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INTEGRAL REPRESENTATIONS OF THE LARGE AND LITTLE SCHRÖDER NUMBERS

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In the paper, the authors establish several integral representations for the generating functions of the large and little Schröder numbers and for the large and little Schröder numbers.

Key words : Large Schröder number; little Schröder numbers; integral representation; generating function; complete monotonicity; Bernstein function; Stieltjes transform.

1. INTRODUCTION

In combinatorics and number theory, there are two kinds of Schröder numbers, the large Schröder numbers S_n and the little Schröder numbers s_n. They are named after the German mathematician Ernst Schröder.

A large Schröder number S_n describes the number of paths from the southwest corner $(0,0)$ of an $n \times n$ grid to the northeast corner (n,n), using only single steps north, northeast, or east, that do not rise above the southwest-northeast diagonal. The first eleven large Schröder numbers S_n for $0 \le n \le 10$ are

$$1, \quad 2, \quad 6, \quad 22, \quad 90, \quad 394, \quad 1806, \quad 8558, \quad 41586, \quad 206098, \quad 1037718.$$

In [3, Theorem 8.5.7], it was proved that the large Schröder numbers S_n have the generating function

$$G(t) = \frac{1 - t - \sqrt{t^2 - 6t + 1}}{2t} = \sum_{n=0}^{\infty} S_n t^n. \tag{1.1}$$

The little Schröder numbers s_n form an integer sequence that can be used to count the number of plane trees with a given set of leaves, the number of ways of inserting parentheses into a sequence, and the number of ways of dissecting a convex polygon into smaller polygons by inserting diagonals. The first eleven little Schröder numbers s_n for $1 \le n \le 11$ are

$$1, \quad 1, \quad 3, \quad 11, \quad 45, \quad 197, \quad 903, \quad 4279, \quad 20793, \quad 103049, \quad 518859.$$

They are also called the small Schröder numbers, the Schröder-Hipparchus numbers, or the Schröder numbers, after Ernst Schröder and the ancient Greek mathematician Hipparchus who appears from evidence in Plutarch to have known of these numbers. They are also called the super-Catalan numbers, after Eugéne Charles Catalan, but different from any generalization of the Catalan numbers in the monograph [8] and the papers [13, 14, 25]. In [3, Theorem 8.5.6], it was proved that the little Schröder numbers s_n have the generating function

$$g(t) = \frac{1 + t - \sqrt{t^2 - 6t + 1}}{4} = \sum_{n=1}^{\infty} s_n t^n. \tag{1.2}$$

For more information on the large Schröder numbers S_n and the little Schröder numbers s_n, please refer to the monograph [3] and plenty of references therein.

Comparing (1.1) with (1.2), we see easily that

$$\sqrt{t^2 - 6t + 1} = 1 + t - 4\sum_{n=1}^{\infty} s_n t^n = 1 - t - 2\sum_{n=0}^{\infty} S_n t^{n+1},$$

that is,

$$1 - 2\sum_{n=1}^{\infty} s_n t^{n-1} = 1 - 2\sum_{n=0}^{\infty} s_{n+1} t^n = -\sum_{n=0}^{\infty} S_n t^n.$$

Accordingly, we acquire

$$S_n = 2s_{n+1}, \quad n \in \mathbb{N}.$$

See also [3, Corollary 8.5.8].

Recently, the following two explicit formulas for the large and little Schröder numbers S_n and s_{n+1} were established in [17]. For $n \in \mathbb{N}$, the large and little Schröder numbers S_n and s_{n+1} can be computed by

$$S_n = 2s_{n+1} = \frac{1}{12}\frac{(-1)^n}{6^n}\sum_{k=\lceil (n+1)/2\rceil}^{n+1}\frac{6^{2k}}{k!}\left\langle\frac{1}{2}\right\rangle_k\binom{k}{n-k+1}$$

and

$$S_n = 2s_{n+1} = \frac{1}{12}\frac{n!}{6^n}\sum_{k=0}^{n}\sum_{r+s=k}\sum_{\ell+m=n}\sum_{q=0}^{s}\sum_{j=0}^{n-r-1}(-1)^{s-q}6^{2(r+j)}\left\langle\frac{1}{2}\right\rangle_k\left\langle\frac{1}{2}-k\right\rangle_j \times\binom{\ell-1}{r-1}\binom{s}{q}\binom{j}{n-r-j+1}\binom{m+2q-1}{2q-1}\frac{1}{r!s!j!},$$

where $\lceil t\rceil$ stands for the ceiling function which gives the smallest integer not less than t and $\langle t\rangle_n$ is the falling factorial defined by

$$\langle t\rangle_n = \prod_{k=0}^{n-1}(t-k) = \begin{cases} t(t-1)\cdots(t-n+1), & n \ge 1,\\ 1, & n = 0.\end{cases}$$

Recall from [9, Chapter XIII], [26, Chapter 1] and [27, Chapter IV] that an infinitely differentiable function f is said to be completely monotonic on an interval I if it satisfies $0 \le (-1)^k f^{(k)}(t) < \infty$ on I for all $k \ge 0$. It is known [27, p. 161, Theorem 12b] that a function f is completely monotonic on $(0,\infty)$ if and only if it is a Laplace transform $f(t) = \int_0^\infty e^{-ts}\,\mathrm{d}\,\mu(s)$ of a positive measure μ defined on $[0,\infty)$ such that the above integral converges on $(0,\infty)$. For more information on this topic, please refer to the chapters [9, Chapter XIII] and [27, Chapter IV] and to the monograph [26].

