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ON A RELATION BETWEEN de RHAM COHOMOLOGY OF $H^1_{(f)}(R)$ AND THE KOSZUL COHOMOLOGY OF $\partial(f)$ IN $R/(f)$

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Let K be a field of characteristic zero, $R = K[X_1, \dots, X_n]$. Let $A_n(K) = K < X_1, \dots, X_n, \partial_1, \dots, \partial_n >$ be the n^{th} Weyl algebra over K. We consider the case when R and $A_n(K)$ is graded by giving $\deg X_i = \omega_i$ and $\deg \partial_i = -\omega_i$ for $i = 1, \dots, n$ (here ω_i are positive integers). Set $\omega = \sum_{k=1}^n \omega_k$. Let I be a graded ideal in R. By a result due to Lyubeznik the local cohomology modules $H^i_I(R)$ are holonomic $A_n(K)$-modules for each $i \geq 0$. In this article we compute the de Rham cohomology modules $H^j(\partial; H^1_{(f)}(R))$ for $j \leq n-2$ when $V(f)$ is a smooth hypersurface in $\mathbb{P}^n$ (equivalently $A = R/(f)$ is an isolated singularity).

Key words : Local cohomology; associated primes; D-modules; Koszul homology.

1. Introduction

Let K be a field of characteristic zero and let $R = K[X_1, \dots, X_n]$. We consider R graded with $\deg X_i = \omega_i$ for $i = 1, \dots, n$; here ω_i are positive integers. Set $\mathfrak{m} = (X_1, \dots, X_n)$. Let I be a graded ideal in R. The local cohomology modules $H^*_I(R)$ are clearly graded R-modules. Let $A_n(K) = K < X_1, \dots, X_n, \partial_1, \dots, \partial_n >$ be the n^{th} Weyl algebra over K. By a result due to Lyubeznik, see [2], the local cohomology modules $H^i_I(R)$ are *holonomic* $A_n(K)$-modules for each $i \geq 0$. We can consider $A_n(K)$ graded by giving $\deg \partial_i = -\omega_i$ for $i = 1, \dots, n$.

Let N be a graded left $A_n(K)$ module. Now $\partial = \partial_1, \dots, \partial_n$ are pairwise commuting K-linear maps. So we can consider the de Rham complex $K(\partial; N)$. Notice that the de Rham cohomology modules $H^*(\partial; N)$ are in general only *graded* K-vector spaces. They are finite dimensional if N is holonomic; [1, Chapter 1, Theorem 6.1]. In particular $H^*(\partial; H^*_I(R))$ are finite dimensional graded K-vector spaces. By [4, Theorem 1] the de Rham cohomology modules $H^*(\partial; H^*_I(R))$ is concentrated in degree $-\omega$, i.e., $H^*(\partial; H^*_I(R))_j = 0$ for $j \neq -\omega$.

Let f be a homogenous polynomial in R and set $A = R/(f)$. By [3, Theorem 2.7] we have $H^0(\partial; H^1_{(f)}(R)) = 0$. In this paper we extend a technique from [4]. In that paper the first author related $H^{n-1}(\partial; H^1_{(f)}(R))$ with $H^{n-1}(\partial(f); A)$. In this paper quite generally we prove that (see, Theorem 3.9):

***Theorem* 1** — *Assume* $H^{i-1}(\partial f; A) = 0$. *Then there exists a filtration* $\{\mathcal{F}_\nu\}_{\nu \geq 0}$ *consisting of* $K-$ *subspaces of* $H^i(\partial; R_f)$ *with* $\mathcal{F}_\nu = H^i(\partial; R_f)$ *for* $\nu \gg 0, \mathcal{F}_\nu \supseteq \mathcal{F}_{\nu-1}$ *and* $\mathcal{F}_0 = 0$ *and* $K-$*linear maps*

$$\eta_\upsilon : \frac{\mathcal{F}_\upsilon}{\mathcal{F}_{\upsilon-1}} \to H^i(\partial f; A)_{(\upsilon+n-i)\deg f - \omega}.$$

such that

(a) η_υ *is injective for all* $\upsilon \geq 2$.

(b) *If* $i \neq 1$. *Then* η_1 *also injective.*

(c) *If* $i = 1$. *Then* $\ker(\eta_1) = K$.

If $H^{n-i}(\partial f; A) = 0$ for $i \geq \alpha + 1$. Then as a corollary we obtain (see, Corollary 3.10)

$$H^{n-i}(\partial; R_f) = \begin{cases} 0 & \text{if } \alpha + 1 \leq i \leq n-2 \\ K & \text{if } i = n-1. \end{cases}$$

When A is an isolated singularity, i.e., A_P is regular for all homogeneous prime ideals $P \neq \mathfrak{m}$. Note that $V(f)$ is a smooth hypersurface in $\mathbb{P}^n$. As a corollary we prove that (see, Corollary 3.11)

$$H^i(\partial; H^1_{(f)}(R)) = \begin{cases} 0 & \text{if } i \leq n-2 \text{ and } i \neq 1 \\ K & \text{if } i = 1. \end{cases}$$

We now describe in brief the contents of the paper. In Section 2 we discuss a few preliminaries that we need. In Section 3 we construct certain functions which we need to define η_ν. In Section 4 we construct our filtration of $H^i(\partial; H^1_{(f)}(R))$ and prove our result.

2. Preliminaries

In this section we discuss a few preliminary results that we need.

Remark 2.1 : Although all the results are stated for de Rham cohomology of a $A_n(K)$-module M, we will actually work with de Rham homology. Note that $H_i(\partial; M) = H^{n-i}(\partial; M)$ for any

$A_n(K)$-module M. Let $S = K[\partial_1, \dots, \partial_n]$. Consider it as a subring of $A_n(K)$. Then note that $H_i(\partial, M)$ is the i^{th} Koszul homology module of M with respect to ∂.

