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THE SIGNLESS LAPLACIAN SPECTRAL RADIUS OF SOME STRONGLY CONNECTED DIGRAPHS[1]

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Let $\overrightarrow{G}$ be a strongly connected digraph and $Q(\overrightarrow{G})$ be the signless Laplacian matrix of $\overrightarrow{G}$. The spectral radius of $Q(\overrightarrow{G})$ is called the signless Lapliacian spectral radius of $\overrightarrow{G}$. Let $\widetilde{\infty}_1$-digraph and $\widetilde{\infty}_2$-digraph be two kinds of generalized strongly connected ∞-digraphs and let $\widetilde{\theta}_1$-digraph and $\widetilde{\theta}_2$-digraph be two kinds of generalized strongly connected θ-digraphs. In this paper, we determine the unique digraph which attains the maximum(or minimum) signless Laplacian spectral radius among all $\widetilde{\infty}_1$-digraphs and $\widetilde{\theta}_1$-digraphs. Furthermore, we characterize the extremal digraph which achieves the maximum signless Laplacian spectral radius among $\widetilde{\infty}_2$-digraphs and $\widetilde{\theta}_2$-digraphs, respectively.

Key words : The signless Laplacian spectral radius; $\widetilde{\infty}_1$-digraph; $\widetilde{\infty}_2$-digraph; $\widetilde{\theta}_1$-digraph; $\widetilde{\theta}_2$-digraph.

1. Introduction

Throughout this article, all digraphs are finite simple strongly connected digraphs, i.e., without loops and multiple arcs. Let $\overrightarrow{G} = (V(\overrightarrow{G}), E(\overrightarrow{G}))$ be a digraph with vertex

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set $V(\overrightarrow{G})$ and arc set $E(\overrightarrow{G})$. Two vertices are called adjacent if they are connected by an arc. If there is an arc from u to v, we indicate this by writing (u,v), call u the head of (u,v), and v the tail of (u,v), respectively. For any vertex v, let $N^+_{\vec{G}}(v) = \{u \in V(\overrightarrow{G}) \mid (v,u) \in E(\overrightarrow{G})\}$ and $N^-_{\vec{G}}(v) = \{u \in V(\overrightarrow{G}) \mid (u,v) \in E(\overrightarrow{G})\}$ denote the sets of out-neighbors and in-neighbors of v, respectively. Let $d^+_{\vec{G}}(v) = |N^+_{\vec{G}}(v)|$ and $d^-_{\vec{G}}(v) = |N^-_{\vec{G}}(v)|$ denote the out-degree and in-degree of v in $\overrightarrow{G}$. The digraph $\overrightarrow{G}$ is called strongly connected if for every pair of vertices $u, v \in V(\overrightarrow{G})$, there exist a directed path from u to v and a directed path from v to u. Let $\overrightarrow{P_l}$ and $\overrightarrow{C_l}$ denote the directed path and directed cycle on l vertices, respectively. Suppose $\overrightarrow{P_k} = v_1v_2\ldots v_k$, we call v_1 the initial vertex and v_k the terminal vertex of the directed path $\overrightarrow{P_k}$, respectively.

Let $A(\overrightarrow{G}) = (a_{ij})_{n\times n}$ denote the adjacency matrix of a digraph $\overrightarrow{G}$, where $a_{ij} = 1$ if $(v_i, v_j) \in E(\overrightarrow{G})$ and $a_{ij} = 0$ otherwise. Let $D(\overrightarrow{G})$ be the diagonal matrix with out-degrees of the vertices of $\overrightarrow{G}$. Then the matrix $Q(\overrightarrow{G}) = D(\overrightarrow{G}) + A(\overrightarrow{G})$ is called the signless Laplacian matrix of $Q(\overrightarrow{G})$, and let $q(\overrightarrow{G})$ denote its signless Laplacian spectral radius, the largest modulus of an eigenvalue of $Q(\overrightarrow{G})$. The polynomial $\phi(\overrightarrow{G}, \lambda) = det(\lambda I - Q(\overrightarrow{G}))$ is defined as the characteristic polynomial with respect to the signless Laplacian matrix $Q(\overrightarrow{G})$. The collection of eigenvalues of $Q(\overrightarrow{G})$ together with multiplicates is called the Q-spectrum of $\overrightarrow{G}$. There are many articles on the signless Laplacian spectrum of undirected graphs [1, 2, 12, 14, 16]. There are also many articles on the topic of adjacency spectrum [3, 6, 9], and the Laplacian spectrum [8, 13]. For additional remarks on this topic we refer the reader to see two excellent surveys [10] and [11]. However, there is not much known about digraphs.

The matrix $Q(\overrightarrow{G})$ is nonnegative and irreducible when $\overrightarrow{G}$ is strongly connected. It follows from the Perron-Frobenius Theorem [7] that $q(\overrightarrow{G})$ is an eigenvalue of the signless Laplacian matrix $Q(\overrightarrow{G})$ and there is a positive unit eigenvector corresponding to $q(\overrightarrow{G})$. The positive unit eigenvector corresponding to $q(\overrightarrow{G})$ is called the Perron vector of $Q(\overrightarrow{G})$. In [15], Xi and Wang characterized the extremal digraphs which achieve the maximum and minimum signless Laplacian spectral radius among strongly connected bicyclic digraphs. In [4], Guo and Liu characterized the extremal digraphs which achieve the maximum and minimum adjacency spectral radius among two kinds of generalized strongly connected bicyclic digraphs. In our paper, we mainly generalize their results and obtain some results about the signless Laplacian spectral radius of generalized $\widetilde{\infty}$ and $\widetilde{\theta}$-digraphs.