An infinitely differentiable and nonnegative function $f : I \to [0,\infty)$ is called a Bernstein function on an interval I if $f'(t)$ is completely monotonic on I. See Definition 1.2 in [6]. Theorem 3.2 in [26] states that a function $f : (0,\infty) \to [0,\infty)$ is a Bernstein function if and only if it admits the Lévy-Khintchine representation

$$f(x) = a + bx + \int_0^\infty\left(1-e^{-xt}\right)\mathrm{d}\,\mu(t), \tag{1.3}$$

where $a, b \ge 0$ and μ is a Lévy measure on $(0,\infty)$, with $\int_0^\infty \min\{1,t\}\,\mathrm{d}\,\mu(t) < \infty$. The Lévy triplet (a,b,μ) determines f uniquely and vice versa.

In this paper, we will establish several integral representations for principal branches of the generating functions $\mathcal{G}(z) = G(-z)$ and $\mathfrak{G}(z) = g(-z)$ on the cut complex plane $\mathbb{C}\setminus\left[-2\sqrt{2}-3, 2\sqrt{2}-3\right]$

and for the large and little Schröder numbers S_n and s_{n+1}. By the way, we will find that the function $\mathcal{G}(t)$ is completely monotonic on $\big(2\sqrt{2}\,-3,\infty\big)$ and the negative of $\mathfrak{G}(t)$ is a Bernstein function on the interval $\big(3-2\sqrt{2}\,,\infty\big)$.

Our main results can be summarized as the following two theorems.

***Theorem* 1.1** — *The principal branch of the generating function*

$$\mathcal{G}(z)=\frac{\sqrt{z^2+6z+1}\,-1-z}{2z} \tag{1.4}$$

on the cut complex plane $\mathbb{C}\setminus\big[-2\sqrt{2}\,-3,2\sqrt{2}\,-3\big]$ *has the integral representations*

$$\mathcal{G}(z)=\frac{\sqrt{2}}{\pi}\int_0^\infty \varrho(s)e^{-(3-2\sqrt{2}\,)s}\int_{1/e}^1 v^{zs-1}\operatorname{d}v\operatorname{d}s, \tag{1.5}$$

$$\mathcal{G}(z)=\frac{2\sqrt{2}}{\pi}\int_0^\infty F(s)e^{-(3-2\sqrt{2}\,)s}\int_{1/e}^1 v^{zs-1}\operatorname{d}v\operatorname{d}s, \tag{1.6}$$

and

$$\begin{aligned}\mathcal{G}(z)&=\frac{1}{2\pi}\int_0^\infty sq(s)\int_{1/e}^1 v^{zs-1}\operatorname{d}v\operatorname{d}s\\&=\frac{1}{2\pi}\int_{3-2\sqrt{2}}^{3+2\sqrt{2}}\frac{\sqrt{-u^2+6u-1}}{u}\frac{1}{u+z}\operatorname{d}u,\end{aligned} \tag{1.7}$$

where

$$\varrho(s)=\int_0^{1/2}\left(\sqrt{\frac{1}{u}-1}-\frac{1}{\sqrt{1/u-1}}\right)\big[e^{-4\sqrt{2}\,su}-e^{-4\sqrt{2}\,s(1-u)}\big]\operatorname{d}u,$$

$$F(s)=\int_0^1\left(\frac{1}{u}-1\right)^{1/2}\left[1-\frac{1}{2(1-u)}\right]e^{-4\sqrt{2}\,su}\operatorname{d}u,$$

and

$$q(s)=\int_{3-2\sqrt{2}}^{3+2\sqrt{2}}\sqrt{-u^2+6u-1}\,e^{-su}\operatorname{d}u$$

are positive. Consequently,

1. *the generating function* $\mathcal{G}(t)$ *is completely monotonic on* $\big(2\sqrt{2}\,-3,\infty\big)$*;*

2. *the large and little Schröder numbers* S_n *and* s_{n+1} *can be represented by the integral represen-*

tations

$$\begin{aligned}S_n = 2s_{n+1} &= \frac{\sqrt{2}}{\pi}\frac{1}{(n+1)!}\int_0^\infty \varrho(s)e^{-(3-2\sqrt{2}\,)s}s^n \,\mathrm{d}\, s\\ &= \frac{\sqrt{2}}{\pi}\frac{1}{n+1}\int_0^{1/2}\left(\sqrt{\frac{1}{u}-1}-\frac{1}{\sqrt{1/u-1}}\right)\\ &\quad\times\left[\frac{1}{(3-2\sqrt{2}\,+4\sqrt{2}\,u)^{n+1}}-\frac{1}{(3+2\sqrt{2}\,-4\sqrt{2}\,u)^{n+1}}\right]\mathrm{d}\, u,\end{aligned} \tag{1.8}$$

$$\begin{aligned}S_n = 2s_{n+1} &= \frac{2\sqrt{2}}{\pi}\frac{1}{(n+1)!}\int_0^\infty F(s)e^{-(3-2\sqrt{2}\,)s}s^n \,\mathrm{d}\, s\\ &= \frac{\sqrt{2}}{\pi}\frac{1}{n+1}\int_0^1\frac{2u-1}{\sqrt{u(1-u)}}\frac{1}{\left(4\sqrt{2}\,u+3-2\sqrt{2}\,\right)^{n+1}}\,\mathrm{d}\, u,\end{aligned} \tag{1.9}$$

and

$$\begin{aligned}S_n = 2s_{n+1} &= \frac{1}{2\pi}\frac{1}{(n+1)!}\int_0^\infty q(s)s^{n+1}\,\mathrm{d}\, s\\ &= \frac{1}{2\pi}\int_{3-2\sqrt{2}}^{3+2\sqrt{2}}\frac{\sqrt{-u^2+6u-1}}{u^{n+2}}\,\mathrm{d}\, u\end{aligned} \tag{1.10}$$

for $n \ge 0$.