2.2 Let A be commutative ring and $a = a_1, \cdots, a_n \in A$. Let $I \subseteq \{1, \cdots, n\}$, $|I| = m$. Say $I = \{i_1 < i_2 < \cdots < i_m\}$. Let

$$e_I := e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_m}.$$

Then the Koszul complex of A with respect to a is

$$\mathbb{K}(a; A) := 0 \to \mathbb{K}_n \xrightarrow{\phi_n} \mathbb{K}_{n-1} \xrightarrow{\phi_{n-1}} \cdots \to \mathbb{K}_1 \xrightarrow{\phi_1} \mathbb{K}_0 \to 0.$$

Here $\mathbb{K}_m = \bigoplus_{|I|=m} Ae_I$.

Let $\xi = \sum_{|I|=m} \xi_I e_I \in \mathbb{K}_m$ we write $\xi = (\xi_I \mid |I| = m)$. For the map $\mathbb{K}_p \xrightarrow{\phi_p} \mathbb{K}_{p-1}$,

say $\phi_p(\xi) = U$. Write $U = (U_J \mid |J| = p - 1)$. Then

$$U_J = \sum_{i \notin J} (-1)^{\sigma(J \cup \{i\})} \left(a_i \xi_{J \cup \{i\}} \right).$$

Here $J = \{j_1 < j_2 < \cdots < j_{p-1}\}$ and

$$\sigma(J \cup \{i\}) = \begin{cases} 0, & \text{if } i < j_1; \\ p, & \text{if } i > j_{p-1} \\ r, & \text{if } j_r < i < j_{r+1}. \end{cases}$$

2.3 Let $f \in R$ be a homogeneous polynomial. We consider elements of R_f^m as column-vectors. For $x \in R_f^m$ we write it as $x = (x_1, \dots, x_m)'$; here $'$ indicates transpose.

2.4 Let $f \in R$ be a homogeneous polynomial. Set $\partial = \partial_1, \cdots, \partial_n$. Consider the commutative subring $S = K[\partial_1, \cdots, \partial_n]$ of $A_n(K)$. The de Rham complex on a holonomic module N is just the Koszul complex $K(\partial; N)$ of N with respect to S. In particular when $N = R_f$ we have,

$$\mathbb{K}(\partial; R_f) = 0 \to \mathbb{K}_n \xrightarrow{\phi_n} \mathbb{K}_{n-1} \to \cdots \mathbb{K}_p \xrightarrow{\phi_p} \mathbb{K}_{p-1} \to \cdots \to \mathbb{K}_1 \xrightarrow{\phi_1} \mathbb{K}_0 \to 0.$$

$$\text{Here} \quad \mathbb{K}_0 = R_f \quad \text{and} \quad \mathbb{K}_p = \bigoplus_{|I|=p} R_f(\omega_{i_1} + \cdots + \omega_{i_p}).$$

The maps $\mathbb{K}_p \xrightarrow{\phi_p} \mathbb{K}_{p-1},$ say $\xi = (\xi_I \mid |I| = p)'$. Then

$$\phi_p(\xi) = \left(\sum_{i \notin J} (-1)^{\sigma(J \cup \{i\})} \left(\frac{\partial}{\partial x_i} \xi_{J \cup \{i\}} \right) : \ |J| = p-1 \right).$$

2.5 Let $f \in R$ be a homogeneous polynomial. Set $A = R/(f)$, and $\partial f = \partial f/\partial x_1, \cdots, \partial f/\partial x_n$. Consider the Koszul complex $\mathbb{K}'(\partial f; A)$ on A with respect to ∂f.

$$\mathbb{K}'(\partial f; A) = 0 \to \mathbb{K}'_n \xrightarrow{\psi_n} \mathbb{K}'_{n-1} \cdots \to \mathbb{K}'_p \xrightarrow{\psi_p} \mathbb{K}'_{p-1} \to \cdots \to \mathbb{K}'_1 \xrightarrow{\psi_1} \mathbb{K}'_0 \to 0.$$

$$\text{Here} \quad \mathbb{K}'_p = \bigoplus_{|I|=p} A(-p \deg f + \omega_{i_1} + \cdots + \omega_{i_p}).$$

2.6 By [4, Theorem 1], $H_i(\partial; R_f)_j = 0$ for $j \neq \omega$, where $\omega = \omega_1 + \cdots + \omega_n$.

2.7 Let $\xi \in R_f^m \setminus R^m$. The element $(a_1/f^i, a_2/f^i, \ldots, a_m/f^i)'$, with $a_j \in R$ for all j, is said to be a *normal form* of ξ if

(1) $\xi = (a_1/f^i, a_2/f^i, \ldots, a_m/f^i)'$.

(2) f does not divide a_j for some j.

(3) $i \geq 1$.

It can be easily shown that normal form of ξ exists and is unique (see [4, Proposition 5.1]).

2.8 Let $\xi \in R_f^m$. We define $L(f)$ as follows.

Case 1 : $\xi \in R_f^m \setminus R^m$.

Let $(a_1/f^i, a_2/f^i, \ldots, a_m/f^i)'$ be the normal form of ξ. Set $L(\xi) = i$. Notice $L(\xi) \geq 1$ in this case.

Case 2 : $\xi \in R^m \setminus \{0\}$.

Set $L(\xi) = 0$.

Case 3 : $\xi = 0$.

Set $L(\xi) = -\infty$.

The following properties of the function L can be easily verified.

Proposition 2.9 — (with hypotheses as above). Let $\xi, \xi_1, \xi_2 \in R_f^m$ and $\alpha, \alpha_1, \alpha_2 \in K$.

(1) If $L(\xi_1) < L(\xi_2)$ then $L(\xi_1 + \xi_2) = L(\xi_2)$.