The rest of this paper is organized as follow. In Section 2, we characterize the extremal digraphs which attain the maximum and minimum signless Laplacian spectral radius among all $\widetilde{\infty}_1$-digraphs. At the same time, we also obtain the maximum signless Laplacian spectral radius among $\widetilde{\infty}_2$-digraphs. In Section 3, we characterize the extremal digraphs which attain the maximum and minimum signless Laplacian spectral radius among all $\widetilde{\theta}_1$-digraphs. Under some conditions, we also obtain the maximum signless Laplacian spectral radius among all $\widetilde{\theta}_2$-digraphs. In Section 4, we determine the extremal digraphs which attain the maximum and minimum signless Laplacian spectral radius among all $\widetilde{\infty}_1$ and $\widetilde{\theta}_1$-digraphs.

2. The signless Laplacian Spectral Radius of $\widetilde{\infty}_1$-digraphs and $\widetilde{\infty}_2$-digraphs

A $\widetilde{\infty}_1$-digraph is a graph consisting of m ($m \geq 2$) directed cycles with just a vertex in common (as shown in Figure 1). We use $\widetilde{\infty}_1(k_1, k_2, \ldots, k_m)$ to denote the $\widetilde{\infty}_1$-digraph such that $\sum_{i=1}^{m} k_i + 1 = n$. Without loss of generality, let $1 \leq k_i \leq k_{i+1}$ for $i = 1, 2, \ldots, m-1$. In this section, we first prove that $\widetilde{\infty}_1(1, \ldots, 1, n-m)$ is the unique digraph which attains the maximum signless Laplacian spectral radius and $\widetilde{\infty}_1(a_1, a_2, \ldots, a_m)$ such that $a_i = \lfloor \frac{n-1}{m} \rfloor$ and $a_j = \lceil \frac{n-1}{m} \rceil$ for any $i \in \{1, 2, \ldots, m-(n-1-m\lfloor \frac{n-1}{m} \rfloor)\}$ and $j \in \{m-(n-1-m\lfloor \frac{n-1}{m} \rfloor)+1, \ldots, m\}$, is the unique digraph which attains the minimum signless Laplacian spectral radius among all $\widetilde{\infty}_1(k_1, k_2, \ldots, k_m)$-digraphs on n vertices.

Lemma 2.1 — [5]. Let A be a nonnegative irreducible matrix with the largest eigenvalue $\varrho(A)$ and row sums $s_1, s_2, \ldots, s_n$, then

$$\min_{1\leq i\leq n} s_i \leq \varrho(A) \leq \max_{1\leq i\leq n} s_i.$$

Moreover, one of the equalities holds if and only if the row sums of A are all equal.

Lemma 2.2 —For any $p, q \in \{1, 2, \ldots, m\}$, if $2 \leq k_p \leq k_q$, then we have

$$q(\widetilde{\infty}_1(k_1, k_2, \ldots, k_{p-1}, k_p, k_{p+1}, \ldots, k_{q-1}, k_q, k_{q+1}, \ldots, k_m)) <$$

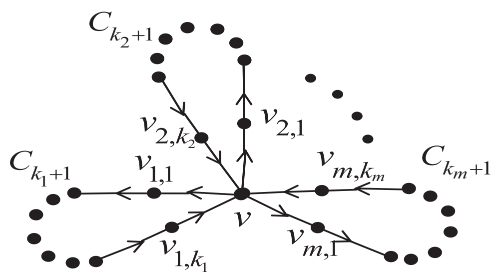

Figure 1: The digraph $\widetilde{\infty}_1(k_1, k_2, \ldots, k_m)$.

$$q(\widetilde{\infty}_1(k_1, k_2, \ldots, k_{p-1}, k_p - 1, k_{p+1}, \ldots, k_{q-1}, k_q + 1, k_{q+1}, \ldots, k_m)).$$

Proof : Let $\overrightarrow{G'} = \widetilde{\infty}_1(k_1, k_2, \ldots, k_{p-1}, k_p, k_{p+1}, \ldots, k_{q-1}, k_q, k_{q+1}, \ldots, k_m)$ and $\overrightarrow{G''} = \widetilde{\infty}_1(k_1, k_2, \ldots, k_{p-1}, k_p - 1, k_{p+1}, \ldots, k_{q-1}, k_q + 1, k_{q+1}, \ldots, k_m)$ in the following. Suppose that $\overrightarrow{x} = (x_v, x_{1,1}, \ldots, x_{1,k_1}; x_{2,1}, \ldots, x_{2,k_2}; \ldots; x_{m,1}, \ldots, x_{m,k_m})^{\intercal}$ is the Perron vector of $Q(\overrightarrow{G'})$ corresponding to $q(\overrightarrow{G'})$, where x_v corresponds to v, $x_{i,j}$ corresponds to $v_{i,j}$ ($i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, k_i$), respectively. Since $Q(\overrightarrow{G'})\overrightarrow{x} = q(\overrightarrow{G'})\overrightarrow{x}$, it is not difficult to see that

$$\begin{cases} q(\overrightarrow{G'})x_{1,i_1} = x_{1,i_1} + x_{1,i_1+1}, & i_1 = 1, 2, \ldots, k_1 - 1, \\ q(\overrightarrow{G'})x_{2,i_2} = x_{2,i_2} + x_{2,i_2+1}, & i_2 = 1, 2, \ldots, k_2 - 1, \\ \qquad\qquad \vdots & \\ q(\overrightarrow{G'})x_{m,i_m} = x_{m,i_m} + x_{m,i_m+1}, & i_m = 1, 2, \ldots, k_m - 1, \\ & \\ q(\overrightarrow{G'})x_v = mx_v + x_{1,1} + x_{2,1} + \ldots + x_{m,1}, & \\ q(\overrightarrow{G'})x_{j,k_j} = x_{j,k_j} + x_v, & j = 1, 2, \ldots, m. \end{cases}$$

Then we have

$$x_{j,k_j} = (q(\overrightarrow{G'}) - 1)^{k_j - 1} x_{j,1}, \qquad j = 1, 2, \ldots, m.$$

Furthermore,

$$x_v = (q(\overrightarrow{G'}) - 1)^{k_j} x_{j,1}, \qquad j = 1, 2, \ldots, m.$$

