \otimes b_j \otimes b_i\Big) - \mu^4_{-2}\Big((-1)^{|a_j||b_i|} a_i \otimes [a_j, b_i] \otimes b_j\Big)$$
$$-\mu^2_{-3}\Big(a_i \otimes a_j \otimes [b_j, b_i]\Big). \tag{5.1}$$

where μ^l_{-k} means that the element of H that appears in its argument in the k-th place acts via the antipode on the element of L located in the l-th entry. For example, $\mu^3_{-1}(h \otimes l \otimes m \otimes n) = l \otimes S(h)m \otimes n$ with $l, m, n \in L, h \in H$.

Definition 5.1 — A coboundary graded Lie H-pseudo-bialgebra $(L, [*], \delta, r)$ consists of a graded Lie H-pseudo-bialgebra $(L, [*], \delta)$ and an element $r = \sum a_i \otimes b_i \in L^{\otimes 2}$ and

$$\delta(a) = a \cdot r = \sum \mu\Big([a, a_i] \otimes b_i + (-1)^{|a_i||a|}\sigma_{12}\big(a_i \otimes [a, b_i]\big)\Big)$$

for all $a \in L$, where $\mu : H \otimes (L \otimes L) \longrightarrow L \otimes L$ given by $\mu(h \otimes m \otimes n) = \Delta(h)(m \otimes n)$.

***Theorem* 5.2** — *Let $(L, [*])$ be a graded Lie H-pseudoalgebra and $\mu : H \otimes (L \otimes L) \longrightarrow L \otimes L$ given by $\mu(h \otimes m \otimes n) = \Delta(h)(m \otimes n)$, where $r = \sum_i a_i \otimes b_i \in L \otimes L$ is an even element. Then the map $\delta_r : L \longrightarrow L \otimes L$ given by ($a \in L$)*

$$\delta_r(a) = a \cdot r = \sum \mu\Big([a, a_i] \otimes b_i + (-1)^{|a||a_i|}\sigma_{12}\big(a_i \otimes [a, b_i]\big)\Big).$$

is the cocommutator of a graded Lie H-pseudo-bialgebra structure on $(L, [*], \alpha)$ if and only if the following conditions are satisfied:

(i) $\delta_{r+r_{21}}(a)=0$, *where* $r_{21}=-\sum_i(-1)^{|a_i||b_i|}b_i\otimes a_i$.

(ii)

$$\mu_3(a\cdot[[r,r]])=0,$$

where $\mu_3(h\otimes m\otimes n\otimes p)=((\Delta\otimes I)\Delta(h))(m\otimes n\otimes p)$ *and*

$$\begin{aligned} a\cdot(b_1\otimes b_2\otimes b_3) &= a\cdot b_1\otimes b_2\otimes b_3+(-1)^{|a||b_1|}b_1\otimes a\cdot b_2\otimes b_3\\ &\quad +(-1)^{|a|(|b_1|+|b_2|)}b_1\otimes b_2\otimes a\cdot b_3. \end{aligned}$$

PROOF : It follows the lines of the proof of the corresponding 'non-super' statements [4] with straightforward modifications. For any $a,b\in L$, we denote $[a*b]=h^{a,b}\otimes 1\otimes_H c_{a,b}$. By the definition of δ_r, we get

$$\begin{aligned} \delta_r(a)=a\cdot r &= \sum_i \mu\Big([a,a_i]\otimes b_i+(-1)^{|a||a_i|}\sigma_{12}(a_i\otimes[a,b_i])\Big)\\ &= \sum_i \mu\Big(h^{a,a_i}\otimes c_{a,a_i}\otimes b_i+(-1)^{|a||a_i|}\sigma_{12}(a_i\otimes h^{a,b_i}\otimes c_{a,b_i})\Big)\\ &= \sum_i h^{a,a_i}\cdot\Big(c_{a,a_i}\otimes b_i\Big)+(-1)^{|a||a_i|}h^{a,b_i}\cdot\Big(a_i\otimes c_{a,b_i}\Big). \end{aligned}$$

Obviously, the graded skew-symmetry of δ_r is equivalent to $\delta_{r+r_{21}}(a)=0$. Now,