(2) If $L(\xi_1) = L(\xi_2)$ then $L(\xi_1 + \xi_2) \leq L(\xi_2)$.

(3) $L(\xi_1 + \xi_2) \leq \max\{L(\xi_1), L(\xi_2)\}$.

(4) If $\alpha \in K^*$ then $L(\alpha\xi) = L(\xi)$.

(5) $L(\alpha\xi) \leq L(\xi)$ for all $\alpha \in K$.

(6) $L(\alpha_1\xi_1 + \alpha_2\xi_2) \leq \max\{L(\xi_1), L(\xi_2)\}$.

(7) Let $\xi_1, \ldots, \xi_r \in R_f^m$ and let $\alpha_1, \ldots, \alpha_r \in K$. Then

$$L\left(\sum_{j=1}^{r} \alpha_j \xi_j\right) \leq \max\{L(\xi_1), L(\xi_2), \ldots, L(\xi_r)\}.$$

3. Construction of Certain Functions

In this section we construct few functions.

We define a function, $\theta : Z_p(\partial; R_f)\backslash R^{\binom{n}{p}} \longrightarrow H_p(\partial f; A)$, as follows.

Let $\xi \in Z_p(\partial; R_f)\backslash R^{\binom{n}{p}}$ and let $(\xi_I/f^c \mid |I| = p)'$ be the normal form of ξ. As $\phi_p(\xi) = 0$. We have for every J such that $|J| = p - 1$,

$$\sum_{i \notin J} (-1)^{\sigma(J\cup\{i\})} \frac{\partial}{\partial x_i}\left(\frac{\xi_{J\cup\{i\}}}{f^c}\right) = 0.$$

This implies $$\sum_{i \notin J} (-1)^{\sigma(J\cup\{i\})} \left(\frac{\frac{\partial}{\partial x_i}(\xi_{J\cup\{i\}})}{f^c} - c\xi_{J\cup\{i\}}\frac{\frac{\partial f}{\partial x_i}}{f^{c+1}}\right) = 0.$$

So $$f.\sum_{i \notin J} (-1)^{\sigma(J\cup\{i\})} \left(\frac{\partial}{\partial x_i}(\xi_{J\cup\{i\}})\right) = c.\sum_{i \notin J} (-1)^{\sigma(J\cup\{i\})} \left(\xi_{J\cup\{i\}}\frac{\partial f}{\partial x_i}\right).$$

Thus f divides $\sum_{i \notin J} (-1)^{\sigma(J\cup\{i\})} \left(\xi_{J\cup\{i\}}\frac{\partial f}{\partial x_i}\right)$. Therefore

$$(\bar{\xi_I} \mid |I| = p)' \in Z_p(\partial f; A).$$

Set $\theta(\xi) = [(\bar{\xi_I} \mid |I| = p)'] \in H_p(\partial f; A)$.

The following Lemma identifies the degree of $\theta(\xi)$.

Lemma 3.1 — Assume $p < n$. Let $\xi \in Z_p(\partial; R_f)_{-\omega}$ be non zero. Then

(a) $\xi \in R_f^{\binom{n}{p}} \backslash R^{\binom{n}{p}}$.

(b) If $L(\xi) = c$. Then $\theta(\xi) \in H_p(\partial f; A)_{(c+p)\deg f - \omega}$.

PROOF : (a) Let $\xi = (\xi_I)'$ be non-zero in $Z_p(\partial; R_f)_{-\omega}$. Note that

$$\xi \in \bigoplus_{|I|=p} (R_f(\omega_{i_1} + \cdots + \omega_{i_p}))_{-\omega}.$$

It follows that

$$\xi_I \in (R_f)_{-\omega + \sum_{s=1}^{p} \omega_{i_s}}.$$

It follows that $\xi \in R_f^{\binom{n}{p}} \backslash R^{\binom{n}{p}}$.

(b) Let $(a_I/f^c \ \mid\ |I| = p)'$ be the normal form of ξ. As $a_I/f^c \in R_f(\omega_{i_1} + \cdots + \omega_{i_p})_{-\omega}$. It follows that

$$\deg(a_I) = c\deg f - \omega + \sum_{s=1}^{p} \omega_{i_s}.$$

As $\theta(\xi) = [(\bar{a}_I \ \mid\ |I| = p)']$. Let

$$\bar{a}_I \in A(-p\deg f + \sum_{s=1}^{p} \omega_{i_s})_t.$$

Then

$$\bar{a}_I \in A_{(-p\deg f + \sum_{s=1}^{p} \omega_{i_s}) + t}.$$

It follows that

$$t = (c+p)\deg f - \omega.$$

Thus $\theta(\xi) \in H_p(\partial f; A)_{(c+p)\deg f - \omega}$. □

A natural condition we want in θ is that it vanishes on boundaries. The following result gives a sufficient condition when this happens.

Proposition 3.2 — Let $p < n$. Assume $H_{p+1}(\partial f; A) = 0$. Then $\theta(B_p(\partial; R_f)_{-\omega} \backslash \{0\}) = 0$.