Thus, we have

$$(q(\overrightarrow{G'})-m)(q(\overrightarrow{G'})-1)^{n-1}x_{m,1}=\sum_{j=1}^{m}(q(\overrightarrow{G'})-1)^{n-1-k_j}x_{m,1}.$$

By Perron-Frobenius Theorem, we have $x_{m,1}>0$, therefore

$$(q(\overrightarrow{G'})-m)(q(\overrightarrow{G'})-1)^{n-1}=\sum_{j=1}^{m}(q(\overrightarrow{G'})-1)^{n-1-k_j}.$$

Similarly, we have

$$\begin{aligned}(q(\overrightarrow{G''})-m)(q(\overrightarrow{G''})-1)^{n-1} &= \sum_{j=1,j\neq p \text{ and } q}^{m}(q(\overrightarrow{G''})-1)^{n-1-k_j}\\ &\quad +(q(\overrightarrow{G''})-1)^{n-k_p}+(q(\overrightarrow{G''})-1)^{n-k_q-2}.\end{aligned}$$

Let $f(x)=(x-m)(x-1)^{n-1}-\sum_{j=1}^{m}(x-1)^{n-1-k_j}$ and $g(x)=(x-m)(x-1)^{n-1}-\sum_{j=1,j\neq p \text{ and } q}^{m}(x-1)^{n-1-k_j}-(x-1)^{n-k_p}-(x-1)^{n-k_q-2}$. It is easy to see that $q(\overrightarrow{G'})$ and $q(\overrightarrow{G''})$ are the largest roots of $f(x)=0$ and $g(x)=0$, respectively. $f(x)-g(x)=((x-1)^{n-k_p-1}-(x-1)^{n-k_q-2})(x-2)>0$, for all $x>2$. Since the minimum row sum of $Q(\overrightarrow{G'})$ is 2, and the row sums of $Q(\overrightarrow{G'})$ are not all equal, then by Lemma 2.1, we have $q(\overrightarrow{G'})>2$. Therefore we have $q(\overrightarrow{G'})<q(\overrightarrow{G''})$. □

Using the above Lemma, we immediately obtain:

***Theorem* 2.3** — *Among all $\widetilde{\infty}_1(k_1,k_2,\ldots,k_m)$-digraphs of order n, the digraph $\widetilde{\infty}_1(1,\ldots,1,n-m)$ is the unique digraph which achieves the maximum signless Laplacian spectral radius, and $\widetilde{\infty}_1(a_1,a_2,\ldots,a_m)$ such that $a_i=\lfloor\frac{n-1}{m}\rfloor$ and $a_j=\lceil\frac{n-1}{m}\rceil$ for any $i\in\{1,2,\ldots,m-(n-1-m\lfloor\frac{n-1}{m}\rfloor)\}$ and $j\in\{m-(n-1-m\lfloor\frac{n-1}{m}\rfloor)+1,\ldots,m\}$, is the unique digraph which achieves the minimum signless Laplacian spectral radius.*

Now, let $\widetilde{\infty}_2$-digraph be a strongly connected tricyclic digraph of order $n(=p+q+s+2)$ (as shown in Figure 2), denoted by $\widetilde{\infty}_2(p,q,s)$, which obtained from three directed cycles C_{p+1}, C_{q+2} and C_{s+1} by identifying a vertex of C_{p+1} with a vertex of C_{q+2} and identifying a vertex of C_{s+1} with a vertex of C_{q+2}. Without loss of generality, we assume that $1\le p\le s$. In the following, we will prove that the digraph $\widetilde{\infty}_2(1,0,n-3)$ is the unique digraph which achieves the maximum spectral radius among all $\widetilde{\infty}_2(p,q,s)$-digraph of order $n(=p+q+s)$ with $q\le p\le s$.

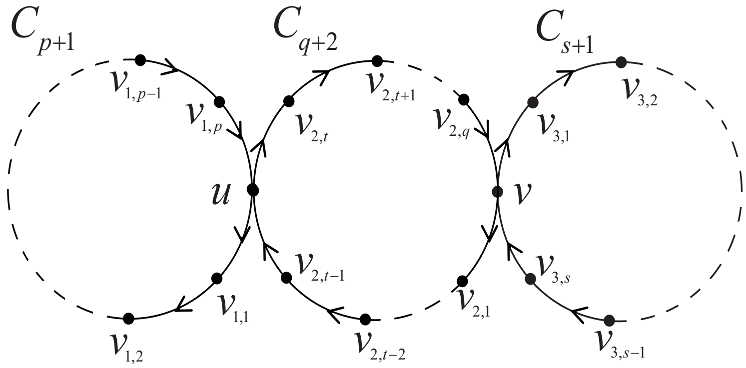

Figure 2: The digraph $\widetilde{\infty}_2(p,q,s)$.

Lemma 2.4 — If $1 \leq p \leq s$, and $p \geq 2$ then $q(\widetilde{\infty}_2(p,q,s)) < q(\widetilde{\infty}_2(p-1,q,s+1))$.