***Theorem* 1.2** — *The principal branch of the generating function*

$$\mathfrak{G}(z) = \frac{1-z-\sqrt{z^2+6z+1}}{4}$$

on the cut complex plane $\mathbb{C}\setminus\left[-2\sqrt{2}\,-3, 2\sqrt{2}\,-3\right]$ *has the integral representations*

$$\mathfrak{G}(z) = -\frac{1}{2}z-\frac{\sqrt{2}}{2\pi}\int_0^\infty\frac{\varrho(s)}{s}e^{-(3-2\sqrt{2}\,)s}(1-e^{-zs})\,\mathrm{d}\, s, \tag{1.11}$$

$$\mathfrak{G}(z) = -\frac{1}{2}z-\frac{\sqrt{2}}{\pi}\int_0^\infty\frac{F(s)}{s}e^{-(3-2\sqrt{2}\,)s}(1-e^{-zs})\,\mathrm{d}\, s, \tag{1.12}$$

and

$$\mathfrak{G}(z) = -\frac{1}{2}z-\frac{\pi}{4}\int_0^\infty q(s)(1-e^{-zs})\,\mathrm{d}\, s. \tag{1.13}$$

Consequently, the negative of $\mathfrak{G}(t)$ is a Bernstein function on $\bigl(3-2\sqrt{2}\,,\infty\bigr)$ and

$$\begin{aligned}
s_n=\frac{1}{2}S_{n-1}&=\frac{\sqrt{2}}{2\pi}\frac{1}{n!}\int_0^\infty \varrho(s)e^{-(3-2\sqrt{2}\,)s}s^{n-1}\,\mathrm{d}\,s\\
&=\frac{\sqrt{2}}{2\pi}\frac{1}{n}\int_0^{1/2}\left(\sqrt{\frac{1}{u}-1}-\frac{1}{\sqrt{1/u-1}}\right)\\
&\quad\times\left[\frac{1}{(3-2\sqrt{2}\,+4\sqrt{2}\,u)^n}-\frac{1}{(3+2\sqrt{2}\,-4\sqrt{2}\,u)^n}\right]\mathrm{d}\,u,
\end{aligned}\tag{1.14}$$

$$\begin{aligned}
s_n=\frac{1}{2}S_{n-1}&=\frac{\sqrt{2}}{\pi}\frac{1}{n!}\int_0^\infty F(s)e^{-(3-2\sqrt{2}\,)s}s^{n-1}\,\mathrm{d}\,s\\
&=\frac{\sqrt{2}}{2\pi}\frac{1}{n}\int_0^1\frac{2u-1}{\sqrt{u(1-u)}}\frac{1}{\bigl(4\sqrt{2}\,u+3-2\sqrt{2}\,\bigr)^n}\,\mathrm{d}\,u,
\end{aligned}\tag{1.15}$$

and

$$s_n=\frac{1}{2}S_{n-1}=\frac{\pi}{4}\frac{1}{n!}\int_0^\infty q(s)s^n\,\mathrm{d}\,s=\frac{1}{4\pi}\int_{3-2\sqrt{2}}^{3+2\sqrt{2}}\frac{\sqrt{-u^2+6u-1}}{u^{n+1}}\,\mathrm{d}\,u.\tag{1.16}$$

for $n\ge2$.

2. Lemmas

In order to prove our main results, we need the following lemmas.

Lemma 2.1 — ([24, Theorem 4.2]). For $a>b>0$ and $z\in\mathbb{C}\setminus[-a,-b]$, the principal branch of the geometric mean

$$G_{a,b}(z)=\sqrt{(a+z)(b+z)}$$

has the integral representation

$$G_{a,b}(z)=G_{a,b}(0)+z+\frac{a-b}{2\pi}\int_0^\infty\frac{\rho((a-b)s)}{s}e^{-bs}(1-e^{-zs})\,\mathrm{d}\,s,$$

where $\rho(s)$ is defined by

$$\rho(s)=\int_0^{1/2}\left(\sqrt{\frac{1}{u}-1}-\frac{1}{\sqrt{1/u-1}}\right)\bigl[1-e^{-(1-2u)s}\bigr]e^{-su}\,\mathrm{d}\,u.$$

Lemma 2.2 — ([23, Theorem 1.1]). For $\lambda\in(0,1)$ and $a>b>0$, the principal branch of the weighted geometric mean

$$G_{a,b;\lambda}(z)=(a+z)^\lambda(b+z)^{1-\lambda}$$

for $\lambda\in(0,1)$ and $z\in\mathbb{C}\setminus[-a,-b]$ has the integral representation

$$G_{a,b;\lambda}(z)=a^\lambda b^{1-\lambda}+z+\frac{\sin(\lambda\pi)}{\pi}(a-b)\int_0^\infty\frac{F(\lambda,(a-b)s)}{s}e^{-bs}(1-e^{-zs})\,\mathrm{d}\,s,$$

where

$$F(\lambda,s)=\int_0^1\biggl(\frac{1}{u}-1\biggr)^{\lambda}\biggl(1-\frac{\lambda}{1-u}\biggr)e^{-su}\,\mathrm{d}\,u$$

is positive for all $(\lambda,s)\in(0,1)\times(0,\infty)$.