PROOF : Let $U \in B_p(\partial; R_f)_{-\omega}$ be non zero. As $B_p(\partial; R_f)_{-\omega} \subset Z_p(\partial; R_f)_{-\omega}$. We get by Lemma 3.1(a) that $U \in R_f^{\binom{n}{p}} \backslash R^{\binom{n}{p}}$. Set

$$c = \min\{\ j \mid j = L(\xi) \text{ where } \phi_{p+1}(\xi) = U \text{ and } \xi \in (\mathbb{K}_{p+1})_{-\omega}\}.$$

Notice $c \geq 1$. Let $\xi \in (\mathbb{K}_{p+1})_{-\omega}$ be such that $L(\xi) = c$ and $\phi_{p+1}(\xi) = U$. Let $(b_G/f^c \mid |G| = p+1)'$ be the normal form of ξ. Let $U = (U_I \mid |I| = p)'$. Then

$$\begin{aligned} U_I =& \sum_{i \notin I} (-1)^{\sigma(I \cup \{i\})} \frac{\partial}{\partial x_i} \left(\frac{b_{I \cup \{i\}}}{f^c} \right) \\ =& \sum_{i \notin I} (-1)^{\sigma(I \cup \{i\})} \left(\frac{\frac{\partial}{\partial x_i} b_{I \cup \{i\}}}{f^c} - c \frac{b_{I \cup \{i\}} \frac{\partial f}{\partial x_i}}{f^{c+1}} \right) \\ =& \frac{f}{f^{c+1}} \sum_{i \notin I} (-1)^{\sigma(I \cup \{i\})} \left(\frac{\partial}{\partial x_i} (b_{I \cup \{i\}}) \right) \\ & - \frac{c}{f^{c+1}} \sum_{i \notin I} (-1)^{\sigma(I \cup \{i\})} \left(b_{I \cup \{i\}} \frac{\partial f}{\partial x_i} \right). \end{aligned}$$

Set

$$V_I = -c \sum_{i \notin I} (-1)^{\sigma(I \cup \{i\})} \left(b_{I \cup \{i\}} \frac{\partial f}{\partial x_i} \right).$$

Then

$$U_I = \frac{f * + V_I}{f^{c+1}}.$$

Claim : f does not divides V_I for some I with $|I| = p$.

First assume the claim. Then $U = (U_I \mid |I| = p)' = ((f * + V_I)/f^{c+1} \mid |I| = p)'$ is the normal form of U. Therefore

$$\theta(U) = [(\bar{V}_I)'] = [\psi_{p+1}(-c\bar{b}_G \mid |G| = p+1)'] = 0.$$

We now prove our claim. Suppose if possible $f | V_I$ for all I with $|I| = p$. Let $b = (b_G \mid |G| = p+1)'$. Then

$$\psi_{p+1}(-c\bar{b}) = (\bar{V}_I)' = (0, 0, \cdots, 0)'.$$

So $-c\bar{b} \in Z_{p+1}(\partial f; A)$. As $H_{p+1}(\partial f; A) = 0$, we get $-c\bar{b} \in B_{p+1}(\partial f; A)$. Thus

$$-c\bar{b} = \psi_{p+2}(\bar{r}), \quad \text{here} \quad r = (r_L \in R \mid |L| = p+2).$$

For $p = n-1$ we have

$$-c\bar{b} = 0$$
$$\Rightarrow \quad cb_G = \tilde{\alpha}_G f \quad \text{for some} \quad \tilde{\alpha}_G \in R.$$

Thus f/b_G for all G. This contradicts that $(b_G/f^c \mid |G| = p+1)'$ is the normal form of ξ. Therefore $p < n-1$. So

$$-cb_G = \sum_{k \notin G} (-1)^{\sigma(G\cup\{k\})} \left(r_{G\cup\{k\}} \frac{\partial f}{\partial x_k} \right) + \tilde{\alpha}_G f. \tag{3.2.1}$$

Now we compute the degrees of r_L. Note that $\xi \in (\mathbb{K}_{p+1})_{-\omega}$. So

$$\frac{b_G}{f^c} \in R_f(\omega_{i_1} + \cdots + \omega_{i_{p+1}})_{-\omega}.$$

It follows that

$$\deg b_G = c \deg f - \omega + \sum_{s=1}^{p+1} \omega_{i_s}. \tag{3.2.2}$$

It can be easily checked that

$$\bar{b}_{I\cup\{i\}} \in A(-(p+1)\deg f + \omega_{i_1} + \cdots + \omega_{i_{p+1}})_{(c+p+1)\deg f - \omega}.$$

So

$$\bar{r}_L \in A(-(p+2)\deg f + \omega_{i_1} + \cdots + \omega_{i_{p+2}})_{(c+p+1)\deg f - \omega}.$$

It follows that

$$\deg r_L = (c-1)\deg f - \omega + \sum_{j=1}^{p+2} \omega_{i_j}. \tag{3.2.3}$$

Case (1) : Let $c = 1$. Then by equation (3.2.1) we get $\tilde{\alpha}_G = 0$. Also notice

$$\deg r_L = -\omega + \sum_{j=1}^{p+2} \omega_{i_j} < 0 \quad \text{if} \quad n > p+2.$$

So if $n > p+2$ we get $r_L = 0$. So $b = 0$ and $\xi = 0$ a contradiction.

Now consider $n = p + 2$. Then $r_L = r_{12\cdots n} =$ constant. Say $r_L = r$. So by equation (3.2.1) it follows

$$b = (b_G \mid |G| = n-1)' = (-r\partial f/\partial x_1, \cdots, (-1)^n r\partial f/\partial x_n)'. \qquad (3.2.4)$$

$$\text{As} \quad U_I = \sum_{i \notin I} (-1)^{\sigma(I\cup\{i\})} \left(\frac{\partial}{\partial x_i} (b_{I\cup\{i\}}/f) \right).$$

Using equation (3.2.4) we get $U_I = 0$ for all I with $|I| = p$. Therefore $U = 0$ a contradiction.