PROOF : Let $\overrightarrow{G'} = \widetilde{\infty}_2(p,q,s)$ and $\overrightarrow{G''} = \widetilde{\infty}_2(p-1,q,s+1)$ in the following. Suppose that $\overrightarrow{x} = (x_u, x_v, x_{1,1}, \ldots, x_{1,p}, x_{2,1}, \ldots, x_{2,q}, x_{3,1}, \ldots, x_{3,s})^{\intercal}$ is the Perron vector of $Q(\overrightarrow{G'})$ corresponding to $q(\overrightarrow{G'})$, where x_u corresponds to u, x_v corresponds to v and $x_{i,j}$ corresponds to $v_{i,j}$ ($i=1, j=1,2,\ldots,p; i=2, j=1,2,\ldots,q; i=3, j=1,2,\ldots,s$), respectively. Since $Q(\overrightarrow{G'})\overrightarrow{x} = q(\overrightarrow{G'})\overrightarrow{x}$, it is not difficult to see that

$$\begin{cases} q(\overrightarrow{G'})x_u = 2x_u + x_{1,1} + x_{2,t}, & \\ q(\overrightarrow{G'})x_v = 2x_v + x_{2,1} + x_{3,1}, & \\ q(\overrightarrow{G'})x_{1,i_1} = x_{1,i_1} + x_{1,i_1+1}, & i_1 = 1,2,\ldots,p-1, \\ q(\overrightarrow{G'})x_{1,p} = x_{1,p} + x_u, & \\ q(\overrightarrow{G'})x_{2,i_2} = x_{2,i_2} + x_{2,i_2+1}, & i_2 = 1,2,\ldots,t-2, \\ q(\overrightarrow{G'})x_{2,t-1} = x_{2,t-1} + x_u, & \\ q(\overrightarrow{G'})x_{2,i_2} = x_{2,i_2} + x_{2,i_2+1}, & i_2 = t,t+1,\ldots,q-1, \\ q(\overrightarrow{G'})x_{2,q} = x_{2,q} + x_v, & \\ q(\overrightarrow{G'})x_{3,i_3} = x_{3,i_3} + x_{3,i_3+1}, & i_3 = 1,2,\ldots,s-1, \\ q(\overrightarrow{G'})x_{3,s} = x_{3,s} + x_v. & \end{cases}$$

Then we have

$$\begin{cases} x_{3,s} = (q(\overrightarrow{G'})-1)x_{3,s-1} = (q(\overrightarrow{G'})-1)^2 x_{3,s-2} = \cdots = (q(\overrightarrow{G'})-1)^{s-1}x_{3,1}, \\ x_v = (q(\overrightarrow{G'})-1)^s x_{3,1}, \\ x_{1,p} = (q(\overrightarrow{G'})-1)x_{1,p-1} = (q(\overrightarrow{G'})-1)^2 x_{1,p-2} = \cdots = (q(\overrightarrow{G'})-1)^{p-1}x_{1,1}, \\ x_u = (q(\overrightarrow{G'})-1)^p x_{1,1}, \\ x_{2,q} = (q(\overrightarrow{G'})-1)x_{2,q-1} = (q(\overrightarrow{G'})-1)^2 x_{2,q-2} = \cdots = (q(\overrightarrow{G'})-1)^{q-t}x_{2,t}, \\ x_{2,t-1} = (q(\overrightarrow{G'})-1)x_{2,t-2} = (q(\overrightarrow{G'})-1)^2 x_{2,t-3} = \cdots = (q(\overrightarrow{G'})-1)^{t-2}x_{2,1}, \\ x_u = (q(\overrightarrow{G'})-1)x_{2,t-1} = (q(\overrightarrow{G'})-1)^{t-1}x_{2,1}, \\ x_v = (q(\overrightarrow{G'})-1)x_{2,q} = (q(\overrightarrow{G'})-1)^{q-t+1}x_{2,t}. \end{cases}$$

Furthermore,

$$\begin{aligned} x_{2,1} &= (q(\overrightarrow{G'})-1)^{-(t-1)}x_u \\ &= (q(\overrightarrow{G'})-1)^{-t}(x_u + x_{1,1} + x_{2,t}) \\ &= (q(\overrightarrow{G'})-1)^{-t}x_u + (q(\overrightarrow{G'})-1)^{-t}x_{1,1} + (q(\overrightarrow{G'})-1)^{-t}x_{2,t} \\ &= (q(\overrightarrow{G'})-1)^{-1}x_{2,1} + (q(\overrightarrow{G'})-1)^{-t-p}x_u + (q(\overrightarrow{G'})-1)^{-t-q+t-1}x_v \\ &= (q(\overrightarrow{G'})-1)^{-1}x_{2,1} + (q(\overrightarrow{G'})-1)^{-1-p}x_{2,1} + (q(\overrightarrow{G'})-1)^{-q-1}x_v. \end{aligned}$$

Then we get

$$x_{2,1} = \frac{(q(\overrightarrow{G'})-1)^{-q-1}}{1-(q(\overrightarrow{G'})-1)^{-1}-(q(\overrightarrow{G'})-1)^{-p-1}}x_v.$$

Then

$$\begin{aligned} q(\overrightarrow{G'})x_v &= 2x_v + x_{2,1} + x_{3,1} \\ &= 2x_v + \frac{(q(\overrightarrow{G'})-1)^{-q-1}}{1-(q(\overrightarrow{G'})-1)^{-1}-(q(\overrightarrow{G'})-1)^{-p-1}}x_v + (q(\overrightarrow{G'})-1)^{-s}x_v. \end{aligned}$$

Thus we deduce that

$$(q(\overrightarrow{G'})-2)^2(q(\overrightarrow{G'})-1)^{n-2}x_v - (q(\overrightarrow{G'})-2)(q(\overrightarrow{G'})-1)^{q+s}x_v$$

$$= (q(\overrightarrow{G'})-1)^{p+q+1}x_v - (q(\overrightarrow{G'})-1)^{p+q}x_v - (q(\overrightarrow{G'})-1)^{q}x_v + (q(\overrightarrow{G'})-1)^{p+s}x_v.$$

By Perron-Frobenius Theorem, we have $x_v > 0$, therefore

$$(q(\overrightarrow{G'})-2)^2(q(\overrightarrow{G'})-1)^{n-2} - (q(\overrightarrow{G'})-2)(q(\overrightarrow{G'})-1)^{q+s}$$
$$= (q(\overrightarrow{G'})-1)^{p+q+1} - (q(\overrightarrow{G'})-1)^{p+q} - (q(\overrightarrow{G'})-1)^{q} + (q(\overrightarrow{G'})-1)^{p+s}.$$