Lemma 2.3 — ([22, Theorem 1.1]). Let $n\in\mathbb{N}$ and $\mathbf{a}=(a_1,a_2,\dots,a_n)$ be a positive and non-decreasing sequence, that is, $0<a_1\le a_2\le\cdots\le a_n$. For $n\ge2$ and $z\in\mathbb{C}\setminus[-a_n,-a_1]$, let

$$G_{n,\boldsymbol{a}}(z)=\Biggl[\prod_{k=1}^{n}(a_k+z)\Biggr]^{1/n}.$$

Then the principal branch of the geometric mean $G_{n,\boldsymbol{a}}(z)$ has the Lévy-Khintchine representation

$$G_{n,\boldsymbol{a}}(z)=G_{n,\boldsymbol{a}}(0)+z+\int_0^\infty Q_{n,\boldsymbol{a}}(u)\bigl(1-e^{-zu}\bigr)\,\mathrm{d}\,u,$$

where

$$Q_{n,\boldsymbol{a}}(u)=\frac{1}{\pi}\sum_{\ell=1}^{n-1}\sin\frac{\ell\pi}{n}\int_{a_\ell}^{a_{\ell+1}}\sqrt[n]{\prod_{k=1}^{n}|a_k-t|}\,e^{-tu}\,\mathrm{d}\,t.$$

3. Proofs of Theorems 1.1 and 1.2

We are now in a position to prove our main results, Theorems 1.1 and 1.2.

Proofs of Theorems 1.1 : It is not difficult to write the function $\mathcal{G}(z)$ as

$$\mathcal{G}(z)=\frac{\sqrt{\bigl(z+3-2\sqrt{2}\,\bigr)\bigl(z+3+2\sqrt{2}\,\bigr)}-\sqrt{\bigl(3-2\sqrt{2}\,\bigr)\bigl(3+2\sqrt{2}\,\bigr)}-z}{2z}.$$

Applying $a=3+2\sqrt{2}$ and $b=3-2\sqrt{2}$ to Lemma 2.1 yields

$$\begin{aligned}\mathcal{G}(z)&=\frac{1}{2z}\bigl[G_{3+2\sqrt{2}\,,3-2\sqrt{2}}(z)-G_{3+2\sqrt{2}\,,3-2\sqrt{2}}(0)-z\bigr]\\&=\frac{1}{2z}\frac{4\sqrt{2}}{2\pi}\int_0^\infty\frac{\rho\bigl(4\sqrt{2}\,s\bigr)}{s}e^{-(3-2\sqrt{2}\,)s}(1-e^{-zs})\,\mathrm{d}\,s\\&=\frac{\sqrt{2}}{\pi}\int_0^\infty\rho\bigl(4\sqrt{2}\,s\bigr)e^{-(3-2\sqrt{2}\,)s}\frac{1-e^{-zs}}{zs}\,\mathrm{d}\,s\\&=\frac{\sqrt{2}}{\pi}\int_0^\infty\varrho(s)e^{-(3-2\sqrt{2}\,)s}\int_{1/e}^{1}v^{zs-1}\,\mathrm{d}\,v\,\mathrm{d}\,s.\end{aligned}$$

The integral representation (1.5) follows immediately.

From the positivity of the functions $\varrho(s)=\rho\bigl(4\sqrt{2}\,s\bigr)$ on $(0,\infty)$, it follows that

$$(-1)^k\mathcal{G}^{(k)}(t)=\frac{\sqrt{2}}{\pi}\int_0^\infty \varrho(s)e^{-(3-2\sqrt{2}\,)s}s^k\int_{1/e}^1\bigl[(-1)^k(\ln v)^k\bigr]v^{ts-1}\,\mathrm{d}\,v\,\mathrm{d}\,s>0$$

for all $k\ge0$. Hence, the function $\mathcal{G}(t)$ is completely monotonic on $\bigl(2\sqrt{2}\,-3,\infty\bigr)$.

Since

$$\mathcal{G}(t)=G(-t)=\frac{\sqrt{t^2+6t+1}\,-1-t}{2t}=\sum_{n=0}^\infty(-1)^nS_nt^n,$$

it is not difficult to see that

$$\begin{aligned}
S_n&=(-1)^n\frac{1}{n!}\mathcal{G}^{(n)}(0)\\
&=\frac{1}{n!}\lim_{t\to0}\frac{\sqrt{2}}{\pi}\int_0^\infty \varrho(s)e^{-(3-2\sqrt{2}\,)s}s^n\int_{1/e}^1\bigl[(-1)^n(\ln v)^n\bigr]v^{ts-1}\,\mathrm{d}\,v\,\mathrm{d}\,s\\
&=\frac{\sqrt{2}}{\pi}\frac{(-1)^n}{n!}\int_0^\infty \varrho(s)e^{-(3-2\sqrt{2}\,)s}s^n\int_{1/e}^1\frac{(\ln v)^n}{v}\,\mathrm{d}\,v\,\mathrm{d}\,s\\
&=\frac{\sqrt{2}}{\pi}\frac{1}{(n+1)!}\int_0^\infty \varrho(s)e^{-(3-2\sqrt{2}\,)s}s^n\,\mathrm{d}\,s\\
&=\frac{\sqrt{2}}{\pi}\frac{1}{(n+1)!}\int_0^{1/2}\left(\sqrt{\frac{1}{u}-1}\,-\frac{1}{\sqrt{1/u-1}\,}\right)\\
&\quad\times\int_0^\infty\bigl[1-e^{-4\sqrt{2}\,(1-2u)s}\bigr]e^{-(4\sqrt{2}\,u+3-2\sqrt{2}\,)s}s^n\,\mathrm{d}\,s\,\mathrm{d}\,u\\
&=\frac{\sqrt{2}}{\pi}\frac{1}{n+1}\int_0^{1/2}\left(\sqrt{\frac{1}{u}-1}\,-\frac{1}{\sqrt{1/u-1}\,}\right)\\
&\quad\times\left[\frac{1}{(3-2\sqrt{2}\,+4\sqrt{2}\,u)^{n+1}}-\frac{1}{(3+2\sqrt{2}\,-4\sqrt{2}\,u)^{n+1}}\right]\mathrm{d}\,u.
\end{aligned}$$

The integral representation (1.8) is thus proved.