Case (2) : Let $c \geq 2$. By equation (3.2.1) we have

$$\frac{-cb_G}{f^c} = \frac{1}{f^c} \sum_{k \notin G} (-1)^{\sigma(G\cup\{k\})} \left(r_{G\cup\{k\}} \frac{\partial f}{\partial x_k} \right) + \frac{\tilde{\alpha}_G}{f^{c-1}}.$$

Notice

$$\frac{r_{G\cup\{k\}} \partial f/\partial x_k}{f^c} = \frac{\partial}{\partial x_k} \left(\frac{r_{G\cup\{k\}}/(1-c)}{f^{c-1}} \right) - \frac{*}{f^{c-1}}.$$

$$\text{Put} \quad \tilde{r}_{G\cup\{k\}} = \frac{r_{G\cup\{k\}}}{c(c-1)}. \quad \text{We obtain}$$

$$\frac{b_G}{f^c} = \sum_{k \notin G} (-1)^{\sigma(G\cup\{k\})} \left(\frac{\partial}{\partial x_k} \left(\frac{\tilde{r}_{G\cup\{k\}}}{f^{c-1}} \right) \right) + \frac{*}{f^{c-1}}.$$

Set

$$\delta = \left(\frac{\tilde{r}_{G\cup\{k\}}}{f^{c-1}} \right) \quad \text{and} \quad \tilde{\xi} = \left(\frac{*}{f^{c-1}} \right).$$

Then

$$\xi = \phi_{p+2}(\delta) + \tilde{\xi}.$$

So we have $U = \phi_{p+1}(\xi) = \phi_{p+1}(\tilde{\xi})$ and $L(\xi) \leq c - 1$. This contradicts our choice of c. $\square$

4. Construction of a Filtration on $H_p(\partial; R_f)$

In this section we construct a filtration of $H_p(\partial; R_f)$. Throughout this section $1 \leq p < n$ and $H_{p+1}(\partial f; A) = 0$.

4.1 By 2.6 we have

$$H_p(\partial; R_f) = H_p(\partial; R_f)_{-\omega} = \frac{Z_p(\partial; R_f)_{-\omega}}{B_p(\partial; R_f)_{-\omega}}.$$

Let $x \in H_p(\partial; R_f)$ be *non-zero*. Define

$$L(x) = min\{\ L(\xi) \mid \quad x = [\xi], \quad \text{where} \quad \xi \in Z_p(\partial; R_f)_{-\omega}\}.$$

Let $\xi = (\xi_I/f^c \quad | \quad |I| = p)' \in Z_p(\partial; R_f)_{-\omega}$ be such that $x = [\xi]$. So $\xi \in (\mathbb{K}_p)_{-\omega}$. Thus $\xi \in R_f^{\binom{n}{p}}(\omega_{i_1} + \cdots + \omega_{i_p})_{-\omega}$. So if $\xi \neq 0$ then $\xi \in R_f^{\binom{n}{p}} \backslash R^{\binom{n}{p}}$. It follows that $L(\xi) \geq 1$. Thus $L(x) \geq 1$. If $x = 0$ set $L(x) = -\infty$.

We now define a function

$$\tilde{\theta} : H_p(\partial; R_f) \to H_p(\partial f; A)$$
$$x \to \begin{cases} 0 & \text{if } x = 0 \\ \theta(\xi) & \text{if } x \neq 0, x = [\xi], \quad \text{and} \quad L(x) = L(\xi). \end{cases}$$

Proposition 4.2 — (with hypothesis as above). $\tilde{\theta}(x)$ is independent of ξ.

PROOF : Suppose $x = [\xi_1] = [\xi_2]$ is non zero and $L(x) = L(\xi_1) = L(\xi_2) = c$. Let $(a_I/f^c)'$ be the normal form of ξ_1 and $(b_I/f^c)'$ be the normal of ξ_2. As $[\xi_1] = [\xi_2]$ it follows that $\xi_1 = \xi_2 + \delta$ for some $\delta \in B_p(\partial; R_f)_{-w}$. We get $j = L(\delta) \leq c$ by 2.9. Let $(c_I/f^j)'$ be the normal form of δ. We consider two cases.

Case (1) : $j < c$. Then note that $a_I = b_I + f^{c-j}c_I$, for $|I| = p$. It follows that

$$\theta(\xi_1) = [(\bar{a_I})'] = [(\bar{b_I})'] = \theta(\xi_2).$$

Case (2) : $j = c$. Note that $a_I = b_I + c_I$ for $|I| = p$. It follows that

$$\theta(\xi_1) = \theta(\xi_2) + \theta(\delta).$$

However by Proposition 3.2 $\theta(\delta) = 0$. So $\theta(\xi_1) = \theta(\xi_2)$. Hence $\tilde{\theta}(x)$ is independent of choice of ξ. □

4.3 We now construct a filtration $\mathcal{F} = \{\mathcal{F}_\upsilon\}_{\upsilon \geq 0}$ of $H_p(\partial; R_f)$. Set

$$\mathcal{F}_\upsilon = \{x \in H_p(\partial; R_f) \quad | \quad L(x) \leq \upsilon\}.$$

Proposition 4.4 — (1) $\mathcal{F}_\upsilon$ is a $K-$subspace of $H_p(\partial; R_f)$.

(2) $\mathcal{F}_{\upsilon-1} \subseteq \mathcal{F}_\upsilon$ for all $\upsilon \geq 1$.

(3) $\mathcal{F}_\upsilon = H_p(\partial; R_f)$ for all $\upsilon \gg 0$.

(4) $\mathcal{F}_0 = 0$.

PROOF : (1) Let $x \in \mathcal{F}_\upsilon$ and let $\alpha \in K$. Let $x = [\xi]$ with $L(x) = L(\xi) \leq \upsilon$. Then $\alpha x = [\alpha\xi]$. So

$$L(\alpha x) \leq L(\alpha\xi) \leq \upsilon.$$

So $\alpha x \in \mathcal{F}_\upsilon$.

Let $x, x' \in \mathcal{F}_\upsilon$ be non-zero. Let $\xi, \xi' \in Z_p(\partial; R_f)$ be such that $x = [\xi]$, $x' = [\xi']$ and $L(x) = L(\xi)$, $L(x') = L(\xi')$. Then $x + x' = [\xi + \xi']$. It follows that

$$L(x + x') \leq L(\xi + \xi') \leq max\{L(\xi), L(\xi')\} \leq \upsilon.$$

Thus $x + x' \in \mathcal{F}_\upsilon$.