Similarly, we have

$$(q(\overrightarrow{G''})-2)^2(q(\overrightarrow{G''})-1)^{n-2} - (q(\overrightarrow{G''})-2)(q(\overrightarrow{G''})-1)^{q+s+1}$$
$$= (q(\overrightarrow{G''})-1)^{p+q} - (q(\overrightarrow{G''})-1)^{p+q-1} - (q(\overrightarrow{G''})-1)^{q} + (q(\overrightarrow{G''})-1)^{p+s}.$$

Let $f(x) = (x-2)^2(x-1)^{n-2} - (x-2)(x-1)^{q+s} - (x-1)^{p+q+1} + (x-1)^{p+q} + (x-1)^q - (x-1)^{p+s}$ and $g(x) = (x-2)^2(x-1)^{n-2} - (x-2)(x-1)^{q+s+1} - (x-1)^{p+q} + (x-1)^{p+q-1} + (x-1)^q - (x-1)^{p+s}$. It's easy to see that $q(\overrightarrow{G'})$ is the largest real root of $f(x) = 0$ and $q(\overrightarrow{G''})$ is the largest real root of $g(x) = 0$. Since $f(x) - g(x) = (x-2)^2((x-1)^{q+s} - (x-1)^{p+q-1}) > 0$ for all $x > 1$. Since the minimum row sum of $Q(\widetilde{\infty}_2(p,q,s))$ is 2, and the row sums of $Q(\widetilde{\infty}_2(p,q,s))$ are not all equal, then by Lemma 2.1, we have $q(\widetilde{\infty}_2(p,q,s)) > 1$. Then we have $q(\widetilde{\infty}_2(p,q,s)) < q(\widetilde{\infty}_2(p-1,q,s+1))$.□

Lemma 2.5 — If $1 \le q \le p \le s$, then $q(\widetilde{\infty}_2(p,q,s)) < q(\widetilde{\infty}_2(p+1,q-1,s))$.

PROOF : Let $\overrightarrow{G'} = \widetilde{\infty}_2(p,q,s)$ and $\overrightarrow{G''} = \widetilde{\infty}_2(p+1,q-1,s)$ in the following. Suppose that $\overrightarrow{x} = (x_u, x_v, x_{1,1}, \ldots, x_{1,p}, x_{2,1}, \ldots, x_{2,q}, x_{3,1}, \ldots, x_{3,s})^\intercal$ is the Perron vector of $Q(\overrightarrow{G'})$ corresponding to $q(\overrightarrow{G'})$, where x_u corresponds to u, x_v corresponds to v and $x_{i,j}$ corresponds to v_{ij} $(i = 1, j = 1, 2, \ldots, p; i = 2, j = 1, 2, \ldots, q; i = 3, j = 1, 2, \ldots, s)$, respectively.

Similar to the proof of Lemma 2.4, we have

$$(q(\overrightarrow{G'})-2)^2(q(\overrightarrow{G'})-1)^{n-2} - (q(\overrightarrow{G'})-2)(q(\overrightarrow{G'})-1)^{q+s}$$
$$= (q(\overrightarrow{G'})-1)^{p+q+1} - (q(\overrightarrow{G'})-1)^{p+q} - (q(\overrightarrow{G'})-1)^{q} + (q(\overrightarrow{G'})-1)^{p+s}.$$

and

$$(q(\overrightarrow{G''})-2)^2(q(\overrightarrow{G''})-1)^{n-2} - (q(\overrightarrow{G''})-2)(q(\overrightarrow{G''})-1)^{q+s-1}$$

$$= (q(\overrightarrow{G''}) - 1)^{p+q+1} - (q(\overrightarrow{G''}) - 1)^{p+q} - (q(\overrightarrow{G''}) - 1)^{q-1} + (q(\overrightarrow{G''}) - 1)^{p+s+1}.$$

Let $f(x) = (x-2)^2(x-1)^{n-2} - (x-2)(x-1)^{q+s} - (x-1)^{p+q+1} + (x-1)^{p+q} + (x-1)^q - (x-1)^{p+s}$ and $g(x) = (x-2)^2(x-1)^{n-2} - (x-2)(x-1)^{q+s-1} - (x-1)^{p+q+1} + (x-1)^{p+q} + (x-1)^{q-1} - (x-1)^{p+s+1}$. It's easy to see that $q(\overrightarrow{G'})$ is the largest real root of $f(x) = 0$ and $q(\overrightarrow{G''})$ is the largest real root of $g(x) = 0$. Since $f(x) - g(x) = (x-2)((x-1)^{q+s-1} + (x-1)^{q-1} + (x-1)^{p+s} - (x-1)^{q+s}) > 0$ for all $x > 2$ when $1 \leq q \leq p \leq s$. Since the minimum row sum of $Q(\widetilde{\infty}_2(p,q,s))$ is 2, and the row sums of $Q(\widetilde{\infty}_2(p,q,s))$ are not all equal, then by Lemma 2.1, we have $q(\widetilde{\infty}_2(p,q,s)) > 2$. Then we have $q(\widetilde{\infty}_2(p,q,s)) < q(\widetilde{\infty}_2(p+1,q-1,s))$. □

Similar to the proof of Lemma 2.5, we can obtain the following result.

Lemma 2.6 — If $1 \leq q \leq p \leq s$, then $q(\widetilde{\infty}_2(p,q,s)) < q(\widetilde{\infty}_2(p,q-1,s+1))$.

Using the above Lemmas, we immediately obtain the following theorem.

***Theorem* 2.7** — *Among all $\widetilde{\infty}_2(p,q,s)$-digraphs of order n with $q \leq p \leq s$, the digraph $\widetilde{\infty}_2(1,0,n-3)$ is the unique digraph which achieves the maximum signless Laplacian spectral radius.*

Problem 2.8 — Among all $\widetilde{\infty}_2(p,q,s)$-digraphs of order $n(= p+q+s+2)$ with $1 \leq p \leq s$, which digraph achieves the maximum (or minimum) signless Laplacian spectral radius?