Substituting $a=3+2\sqrt{2}\,$, $b=3-2\sqrt{2}\,$, and $\lambda=\frac{1}{2}$ into Lemma 2.2 results in

$$\begin{aligned}
\mathcal{G}(z)&=\frac{1}{2z}\Big[G_{3+2\sqrt{2}\,,3-2\sqrt{2}\,;1/2}(z)-\big(3+2\sqrt{2}\,\big)^{1/2}\big(3+2\sqrt{2}\,\big)^{1-1/2}-z\Big]\\
&=\frac{1}{2z}\frac{\sin(\pi/2)}{\pi}4\sqrt{2}\int_0^\infty\frac{F\big(1/2,4\sqrt{2}\,s\big)}{s}e^{-(3-2\sqrt{2}\,)s}(1-e^{-zs})\,\mathrm{d}\,s\\
&=\frac{2\sqrt{2}}{\pi}\int_0^\infty F\bigg(\frac{1}{2},4\sqrt{2}\,s\bigg)e^{-(3-2\sqrt{2}\,)s}\frac{1-e^{-zs}}{zs}\,\mathrm{d}\,s\\
&=\frac{2\sqrt{2}}{\pi}\int_0^\infty F\bigg(\frac{1}{2},4\sqrt{2}\,s\bigg)e^{-(3-2\sqrt{2}\,)s}\int_{1/e}^1 v^{zs-1}\,\mathrm{d}\,v\,\mathrm{d}\,s\\
&=\frac{2\sqrt{2}}{\pi}\int_0^\infty F(s)e^{-(3-2\sqrt{2}\,)s}\int_{1/e}^1 v^{zs-1}\,\mathrm{d}\,v\,\mathrm{d}\,s.
\end{aligned}$$

The integral representation (1.6) follows.

Since $F(s)>0$, as stated in Lemma 2.2, we have

$$(-1)^k\mathcal{G}^{(k)}(t)=\frac{2\sqrt{2}}{\pi}\int_0^\infty F(s)e^{-(3-2\sqrt{2}\,)s}s^k\int_{1/e}^1\big[(-1)^k(\ln v)^k\big]v^{ts-1}\,\mathrm{d}\,v\,\mathrm{d}\,s>0.$$

By definition, the complete monotonicity of the function $\mathcal{G}(t)$ is verified again.

It is obvious that

$$\begin{aligned}
S_n&=(-1)^n\frac{1}{n!}\mathcal{G}^{(n)}(0)\\
&=\frac{2\sqrt{2}}{\pi}\frac{(-1)^n}{n!}\int_0^\infty F(s)e^{-(3-2\sqrt{2}\,)s}s^n\int_{1/e}^1\frac{(\ln v)^n}{v}\,\mathrm{d}\,v\,\mathrm{d}\,s\\
&=\frac{2\sqrt{2}}{\pi}\frac{1}{(n+1)!}\int_0^\infty F(s)e^{-(3-2\sqrt{2}\,)s}s^n\,\mathrm{d}\,s\\
&=\frac{2\sqrt{2}}{\pi}\frac{1}{(n+1)!}\int_0^1\bigg(\frac{1}{u}-1\bigg)^{1/2}\bigg[1-\frac{1}{2(1-u)}\bigg]\int_0^\infty e^{-(4\sqrt{2}\,u+3-2\sqrt{2}\,)s}s^n\,\mathrm{d}\,s\,\mathrm{d}\,u\\
&=\frac{2\sqrt{2}}{\pi}\frac{1}{n+1}\int_0^1\bigg(\frac{1}{u}-1\bigg)^{1/2}\bigg[1-\frac{1}{2(1-u)}\bigg]\frac{1}{\big(4\sqrt{2}\,u+3-2\sqrt{2}\,\big)^{n+1}}\,\mathrm{d}\,u.
\end{aligned}$$

The integral representation (1.9) is readily acquired.

Setting $n=2$, $a_1=3-2\sqrt{2}\,$, and $a_2=3+2\sqrt{2}$ in Lemma 2.3 leads to

$$\begin{aligned}
\mathcal{G}(z)&=\frac{1}{2z}\big[G_{2,(3-2\sqrt{2}\,,3+2\sqrt{2}\,)}(z)-G_{2,(3-2\sqrt{2}\,,3+2\sqrt{2}\,)}(0)-z\big]\\
&=\frac{1}{2z}\int_0^\infty Q_{2,(3-2\sqrt{2}\,,3+2\sqrt{2}\,)}(s)(1-e^{-zs})\,\mathrm{d}\,s,
\end{aligned}$$

where

$$Q_{2,(3-2\sqrt{2},3+2\sqrt{2})}(s)=\frac{1}{\pi}\int_{3-2\sqrt{2}}^{3+2\sqrt{2}}\sqrt{\bigl(t-3+2\sqrt{2}\,\bigr)\bigl(3+2\sqrt{2}\,-t\bigr)}\,e^{-st}\,\mathrm{d}\,t=\frac{q(s)}{\pi}$$