(2) This is clear from the definition.

(3) Let $\mathcal{B} = \{x_1, \cdots, x_l\}$ be a $K-$ basis of $H_p(\partial; R_f) = H_p(\partial; R_f)_{-\omega}$. Let

$$c = max\{L(x_i) \quad | \quad i = 1, \cdots, l\}.$$

We claim that

$$\mathcal{F}_\upsilon = H_p(\partial; R_f) \quad \text{for all} \quad \upsilon \geq c.$$

Fix $\upsilon \geq c$. Let $\xi_i \in Z_p(\partial; R_f)_{-\omega}$ be such that $x_i = [\xi_i]$ and $L(x_i) = L(\xi_i)$ for $i = 1, \cdots, l$.

Let $u \in H_p(\partial; R_f)$. Say $u = \sum_{i=1}^{l} \alpha_i x_i$ for some $\alpha_1, \cdots, \alpha_l \in K$. Then $u = [\sum_{i=1}^{l} \alpha_i \xi_i]$. It follows that

$$L(u) \leq L(\sum_{i=1}^{l} \alpha_i \xi_i) \leq max\{L(\xi) \quad | \quad i = 1, \cdots, l\} = c \leq \upsilon.$$

So $u \in \mathcal{F}_\upsilon$. Hence $\mathcal{F}_\upsilon = H_p(\partial; R_f)$.

(4) If $x \in H_p(\partial; R_f)$ is non-zero then $L(x) \geq 1$. Therefore $\mathcal{F}_0 = 0$. □

4.5 Let $\mathcal{G} = \bigoplus_{\upsilon \geq 1} \mathcal{F}_\upsilon / \mathcal{F}_{\upsilon-1}$. For $\upsilon \geq 1$ we define

$$\eta_\upsilon : \frac{\mathcal{F}_\upsilon}{\mathcal{F}_{\upsilon-1}} \to H_p(\partial f; A)_{(\upsilon+p)\deg f - \omega}$$

$$u \to \begin{cases} 0 & \text{if } u = 0 \\ \tilde{\theta}(x) & \text{if } u = x + \mathcal{F}_{\upsilon-1} \quad \text{is non-zero.} \end{cases}$$

Proposition 4.6 — (with hypothesis as above). $\eta_\upsilon(u)$ is independent of choice of x.

PROOF : Suppose $u = x + \mathcal{F}_{\upsilon-1} = x' + \mathcal{F}_{\upsilon-1}$ be non-zero. Then $x = x' + y$ where $y \in \mathcal{F}_{\upsilon-1}$. As $u \neq 0$ we have $x, x' \in \mathcal{F}_\upsilon \backslash \mathcal{F}_{\upsilon-1}$. So $L(x) = L(x') = \upsilon$. Say $x = [\xi]$, $x' = [\xi']$, and $y = [\delta]$ where $\xi, \xi', \delta \in Z_p(\partial; R_f)_{-\omega}$ with $L(\xi) = L(\xi') = \upsilon$ and $L(\delta) = L(y) = k \leq \upsilon - 1$. So we have $\xi = \xi' + \delta + \alpha$ where $\alpha \in B_p(\partial; R_f)_{-\omega}$. Let $L(\alpha) = r$. Not that $r \leq \upsilon$.

Let $(a_I/f^\upsilon)'$, $(a'_I/f^\upsilon)'$, $(b_I/f^k)'$ and $(c_I/f^r)'$ be normal forms of ξ, ξ', δ and α respectively, here $|I| = p$. So we have

$$a_I = a'_I + f^{\upsilon-k} b_I + f^{\upsilon-r} c_I \quad \text{with} \quad |I| = p.$$

Case (1) : $r < \upsilon$. In this case we have that $\bar{a}_I = \bar{a'}_I$ in A. So $\theta(\xi) = \theta(\xi')$. Thus $\tilde{\theta}(x) = \tilde{\theta}(x')$.

Case (2) : $r = \upsilon$. In this case we have $\bar{a}_I = \bar{a'}_I + \bar{c}_I$ in A. So $\theta(\xi) = \theta(\xi') + \theta(\alpha)$. However $\theta(\alpha) = 0$ as $\alpha \in B_p(\partial; R_f)_{-\omega}$. Thus $\tilde{\theta}(x) = \tilde{\theta}(x')$. □

Proposition 4.7 — (with notation as above). For all $\upsilon \geq 1$, η_υ is $K-$linear.

PROOF : Let $u, u' \in \mathcal{F}_\upsilon/\mathcal{F}_{\upsilon-1}$. We first show that $\eta_\upsilon(\alpha u) = \alpha\eta_\upsilon(u)$. If $\alpha = 0$ or $u = 0$ we have nothing to show. So assume $\alpha \neq 0$ and $u \neq 0$. Say $u = x + \mathcal{F}_{\upsilon-1}$. Then $\alpha u = \alpha x + \mathcal{F}_{\upsilon-1}$. It can be easily shown that $\tilde{\theta}(\alpha x) = \alpha\tilde{\theta}(x)$. So we get the result.