3. The signless Laplacian spectral radius of $\widetilde{\theta}_1$-digraphs and $\widetilde{\theta}_2$-digraphs

A θ-graph is a graph consisting of three paths which have the same end-vertices. In [4], the authors defined the generalized strongly connected θ-digraph as follow. For convenience, we use the abbreviation $\widetilde{\theta}$-digraph for the generalized strongly connected θ-digraph. The $\widetilde{\theta}$-digraph consists of $s+t$ ($s \geq 2, t \geq 1$ and $st = m$) directed paths $P_{k_1+2}, P_{k_2+2}, \ldots, P_{k_s+2}$ and $P_{l_1+2}, P_{l_2+2}, \ldots, P_{l_t+2}$ such that the initial vertex of $P_{k_1+2}, P_{k_2+2}, \ldots, P_{k_s+2}$ is the terminal vertex of $P_{l_1+2}, P_{l_2+2}, \ldots, P_{l_t+2}$, and the initial vertex of $P_{l_1+2}, P_{l_2+2}, \ldots, P_{l_t+2}$ is the terminal vertex of $P_{k_1+2}, P_{k_2+2}, \ldots, P_{k_s+2}$ (as shown in Figure 3), denoted by $\widetilde{\theta}(k_1, k_2, \ldots, k_s; l_1, l_2, \ldots, l_t)$ such that $\sum_{i=1}^{s} k_i +$

$\sum_{j=1}^{t} l_j + 2 = n$. Without loss of generality, let $k_i \leq k_{i+1}$ for $i = 1, 2, \ldots, s-1$, and $l_j \leq l_{j+1}$ for $j = 1, 2, \ldots, t-1$.

In particularly, we write $\widetilde{\theta}(k_1, k_2, \ldots, k_s; l_1, l_2, \ldots, l_t)$ as $\widetilde{\theta}_1(k_1, k_2, \ldots, k_m; l_1)$(as shown in Figure 4) when $s \geq 2, t = 1$, and $st = m$. In the following, we first prove that $\widetilde{\theta}_1(0, 1, \ldots, 1, n-m; 0)$ is the unique digraph which achieves the maximum signless Laplacian spectral radius and $\widetilde{\theta}_1(0, 1, \ldots, 1; n-m-1)$ is the unique digraph which achieves the minimum signless Laplacian spectral radius among all $\widetilde{\theta}_1(k_1, k_2, \ldots, k_m; l_1)$-digraphs for fixed n.

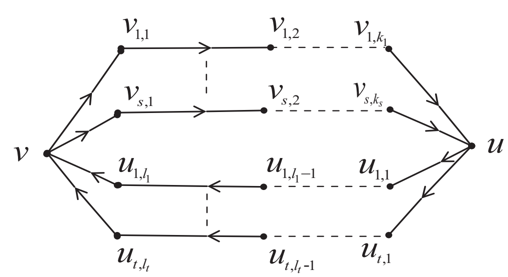

Figure 3: The digraph $\widetilde{\theta}(k_1, k_2, \ldots, k_s; l_1, l_2, \ldots, l_t)$.

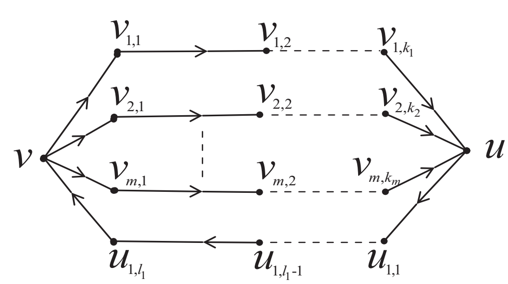

Figure 4: The digraph $\widetilde{\theta}_1(k_1, k_2, \ldots, k_m; l_1)$.

Lemma 3.1 — For any $p, q \in \{1, 2, \ldots, m\}$, if $1 \leq k_p \leq k_q$, then we have

$$q(\widetilde{\theta}_1(k_1, k_2, \ldots, k_{p-1}, k_p, k_{p+1}, \ldots, k_{q-1}, k_q, k_{q+1}, \ldots, k_m; l_1)) <$$

$$q(\widetilde{\theta}_1(k_1, k_2, \ldots, k_{p-1}, k_p - 1, k_{p+1}, \ldots, k_{q-1}, k_q + 1, k_{q+1}, \ldots, k_m; l_1)).$$

PROOF : Let $\overrightarrow{G_1} = \widetilde{\theta}_1(k_1, k_2, \ldots, k_{p-1}, k_p, k_{p+1}, \ldots, k_{q-1}, k_q, k_{q+1}, \ldots, k_m; l_1)$ and $\overrightarrow{G_2} = \widetilde{\theta}_1(k_1, k_2, \ldots, k_{p-1}, k_p - 1, k_{p+1}, \ldots, k_{q-1}, k_q + 1, k_{q+1}, \ldots, k_m; l_1)$ in the following. Suppose that $\overrightarrow{x} = (x_v, x_u, x_{1,1}, \ldots, x_{1,k_1}; x_{2,1}, \ldots, x_{2,k_2}; \ldots; x_{m,1}, \ldots, x_{m,k_m}; x_{l_1,1}, \ldots, x_{l_1,l_1})^{\mathsf{T}}$ is the Perron vector of $\overrightarrow{G_1}$ corresponding to $q(\overrightarrow{G_1})$, where x_v and x_u correspond to v and u, $x_{i,j}$ corresponds to $v_{i,j}$ ($i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, k_i$), $x_{l_1,j}$ corresponds to $u_{1,j}$ ($j = 1, 2, \ldots, l_1$), respectively. Since $Q(\overrightarrow{G_1})\overrightarrow{x} = q(\overrightarrow{G_1})\overrightarrow{x}$, we have