$$\begin{aligned} (\delta_r\otimes I)\delta_r(a) &= \sum_i(\delta_r\otimes I)\Big(h^{a,a_i}\cdot(c_{a,a_i}\otimes b_i)+(-1)^{|a||a_i|}h^{a,b_i}\cdot(a_i\otimes c_{a,b_i})\Big)\\ &= \sum_i h^{a,a_i}_{(1)}\delta_r(c_{a,a_i})\otimes h^{a,a_i}_{(2)}b_i+(-1)^{|a||a_i|}h^{a,b_i}_{(1)}\delta_r(a_i)\otimes h^{a,b_i}_{(2)}c_{a,b_i}\\ &= \sum_i h^{a,a_i}\cdot\Big(\delta_r(c_{a,a_i})\otimes b_i\Big)+(-1)^{|a||a_i|}h^{a,b_i}\cdot\Big(\delta_r(a_i)\otimes c_{a,b_i}\Big)\\ &= \sum_{i,j} h^{a,a_i}\cdot\Big(h^{c_{a,a_i},a_j}\cdot(c_{c_{a,a_i},a_j}\otimes b_j)\otimes b_i+(-1)^{|a_j|(|a|+|a_i|)}h^{c_{a,a_i},b_j}\cdot(a_j\\ &\quad \otimes c_{c_{a,a_i},b_j})\otimes b_i\Big)+(-1)^{|a||a_i|}h^{a,b_i}\cdot\Big(h^{a_i,a_j}\cdot(c_{a_i,a_j}\otimes b_j)\otimes c_{a,b_i}\\ &\quad +(-1)^{|a_i|(|a|+|a_j|)}h^{a_i,b_j}\cdot(a_j\otimes c_{a_i,b_j})\otimes c_{a,b_i}\Big)\\ &= \sum_{i,j}\mu_1\Big(\mu_2^{3,4}([[a,a_i],a_j]\otimes b_j\otimes b_i)\Big)+(-1)^{|a_j|(|a|+|a_i|)}\mu_2\Big(\mu_3^{1,4}(a_j\\ &\quad \otimes[[a,a_i],b_j]\otimes b_i)\Big)+(-1)^{(|a||a_i|)}\mu_3\Big(\mu_1^{2,3}([a_i,a_j]\otimes b_j\otimes[a,b_i])\Big)\\ &\quad +(-1)^{|a|(|a_j|+|a_i|)}\mu_3\Big(\mu_2^{1,3}(a_j\otimes[a_i,b_j]\otimes[a,b_i])\Big), \end{aligned}$$

where $\mu_k^{r,s}$ means that the element of H that appears in its argument in the k-th place acts on the elements of $L\otimes L$ formed by the elements in the r and s-th entries; μ_k represents the action of the element of H in the k-th place acting in the element of $L\otimes L\otimes L$ formed by the remaining elements in its argument. For example, for any $m,\ n,\ p\in L,\ f,\ g\in H$,

$$\mu_2^{1,3}(m\otimes f\otimes n\otimes g\otimes p)=f_{(1)}m\otimes f_{(2)}n\otimes g\otimes p,$$

and

$$\mu_3(m\otimes n\otimes f\otimes p)=(\Delta\otimes I)\Delta(f)(m\otimes n\otimes p).$$

Let $\sum_{c.p.}$ be cyclic permutations of the factors in $L\otimes L\otimes L$. Then

$$\begin{aligned}
\sum_{c.p.}(\delta_r\otimes I)\delta_r(a) \quad = \quad & \sum_{i,j}\mu_1\Big(\mu_2^{3,4}([[a,a_i],a_j]\otimes b_j\otimes b_i)\Big) && (5.2)\\
&+(-1)^{(|a|+|a_i|)|a_j|}\mu_2\Big(\mu_3^{1,4}(a_j\otimes[[a,a_i],b_j]\otimes b_i)\Big) && (5.3)\\
&+\mu_3\Big(\mu_1^{2,3}\big((-1)^{|a||a_i|}[a_i,a_j]\otimes b_j\otimes[a,b_i]\big)\Big) && (5.4)\\
&+(-1)^{(|a|+|a_j|)|a_i|}\mu_3\Big(\mu_2^{1,3}(a_j\otimes[a_i,b_j]\otimes[a,b_i])\Big) && (5.5)\\
&+(-1)^{(|a|+|a_i|)|b_i|}\mu_2\Big(\mu_3^{4,5}(b_i\otimes[[a,a_i],a_j]\otimes b_j)\Big) && (5.6)\\
&+(-1)^{(|a|+|a_i|)(|a_j|+|b_i|)}\mu_3\Big(\mu_4^{2,5}(b_i\otimes a_j\otimes[[a,a_i],b_j])\Big) && (5.7)\\
&+(-1)^{|a_i||b_i|}\mu_1\Big(\mu_3^{4,5}([a,b_i]\otimes[a_i,a_j]\otimes b_j)\Big) && (5.8)\\
&+(-1)^{|a_i|(|a_j|+|b_i|)}\mu_1\Big(\mu_4^{3,5}([a,b_i]\otimes a_j\otimes[a_i,b_j])\Big) && (5.9)\\
&+(-1)^{(|a|+|a_i|+|a_j|)(|b_i|+|b_j|)}\mu_3\Big(\mu_4^{1,5}(b_j\otimes b_i\otimes[[a,a_i],a_j])\Big) && (5.10)\\
&+(-1)^{(|a_i|+|b_j|)|a_j|)}\mu_1\Big(\mu_2^{3,5}([[a,a_i],b_j]\otimes b_i\otimes a_j)\Big) && (5.11)\\
&+(-1)^{|a||a_i|}(-1)^{(|a_i|+|a_j|)(|b_j|+|a|+|b_i|)}\mu_2\\
&\Big(\mu_4^{1,5}(b_j\otimes[a,b_i]\otimes[a_i,a_j])\Big) && (5.12)\\
&+(-1)^{|a||a_i|}(-1)^{|a_i||a_j|}(-1)^{|a_j|(|b_j|+|a|)}\mu_2\\
&\Big(\mu_1^{2,5}([a_i,b_j]\otimes[a,b_i]\otimes a_j)\Big). && (5.13)
\end{aligned}$$