Next we show that $\eta_\upsilon(u+u') = \eta_\upsilon(u) + \eta_\upsilon(u')$. We have nothing to show if u or u' is zero. Now consider the case when $u + u' = 0$. Then $u = -u'$. So $\eta_\upsilon(u) = -\eta_\upsilon(u')$. Thus in this case

$$\eta_\upsilon(u + u') = 0 = \eta_\upsilon(u) + \eta_\upsilon(u').$$

Now consider the case when u, u' are non-zero and $u + u'$ non-zero. Say $u = x + \mathcal{F}_{\upsilon-1}$ and $u' = x' + \mathcal{F}_{\upsilon-1}$. Note that as $u + u'$ is non-zero $x + x' \in \mathcal{F}_\upsilon \backslash \mathcal{F}_{\upsilon-1}$. Let $x = [\xi]$ and $x' = [\xi']$ where $\xi, \xi' \in Z_p(\partial; R_f)_{-\omega}$ and $L(\xi) = L(\xi') = \upsilon$. Then $x + x' = [\xi + \xi']$. Note that $L(\xi + \xi') \leq \upsilon$. But $L(x+x') = \upsilon$. So $L(\xi+\xi') = \upsilon$. Let $(a_I/f^\upsilon)'$, $(a'_I/f^\upsilon)'$ be the normal forms of ξ, and ξ' respectively. Note that $((a_I + a'_I)/f^\upsilon)'$ is the normal form of $\xi + \xi'$. It follows that $\theta(\xi + \xi') = \theta(\xi) + \theta(\xi')$. Thus $\tilde{\theta}(x + x') = \tilde{\theta}(x) + \tilde{\theta}(x')$. Therefore

$$\eta_\upsilon(u + u') = \eta_\upsilon(u) + \eta_\upsilon(u'). \square$$

Surprisingly the following result holds.

Proposition 4.8 — (with notation as above).

(a) η_υ is injective for all $\upsilon \geq 2$.

(b) If $p \neq n-1$. Then η_1 also injective.

(c) If $p = n-1$. Then $\ker(\eta_1) = K$.

PROOF : Suppose if possible η_ν is not injective. Then there exists non-zero $u \in \mathcal{F}_\nu/\mathcal{F}_{\nu-1}$ with $\eta_\nu(u) = 0$. Say $u = x + \mathcal{F}_{\nu-1}$. Also let $x = [\xi]$ where $\xi \in Z_p(\partial; R_f)_{-\omega}$ and $L(\xi) = L(x) = \nu$. Let $(a_I/f^\upsilon \mid |I| = p)'$ be the normal form of ξ. So we have

$$0 = \eta_\nu(u) = \widetilde{\theta}(x) = \theta(\xi) = [(\overline{a_I})'].$$

It follows that $(\overline{a_I})' = \psi_{p+1}(\bar{b})$, where $\bar{b} = (b_G \mid |G| = p+1)'$. It follows that

$$\overline{a_I} = \sum_{i \notin I} (-1)^{\sigma(I\cup\{i\})} \left(\bar{b}_{I\cup\{i\}} \frac{\bar{\partial} f}{\partial x_i} \right).$$

It follows that for $|I| = p$ we have the following equation in R:

$$a_I = \sum_{i \notin I} (-1)^{\sigma(I\cup\{i\})} \left(b_{I\cup\{i\}} \frac{\partial f}{\partial x_i} \right) + d_I f, \tag{4.8.1}$$

for some $d_I \in R$. Note that the above equation is of homogeneous elements in R. So we have the following

$$\frac{a_I}{f^\upsilon} = \frac{\sum_{i \notin I} (-1)^{\sigma(I\cup\{i\})} b_{I\cup\{i\}} \frac{\partial f}{\partial x_i}}{f^\upsilon} + \frac{d_I}{f^{\upsilon-1}}. \tag{4.8.2}$$

We consider two cases:
(a): Let $\nu \geq 2$. Set $\widetilde{b}_{I\cup\{i\}} = -b_{I\cup\{i\}}/(\upsilon - 1)$. Then note that

$$\frac{b_{I\cup\{i\}} \frac{\partial f}{\partial x_i}}{f^\nu} = \frac{\partial}{\partial x_i} \left(\frac{\widetilde{b}_{I\cup\{i\}}}{f^{\nu-1}} \right) - \frac{*}{f^{\nu-1}}.$$

By equation (4.8.2) we have

$$\frac{a_I}{f^\nu} = \sum_{i \notin I} (-1)^{\sigma(I\cup\{i\})} \frac{\partial}{\partial x_i} \left(\frac{\widetilde{b}_{I\cup\{i\}}}{f^{\upsilon-1}} \right) - \frac{*}{f^{\nu-1}}.$$

Put $\xi' = \left(*/f^{\nu-1} : |I| = p \right)'$ and $\delta = \left(\widetilde{b}_{I\cup\{i\}}/f^{\nu-1} \mid i \notin I, |I| = p \right)'$. Then we have

$$\xi = \phi_{p+1}(\delta) + \xi'.$$

So we have $x = [\xi] = [\xi']$. This yields $L(x) \leq L(\xi') \leq \nu - 1$. This is a contradiction.

(b): Let $\nu = 1$ and $p \neq n-1$. Note that $n \geq p+2$. Also note that $\xi \in (\mathbb{K}_p)_{-\omega}$. Thus for $|I| = p$ we have

$$\frac{a_I}{f} \in (R_f(\omega_{i_1} + \cdots + \omega_{i_p})_{-\omega}.$$

It follows that

$$\deg a_I = \deg f - \omega + (\omega_{i_1} + \cdots + \omega_{i_p}).$$

Also note that $\deg \partial f/\partial x_i = \deg f - \omega_i$. By comparing degrees in equation (4.8.1) we get $a_I = 0$ for all I with $|I| = p$. Thus $\xi = 0$. So $x = 0$. Therefore $u = 0$ a contradiction.

(c): Let $p = n-1$. By comparing degrees in equation (4.8.1) we get $d_I = 0$ and $b_{I\cup\{i\}}$ =constant. But

$$\xi = \left(\frac{\partial f}{\partial x_n}/f, -\frac{\partial f}{\partial x_{n-1}}/f, \cdots, (-1)^{n-1}\frac{\partial f}{\partial x_1}/f\right)'.$$

It is easily verified that $\xi \in Z_{n-1}(\partial; R_f)$ and that if $x = [\xi]$ then $\eta_1(x) = 0$.