$$\begin{cases} q(\overrightarrow{G_1})x_{1,i_1} = x_{1,i_1} + x_{1,i_1+1}, & i_1 = 1, 2, \ldots, k_1 - 1, \\ q(\overrightarrow{G_1})x_{2,i_2} = x_{2,i_2} + x_{2,i_2+1}, & i_2 = 1, 2, \ldots, k_2 - 1, \\ \qquad \vdots & \\ q(\overrightarrow{G_1})x_{m,i_m} = x_{m,i_m} + x_{m,i_m+1}, & i_m = 1, 2, \ldots, k_m - 1, \\ & \\ q(\overrightarrow{G_1})x_{1,k_1} = x_{1,k_1} + x_u, & \\ q(\overrightarrow{G_1})x_{2,k_2} = x_{2,k_2} + x_u, & \\ \qquad \vdots & \\ q(\overrightarrow{G_1})x_{m,k_m} = x_{m,k_m} + x_u, & \\ & \\ q(\overrightarrow{G_1})x_v = mx_v + x_{1,1} + x_{2,1} + \ldots + x_{m,1}, & \\ q(\overrightarrow{G_1})x_u = x_u + x_{l_1,1} & \\ q(\overrightarrow{G_1})x_{l_1,j} = x_{l_1,j} + x_{l_1,j+1}, & j = 1, 2, \ldots, l_1 - 1, \\ q(\overrightarrow{G_1})x_{l_1,l_1} = x_{l_1,l_1} + x_v. & \end{cases}$$

Then we get

$$\begin{aligned} x_{j,k_j} &= (q(\overrightarrow{G_1}) - 1)^{k_j - 1} x_{j,1}, \qquad j = 1, 2, \ldots, m, \\ x_{l_1,l_1} &= (q(\overrightarrow{G_1}) - 1)^{l_1 - 1} x_{l_1,1}, \\ x_u &= (q(\overrightarrow{G_1}) - 1)^{k_j} x_{j,1}, \qquad j = 1, 2, \ldots, m. \end{aligned}$$

Furthermore

$$x_v = (q(\overrightarrow{G_1}) - 1)^{l_1} x_{l_1,1} = (q(\overrightarrow{G_1}) - 1)^{l_1+1} x_u = (q(\overrightarrow{G_1}) - 1)^{l_1+k_m+1} x_{m,1}.$$

Thus we deduce that

$$(q(\overrightarrow{G_1})-m)(q(\overrightarrow{G_1})-1)^{n-1}x_{m,1}=\sum_{i=1}^{m}(q(\overrightarrow{G_1})-1)^{n-2-l_1-k_i}x_{m,1}.$$

By Perron-Frobenius Theorem, we have $x_{m,1}>0$, therefore

$$(q(\overrightarrow{G_1})-m)(q(\overrightarrow{G_1})-1)^{n-1}=\sum_{i=1}^{m}(q(\overrightarrow{G_1})-1)^{n-2-l_1-k_i}.$$

Similarly, we have

$$(q(\overrightarrow{G_2})-m)(q(\overrightarrow{G_2})-1)^{n-1}=\sum_{i=1,i\neq p \text{ and } q}^{m}(q(\overrightarrow{G_2})-1)^{n-2-l_1-k_i}$$
$$+(q(\overrightarrow{G_2})-1)^{n-1-l_1-k_p}+(q(\overrightarrow{G_2})-1)^{n-3-l_1-k_q}.$$

Let $f(x)=(x-m)(x-1)^{n-1}-\sum_{i=1}^{m}(x-1)^{n-2-l_1-k_i}$ and $g(x)=(x-m)(x-1)^{n-1}-\sum_{i=1,i\neq p \text{ and } q}^{m}(x-1)^{n-2-l_1-k_i}-(x-1)^{n-1-l_1-k_p}-(x-1)^{n-3-l_1-k_q}$. It is easy to see that $q(\overrightarrow{G_1})$ and $q(\overrightarrow{G_2})$ are the largest roots of $f(x)=0$ and $g(x)=0$, respectively. $f(x)-g(x)=((x-1)^{n-2-l_1-k_p}-(x-1)^{n-3-l_1-k_q})(x-2)>0$, for all $x>2$. Since the minimum row sum of $Q(\overrightarrow{G_1})$ is 2, and the row sums of $Q(\overrightarrow{G_1})$ are not all equal, then we have $q(\overrightarrow{G_1})>2$. Then we have $q(\overrightarrow{G_2})>q(\overrightarrow{G_1})$. □

Similar to the proof of Lemma 3.1, we can obtain the following lemma.

Lemma 3.2 — For any $p\in\{1,2,\dots,m\}$, we have

$$q(\widetilde{\theta}_1(k_1,k_2,\dots,k_{p-1},k_p,k_{p+1},\dots,k_m;l_1))<$$
$$q(\widetilde{\theta}_1(k_1,k_2,\dots,k_{p-1},k_p+1,k_{p+1},\dots,k_m;l_1-1)).$$

Combining Lemmas 3.1 and 3.2, we have the following theorem.

Theorem 3.3 — *Among all $\widetilde{\theta}_1(k_1,k_2,\dots,k_m;l_1)$-digraphs of order n, $\widetilde{\theta}_1(0,1,\dots,1,n-m;0)$ is the unique digraph which achieves the maximum signless Laplacian spectral radius and $\widetilde{\theta}_1(0,1,\dots,1;n-m-1)$ is the unique digraph which achieves the minimum signless Laplacian spectral radius.*

Now, when $s\geq 2, t\geq 1$, and $st=m$, we write $\widetilde{\theta}(k_1,k_2,\dots,k_s;l_1,l_2,\dots,l_t)$ as $\widetilde{\theta}_2(k_1,k_2,\dots,k_m;l_1,l_2,\dots,l_t)$. In the following, we shall prove that $\widetilde{\theta}_2(0,1,\dots,1;0,1,\dots,1,n-$

$s-t+1$) is the unique digraph which achieves the maximum signless Laplacian spectral radius among all $\widetilde{\theta}_2(k_1, k_2, \ldots, k_m; l_1, l_2, \ldots, l_t)$ -digraphs for fixed n.