is positive. Accordingly, we obtain

$$\begin{aligned}\mathcal{G}(z)&=\frac{1}{2\pi}\int_0^\infty s\int_{3-2\sqrt{2}}^{3+2\sqrt{2}}\sqrt{-u^2+6u-1}\,e^{-su}\,\mathrm{d}\,u\int_{1/e}^1 v^{zs-1}\,\mathrm{d}\,v\,\mathrm{d}\,s\\&=\frac{1}{2\pi}\int_{3-2\sqrt{2}}^{3+2\sqrt{2}}\sqrt{-u^2+6u-1}\int_0^\infty se^{-su}\int_{1/e}^1 v^{zs-1}\,\mathrm{d}\,v\,\mathrm{d}\,s\,\mathrm{d}\,u\\&=\frac{1}{2\pi}\int_{3-2\sqrt{2}}^{3+2\sqrt{2}}\frac{\sqrt{-u^2+6u-1}}{u(u+z)}\,\mathrm{d}\,u.\end{aligned}$$

The integral representation (1.7) follows.

Since

$$(-1)^k\mathcal{G}^{(k)}(t)=\frac{k!}{2\pi}\int_{3-2\sqrt{2}}^{3+2\sqrt{2}}\frac{\sqrt{-u^2+6u-1}}{u(u+t)^{k+1}}\,\mathrm{d}\,u>0$$

for all $k\ge0$, we recover the complete monotonicity of $\mathcal{G}(t)$ once again. On the other hand, it is easy to see that

$$S_n=(-1)^n\frac{1}{n!}\mathcal{G}^{(n)}(0)=\frac{1}{2\pi}\int_{3-2\sqrt{2}}^{3+2\sqrt{2}}\frac{\sqrt{-u^2+6u-1}}{u^{n+2}}\,\mathrm{d}\,u.$$

The integral representation (1.10) is thus proved.

The function (1.4) can be rearranged as

$$\sqrt{z^2+6z+1}-1-z=2\sum_{n=0}^\infty(-1)^nS_nz^{n+1}$$

which implies that

$$\begin{aligned}S_n&=\frac{1}{2}\frac{(-1)^n}{(n+1)!}\lim_{z\to0}\bigl(\sqrt{z^2+6z+1}\,\bigr)^{(n+1)}\\&=\frac{1}{2}\frac{(-1)^n}{(n+1)!}\lim_{z\to0}\biggl[\sqrt{\bigl(z+3-2\sqrt{2}\,\bigr)\bigl(z+3+2\sqrt{2}\,\bigr)}\,\biggr]^{(n+1)}\end{aligned}$$

for $n\in\mathbb{N}$. As a result, utilizing Lemmas 2.1 to 2.3 and interchanging the orders of integrals respectively, we obtain

$$
\begin{aligned}
S_n &= \frac{1}{2}\frac{(-1)^n}{(n+1)!}\lim_{z\to 0}\Bigg[G_{3+2\sqrt{2}\,,3-2\sqrt{2}\,}(0)+z \\
&\quad + \frac{4\sqrt{2}}{2\pi}\int_0^\infty \frac{\rho\big(4\sqrt{2}\,s\big)}{s}e^{-(3-2\sqrt{2}\,)s}(1-e^{-zs})\,\mathrm{d}\,s\Bigg]^{(n+1)} \\
&= \frac{\sqrt{2}}{\pi}\frac{1}{(n+1)!}\int_0^\infty \frac{\rho\big(4\sqrt{2}\,s\big)}{s}e^{-(3-2\sqrt{2}\,)s}s^{n+1}\,\mathrm{d}\,s \\
&= \frac{\sqrt{2}}{\pi}\frac{1}{(n+1)!}\int_0^\infty \varrho(s)e^{-(3-2\sqrt{2}\,)s}s^{n}\,\mathrm{d}\,s \\
&= \frac{2\sqrt{2}}{\pi}\frac{1}{(n+1)!}\int_0^\infty F(s)e^{-(3-2\sqrt{2}\,)s}s^{n}\,\mathrm{d}\,s \\
&= \frac{\sqrt{2}}{\pi}\frac{1}{n+1}\int_0^{1/2}\left(\sqrt{\frac{1}{u}-1}-\frac{1}{\sqrt{1/u-1}}\right) \\
&\quad \times\left[\frac{1}{(3-2\sqrt{2}\,+4\sqrt{2}\,u)^{n+1}}-\frac{1}{(3+2\sqrt{2}\,-4\sqrt{2}\,u)^{n+1}}\right]\mathrm{d}\,u, \\
S_n &= \frac{1}{2}\frac{(-1)^n}{(n+1)!}\lim_{z\to 0}\Bigg[\big(3+2\sqrt{2}\,\big)^{1/2}\big(3-2\sqrt{2}\,\big)^{1-1/2}+z \\
&\quad + \frac{\sin(\pi/2)}{\pi}4\sqrt{2}\int_0^\infty \frac{F(1/2,4\sqrt{2}\,s)}{s}e^{-(3-2\sqrt{2}\,)s}(1-e^{-zs})\,\mathrm{d}\,s\Bigg]^{(n+1)} \\
&= \frac{2\sqrt{2}}{\pi}\frac{(-1)^n}{(n+1)!}\lim_{z\to 0}\int_0^\infty \frac{F(1/2,4\sqrt{2}\,s)}{s}e^{-(3-2\sqrt{2}\,)s}(1-e^{-zs})^{(n+1)}\,\mathrm{d}\,s \\
&= \frac{\sqrt{2}}{\pi}\frac{1}{n+1}\int_0^1\sqrt{\frac{1}{u}-1}\,\frac{2u-1}{u-1}\frac{1}{\big(4\sqrt{2}\,u+3-2\sqrt{2}\,\big)^{n+1}}\,\mathrm{d}\,u,
\end{aligned}
$$