In the light of the graded skew commutativity in (2.1), we obtain

$$\begin{aligned}
\mu_3(a\cdot[[r,r]]) &= \sum_{i,j}\mu^3_{-1}\Big((-1)^{|a_i||b_j|}[a_j,a_i]\otimes b_j\otimes b_i\Big)-\mu^4_{-2}\Big((-1)^{|a_j||b_i|}a_i\otimes[a_j,b_i]\otimes b_j\Big)\\
&\quad -\mu^2_{-3}\Big(a_i\otimes a_j\otimes[b_j,b_i]\Big)\\
&= -\mu_1\Big(\mu_2^{3,4}(\mathcal{F}\otimes I\otimes I\otimes I)([a,[a_i,a_j]]\otimes b_j\otimes b_i)\Big) && (5.14)\\
&\quad +\mu_2\Big(\mu_1^{2,5}((-1)^{|a_j||b_i|}(-1)^{|a|(|a_i|+|a_j|)}[a_i,a_j]\otimes[a,b_i]\otimes b_j)\Big) && (5.15)\\
&\quad +\mu_3\Big((\mu^3_{-1}((-1)^{|a_i|(|b_j|+|a|)}[a_j,a_i]\otimes b_j\otimes[a,b_i]))\Big) && (5.16)\\
&\quad +\mu_1\Big(\mu_3^{4,5}([a,a_i]\otimes[b_i,a_j]\otimes b_j)\Big) && (5.17)\\
&\quad +\mu_2\Big(\mu_3^{4,5}(I\otimes\mathcal{F}\otimes I\otimes I)((-1)^{|a_j||a|}a_j\otimes[a,[b_j,a_i]]\otimes b_i)\Big) && (5.18)\\
&\quad +\mu_3\Big(\mu^1_{-2}((-1)^{|a_i||a|}a_j\otimes[b_j,a_i]\otimes[a,b_i])\Big) && (5.19)\\
&\quad +\mu_1\Big(\mu_4^{3,5}((-1)^{|b_i||b_j|}[a,a_i]\otimes a_j\otimes[b_i,b_j])\Big) && (5.20)\\
&\quad -\mu_3^{14}\Big(a_i\otimes(I\otimes\rho(b_i))(\mu_1^{2,3}((-1)^{|a_i||a|}[a,a_j]\otimes b_j)\Big) && (5.21)\\
&\quad +\mu_3\Big(\mu_4^{2,5}(I\otimes I\otimes\mathcal{F}\otimes I)((-1)^{|b_i||b_j|}(-1)^{(|a_i|+|a_j|)|a|}a_i\otimes a_j)\\
&\quad \otimes[a,[b_i,b_j]])\Big), && (5.22)
\end{aligned}$$

where $\mathcal{F}$ is the Fourier transform defined and $\rho(b)(a)=[a,b]$ for all $a,\ b\in L$. Subsequently, we verify that

$$\mu_3(a\cdot[[r,r]])+\sum_{c.p.}(\delta_r\otimes I)\delta_r(a)=0.$$

First, observe that $(5.4)+(5.16)=0$. Indeed, by the graded skew-symmetry in (2.1),

$$\begin{aligned}
(5.4) &= \mu_3\Big(\mu_1^{2,3}((-1)^{|a||a_i|}[a_i,a_j]\otimes b_j\otimes[a,b_i])\Big)\\
&= -\mu_3\Big(\mu_1^{2,3}((-1)^{|a||a_i|}(-1)^{|a_j||a_i|}h^{a_j,a_i}_{(-1)}\otimes h^{a_j,a_i}_{(2)}c_{a_j,a_i}\otimes b_j\otimes[a,b_i])\Big)\\
&= -\mu_3\Big((-1)^{(|a|+|a_j|)|a_i|}h^{a_j,a_i}_{(-1)_{(1)}}\,h^{a_j,a_i}_{(2)}c_{a_j,a_i}\otimes h^{a_j,a_i}_{(-1)_{(2)}}b_j\otimes[a,b_i]\Big)\\
&= -\mu_3\Big((-1)^{(|a|+|a_j|)|a_i|}c_{a_j,a_i}\otimes S(h^{a_j,a_i})b_j\otimes[a,b_i]\Big)\\
&= -\mu_3\Big(\mu^3_{-1}((-1)^{(|a|+|a_j|)|a_i|}[a_j,a_i]\otimes b_j\otimes[a,b_i])\Big)\\
&= -(5.16).
\end{aligned}$$

The remaining part is similar to that Theorem 5.3 in [4]. We omit the details. It follows that

$$\mu_3(a\cdot[[r,r]])+\sum_{c.p.}(\delta_r\otimes I)\delta_r(a)=0.\square$$

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