We prove that $\xi \notin B_{n-1}(\partial; R_f)$. Suppose if possible let $g \in R$, g.c.d$(g, f) = 1$ and

$$\left(\frac{\partial}{\partial x_n}, -\frac{\partial}{\partial x_{n-1}}, \cdots, (-1)^{n-1}\frac{\partial}{\partial x_1}\right)'(g/f^c) = \xi.$$

Thus

$$\frac{\partial}{\partial x_i}(g/f^c) = \frac{\partial f}{\partial x_i}/f \quad \text{for} \quad i = 1, \cdots, n.$$

By computing left hand side we see that f divides $g\partial f/\partial x_i$ for $i = 1, \cdots, n$.

$$\text{Let} \quad g\frac{\partial f}{\partial x_i} = fh_i.$$

Let

$$f = f_1^{a_1} f_2^{a_2} \cdots f_s^{a_s}, \quad f_j \text{ irreducible and } a_j \geq 1.$$

Then

$$\frac{\partial f}{\partial x_i} = \sum_{j=1}^{s} f_1^{a_1} \cdots f_{j-1}^{a_{j-1}}(a_j f_j^{a_j-1}\frac{\partial f_j}{\partial x_i}) f_{j+1}^{a_{j+1}} \cdots f_s^{a_s}.$$

If $a_j = 1 \Rightarrow f_j$ does not divides $\partial f_j/\partial x_j$ so f_j does not divides $\partial f/\partial x_i$.

If $a_j \geq 2 \Rightarrow f_j^{a_j-1}$ divides $\partial f/\partial x_i$ and $f_j^{a_j}$ does not divides $\partial f/\partial x_i$. So we can write

$$\frac{\partial f}{\partial x_i} = f_1^{a_1-1} f_2^{a_2-1} \cdots f_s^{a_s-1} V_i, \quad \text{where } f_j \text{ does not divides } V_i \; \forall \, j.$$

Let $U = f_1 f_2 \cdots f_s$. Then we have $gV_i = Uh_i$. As g.c.d$(g, f) = 1$ so g.c.d$(g, U) = 1$. So f_j divides V_i a contradiction. Therefore $\xi \notin B_{n-1}(\partial; R_f)$. Hence $\ker(\eta_1) = K$. □

By summarizing the above results, we obtain our main Theorem:

***Theorem* 4.9** — *Assume $H_{p+1}(\partial f; A) = 0$. Then there exists a filtration $\{\mathcal{F}_\nu\}_{\nu \geq 0}$ consisting of $K-$ subspaces of $H_p(\partial; R_f)$ with $\mathcal{F}_\nu = H_p(\partial; R_f)$ for $\nu \gg 0, \mathcal{F}_\nu \supseteq \mathcal{F}_{\nu-1}$ and $\mathcal{F}_0 = 0$ and $K-$linear maps*

$$\eta_\upsilon : \frac{\mathcal{F}_\upsilon}{\mathcal{F}_{\upsilon-1}} \to H_p(\partial f; A)_{(\upsilon+p)\deg f - \omega}.$$

such that

(a) η_υ is injective for all $\upsilon \geq 2$.

(b) If $p \neq n-1$. Then η_1 also injective.

(c) If $p = n-1$. Then $\ker(\eta_1) = K$.

Corollary 4.10 — If $H_i(\partial f; A) = 0$ for $i \geq \alpha + 1$. Then

$$H_i(\partial; R_f) = \begin{cases} 0 & \text{if } \alpha + 1 \leq i \leq n-2 \\ K & \text{if } i = n-1. \end{cases}$$

PROOF : Let $\alpha + 1 \leq i \leq n-2$.. By Theorem 4.9 there exist a filtration $\{\mathcal{F}_\upsilon\}_{\upsilon \geq 0}$ of $H_i(\partial; R_f)$ and injective maps

$$\eta_\upsilon : \frac{\mathcal{F}_\upsilon}{\mathcal{F}_{\upsilon-1}} \longrightarrow H_i(\partial f; A).$$

Note that $\mathcal{F}_0 = 0$. As $H_i(\partial f; A) = 0$ we get $\mathcal{F}_1 = 0$. Continuing this way we get $\mathcal{F}_\upsilon = 0$ for all υ. As $\mathcal{F}_\upsilon = H_i(\partial; R_f)$ for $\upsilon \gg 0$. Hence $H_i(\partial; R_f) = 0$.

Let $i = n-1$. Then $\ker(\eta_1) = K$. So $\mathcal{F}_1/K = 0$. Thus $\mathcal{F}_1 = K$.

$$\text{As} \quad \dim_K H_{n-1}(\partial; R_f) = \sum \dim_K(\mathcal{F}_\upsilon/\mathcal{F}_{\upsilon-1}) = \dim_K \mathcal{F}_1 = 1. \square$$

When A is smooth we get:

Corollary 4.11 — Let $f \in R$ be quasi homogeneous. Let $A = R/(f)$ be smooth. Then

$$H_i(\partial; H^1_{(f)}(R)) = \begin{cases} 0 & \text{if} \quad 2 \leq i \leq n-2 \text{ or } i = n \\ K & \text{if} \quad i = n-1. \end{cases}$$

PROOF : As A is smooth, so $H_i(\partial f; A) = 0$ for $i \geq 2$. Therefore by Corollary 4.10.

$$H_i(\partial; R_f) = \begin{cases} 0 & \text{if } \alpha + 1 \leq i \leq n-2 \\ K & \text{if } i = n-1. \end{cases}$$

By [3, Theorem 2.7]

$$H_n(\partial; H^1_{(f)}(R)) = 0 \quad \text{and} \quad H_i(\partial; R_f) \equiv H_i(\partial; H^1_{(f)}(R)) \text{ for } \quad i < n.$$

Hence the result. □

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