and

$$
\begin{aligned}
S_n &= \frac{1}{2}\frac{(-1)^n}{(n+1)!}\lim_{z\to 0}\Bigg[G_{2,(3-2\sqrt{2}\,,3+2\sqrt{2}\,)}(0)+z \\
&\quad + \int_0^\infty Q_{2,(3-2\sqrt{2}\,,3+2\sqrt{2}\,)}(s)(1-e^{-zs})\,\mathrm{d}\,s\Bigg]^{(n+1)} \\
&= \frac{1}{2}\frac{(-1)^n}{(n+1)!}\lim_{z\to 0}\int_0^\infty Q_{2,(3-2\sqrt{2}\,,3+2\sqrt{2}\,)}(s)(1-e^{-zs})^{(n+1)}\,\mathrm{d}\,s \\
&= \frac{1}{2}\frac{1}{(n+1)!}\int_0^\infty Q_{2,(3-2\sqrt{2}\,,3+2\sqrt{2}\,)}(s)s^{n+1}\,\mathrm{d}\,s \\
&= \frac{1}{2\pi}\frac{1}{(n+1)!}\int_0^\infty q(s)s^{n+1}\,\mathrm{d}\,s \\
&= \frac{1}{2\pi}\int_{3-2\sqrt{2}}^{3+2\sqrt{2}}\frac{\sqrt{-u^2+6u-1}}{u^{n+2}}\,\mathrm{d}\,u.
\end{aligned}
$$

Consequently, the integral representations (1.8) to (1.10) follow respectively once again. The proof of Theorem 1.1 is complete. □

PROOF OF THEOREM 1.2 : By Lemma 2.1, we have

$$
\begin{aligned}
\mathfrak{G}(z) &= \frac{1}{4}\Big[1 - z - \sqrt{\big(z+3-2\sqrt{2}\,\big)\big(z+3+2\sqrt{2}\,\big)}\,\Big] \\
&= -\frac{1}{2}\bigg[z + \frac{\sqrt{2}}{\pi}\int_0^\infty \frac{\rho\big(4\sqrt{2}\,s\big)}{s} e^{-(3-2\sqrt{2}\,)s}(1-e^{-zs})\,\mathrm{d}\,s\bigg] \\
&= -\frac{1}{2}\bigg[z + \frac{\sqrt{2}}{\pi}\int_0^\infty \frac{\varrho(s)}{s} e^{-(3-2\sqrt{2}\,)s}(1-e^{-zs})\,\mathrm{d}\,s\bigg].
\end{aligned}
$$

Similarly, by Lemmas 2.2 and 2.3, we obtain

$$
\begin{aligned}
\mathfrak{G}(z) &= \frac{1}{4}\bigg[-2z - \frac{1}{\pi}4\sqrt{2}\int_0^\infty \frac{F\big(1/2,4\sqrt{2}\,s\big)}{s} e^{-(3-2\sqrt{2}\,)s}(1-e^{-zs})\,\mathrm{d}\,s\bigg] \\
&= -\frac{1}{2}\bigg[z + \frac{2\sqrt{2}}{\pi}\int_0^\infty \frac{F\big(1/2,4\sqrt{2}\,s\big)}{s} e^{-(3-2\sqrt{2}\,)s}(1-e^{-zs})\,\mathrm{d}\,s\bigg] \\
&= -\frac{1}{2}\bigg[z + \frac{2\sqrt{2}}{\pi}\int_0^\infty \frac{F(s)}{s} e^{-(3-2\sqrt{2}\,)s}(1-e^{-zs})\,\mathrm{d}\,s\bigg]
\end{aligned}
$$

and

$$
\begin{aligned}
\mathfrak{G}(z) &= \frac{1}{4}\bigg[-2z - \int_0^\infty Q_{2,(3-2\sqrt{2}\,,3+2\sqrt{2}\,)}(s)(1-e^{-su})\,\mathrm{d}\,s\bigg] \\
&= -\frac{1}{2}z - \frac{1}{4}\pi\int_0^\infty q(s)(1-e^{-zs})\,\mathrm{d}\,s.
\end{aligned}
$$

The integral representations (1.11) to (1.13) are thus acquired.

Comparing (1.3) with the integral representations (1.11) to (1.13) respectively, we easily see that the negative of $\mathfrak{G}(t)$ is a Bernstein function on $\big(3-2\sqrt{2}\,,\infty\big)$.

From (1.2), it follows that

$$
\mathfrak{G}(z) = \sum_{n=1}^\infty (-1)^n s_n z^n,
$$

which means that

$$
s_n = \frac{(-1)^n}{n!}\mathfrak{G}^{(n)}(0).
$$

Combining this with the integral representations (1.11) to (1.13) and interchanging the orders of integrals respectively reveal

$$s_n = \frac{(-1)^n}{n!} \lim_{z\to 0} \mathfrak{G}^{(n)}(z) = \frac{1}{n!} \frac{\sqrt{2}}{2\pi} \int_0^\infty \varrho(s) e^{-(3-2\sqrt{2}\,)s} s^{n-1} \,\mathrm{d}\, s,$$

$$s_n = \frac{(-1)^n}{n!} \lim_{z\to 0} \mathfrak{G}^{(n)}(z) = \frac{\sqrt{2}}{\pi} \frac{1}{n!} \int_0^\infty F(s) e^{-(3-2\sqrt{2}\,)s} s^{n-1} \,\mathrm{d}\, s,$$

and

$$s_n = \frac{(-1)^n}{n!} \lim_{z\to 0} \mathfrak{G}^{(n)}(z) = \frac{\pi}{4} \frac{1}{n!} \int_0^\infty q(s) s^n \,\mathrm{d}\, s.$$

The proof of Theorem 1.2 is complete. □

4. Remarks

Finally, we list several remarks.

Remark 4.1 : In [1, 11], it was defined implicitly and explicitly that an infinitely differentiable and positive function f is said to be logarithmically completely monotonic on an interval I if the inequality $(-1)^k[\ln f(x)]^{(k)} \ge 0$ holds on I for all $k \in \mathbb{N}$. In [4, pp. 161-162, Theorem 3] and [26, Proposition 5.25], it was proved that the reciprocal of a Bernstein function is logarithmically completely monotonic. In [2, Theorem 1.1], [5, Theorem 4], [10, Theorem 1], and [11, Theorem 4], it was found and verified once again that a logarithmically completely monotonic function must be completely monotonic. By these conclusions and the relation $-\frac{1}{\mathfrak{G}(t)} = \mathcal{G}(t)$, we can obtain the (logarithmically) complete monotonicity of the function $\mathcal{G}(t)$ readily.

Remark 4.2 : A Stieltjes transform is a function $f : (0,\infty) \to [0,\infty)$ which can be written in the form

$$f(t) = \frac{a}{t} + b + \int_0^\infty \frac{1}{u+t} \,\mathrm{d}\, \mu(u), \tag{4.1}$$

where $a, b \ge 0$ are constants and μ is a nonnegative measure on $(0,\infty)$ such that $\int_0^\infty \frac{1}{1+s} \,\mathrm{d}\, \mu(s) < \infty$. See [26, Definition 2.1]. In [2, Theorem 1.2], it was proved that a positive Stieltjes transform must be a logarithmically completely monotonic function on $(0,\infty)$, but not conversely. Hence, it is natural to ask a question: is the logarithmically completely monotonic function $\mathcal{G}(t)$ on $\big(2\sqrt{2}-3,\infty\big) \supset (0,\infty)$ a Stieltjes transform? Comparing the second integral representation in (1.7) with (4.1) gives an answer to this question: the function $\mathcal{G}(t)$ is a Stieltjes transform, with $a = 0$, $b = 0$, and the positive measure

$$\mathrm{d}\, \mu(u) = \begin{cases} \dfrac{1}{2\pi} \dfrac{\sqrt{-u^2+6u-1}}{u} \,\mathrm{d}\, u, & u \in \big(3-2\sqrt{2}\,, 3+2\sqrt{2}\,\big), \\ 0, & x \notin \big(3-2\sqrt{2}\,, 3+2\sqrt{2}\,\big). \end{cases}$$

Remark 4.3 : In [2, Remark 1.4], it was pointed out that, if $f(t)$ is a positive Stieltjes transform, then the functions $\frac{1}{f(1/t)}$ and $\frac{1}{tf(t)}$ are again Stieltjes transforms. As a result, applying this conclusion to the Stieltjes transform $\mathcal{G}(t)$ reveals that the function

$$\begin{aligned}\frac{1}{\sqrt{t^2+6t+1}-t-1} &= \frac{\sqrt{t^2+6t+1}+t+1}{4t} \\ &= \frac{1}{2t}+\frac{1}{2}+\frac{1}{4\pi}\int_{3-2\sqrt{2}}^{3+2\sqrt{2}} \frac{\sqrt{-u^2+6u-1}}{u}\frac{1}{u+t}\,\mathrm{d}\,u\end{aligned}$$

is also a Stieltjes transform.

Remark 4.4 : By making use of integral representations for the generating functions $\mathcal{G}(t)$ and $\mathfrak{G}(t)$ and for the large and little Schröder numbers S_n and s_{n+1}, we can discover some properties, including convexity, complete monotonicity, and inequalities, of the large and little Schröder numbers S_n and s_{n+1}. Due to limitation on the length of the paper, we will leave the work to a subsequent paper.

Remark 4.5 : The weighted form of Lemma 2.3 has been obtained and restated in [6, Theorem 2.2] and [21, Theorems 3.1 and 4.6] as follows. Let $1 > w_k > 0$ and $\sum_{k=1}^{n} w_k = 1$ for $1 \le k \le n$ and $n \ge 2$. If $a = (a_1, a_2, \dots, a_n)$ is a positive and non-decreasing sequence, that is, $0 < a_1 \le a_2 \le \cdots \le a_n$, then the principal branch of the weighted geometric mean

$$G_{w,n}(a+z) = \prod_{k=1}^{n}(a_k+z)^{w_k}, \quad z \in \mathbb{C} \setminus [-a_n, -a_1]$$

has the Lévy-Khintchine representation

$$G_{w,n}(a+z) = G_{w,n}(a) + z + \int_0^\infty m_{a,w,n}(u)(1-e^{-zu})\,\mathrm{d}\,u,$$

where the density

$$m_{a,w,n}(u) = \frac{1}{\pi}\sum_{\ell=1}^{n-1}\sin\left[\left(\sum_{j=1}^{\ell} w_j\right)\pi\right]\int_{a_\ell}^{a_{\ell+1}}\prod_{k=1}^{n}|a_k-t|^{w_k}e^{-ut}\,\mathrm{d}\,t.$$

This result can also be used to derive some conclusions in Theorems 1.1 and 1.2, as Lemma 2.3 did.

Remark 4.6 : This paper can be regarded as applications of conclusions in the papers [15-17, 21-24] to combinatorics and computational number theory.

Remark 4.7 : This paper is a revised version of the preprint [18] and a companion of the formally published papers [7, 12, 19, 20].

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