Similar to the proof of Lemma 3.1, we obtain the following results.

Lemma 3.4 — For any $p, q \in \{1, 2, \ldots, s\}$, if $1 \leq k_p \leq k_q$, then we have

$$q(\widetilde{\theta}_2(k_1, k_2, \ldots, k_{p-1}, k_p, k_{p+1}, \ldots, k_{q-1}, k_q, k_{q+1}, \ldots, k_s; l_1, l_2, \ldots, l_t)) <$$

$$q(\widetilde{\theta}_2(k_1, k_2, \ldots, k_{p-1}, k_p - 1, k_{p+1}, \ldots, k_{q-1}, k_q + 1, k_{q+1}, \ldots, k_s; l_1, l_2, \ldots, l_t)).$$

Lemma 3.5 — For any $p, q \in \{1, 2, \ldots, t\}$, if $1 \leq l_p \leq l_q$, then we have

$$q(\widetilde{\theta}_2(k_1, k_2, \ldots, k_s; l_1, l_2, \ldots, l_{p-1}, l_p, l_{p+1}, \ldots, l_{q-1}, l_q, l_{q+1}, \ldots, l_t)) <$$

$$q(\widetilde{\theta}_2(k_1, k_2, \ldots, k_s; l_1, l_2, \ldots, l_{p-1}, l_p - 1, l_{p+1}, \ldots, l_{q-1}, l_q + 1, l_{q+1}, \ldots, l_t)).$$

Lemma 3.6 — If $s \leq t, k_i = l_i$ $(i = 1, 2, \ldots, s-1)$ and $k_s \leq l_t$, then

$$q(\widetilde{\theta}_2(k_1, k_2, \ldots, k_s - 1; l_1, l_2, \ldots, l_t + 1)) > q(\widetilde{\theta}_2(k_1, k_2, \ldots, k_s; l_1, l_2, \ldots, l_t)).$$

Combining Lemmas 3.4, 3.5 and 3.6, we have the following theorem.

***Theorem* 3.7** — *If $s \leq t$ and $\sum_{i=1}^{s} k_i \leq \sum_{j=1}^{t} l_j$, then among all $\widetilde{\theta}_2(k_1, k_2, \ldots, k_m; l_1, l_2, \ldots, l_t)$-digraphs, the digraph $\widetilde{\theta}_2(0, 1, \ldots, 1; 0, 1, \ldots, 1, n-s-t+1)$ is the unique digraph which achieves the maximum signless Laplacian spectral radius.*

4. The Signless Laplacian Spectral Radius of $\widetilde{\infty}_1$-digraphs and $\widetilde{\theta}_1$-digraphs

In the following, we will discuss the digraph which achieves the maximum and minimum signless Laplacian spectral radius among all $\widetilde{\infty}_1$-digraphs and $\widetilde{\theta}_1$-digraphs.

The following well-known theorem can be found in [5].

***Theorem* 4.1** — *[5]. Let $\overrightarrow{G} = (V(\overrightarrow{G}), E(\overrightarrow{G}))$ be a simple digraph on n vertices, u, v, w distinct vertices of $V(\overrightarrow{G})$, $(u, v) \in E(\overrightarrow{G})$ and $\overrightarrow{x} = (x_1, x_2, \ldots, x_n)^{\intercal}$ be the unique positive unit eigenvector corresponding to the signless Laplacian spectral radius of $q(\overrightarrow{G})$, where x_i corresponds to the vertex i. Let $\overrightarrow{H} = \overrightarrow{G} - \{(u, v)\} + \{(u, w)\}$ (Noting* *that if* $(u, w) \in E(\overrightarrow{G})$, *then* $\overrightarrow{H}$ *has multiple arc* (u, w)). *If* $x_w \geq x_v$, *then* $q(\overrightarrow{H}) \geq q(\overrightarrow{G})$. *Furthermore, if* $\overrightarrow{H}$ *is strongly connected and* $x_w \geq x_v$, *then* $q(\overrightarrow{H}) > q(\overrightarrow{G})$.

Lemma 4.2 — For any $\widetilde{\theta}_1(k_1, k_2, \ldots, k_m; l_1)$- digraph, there exists a $\widetilde{\infty}_1(k_1, k_2, \ldots, k_{m-1}, k_m + l_1 + 1)$- digraph such that $q(\widetilde{\theta}_1(k_1, k_2, \ldots, k_m; l_1)) < q(\widetilde{\infty}_1(k_1, k_2, \ldots, k_{m-1}, k_m + l_1 + 1))$.

Proof : By lemma 3.1, we know that $x_v = (q(\widetilde{\theta}_1(k_1, k_2, \ldots, k_m; l_1)) - 1)^{l_1+1} x_u > x_u$. It is easy to see that $\widetilde{\infty}_1(k_1, k_2, \ldots, k_{m-1}, k_m + l_1 + 1) \cong \widetilde{\theta}_1(k_1, k_2, \ldots, k_m; l_1) - \{(v_{1k_1}, u), (v_{2k_2}, u), \ldots, (v_{m-1k_{m-1}}, u)\} + \{(v_{1k_1}, v), (v_{2k_2}, v), \ldots, (v_{m-1k_{m-1}}, v)\}$. Thus we have $q(\widetilde{\theta}_1(k_1, k_2, \ldots, k_m; l_1)) < q(\widetilde{\infty}_1(k_1, k_2, \ldots, k_{m-1}, k_m + l_1 + 1))$. □

Lemma 4.3 — For any $\widetilde{\infty}_1(k_1, k_2, \ldots, k_{m-1}, k_m)$- digraph, there exists a $\widetilde{\theta}_1(k_1, k_2, \ldots, k_{m-1}, k_m - 1; 0)$- digraph such that $q(\widetilde{\theta}_1(k_1, k_2, \ldots, k_{m-1}, k_m - 1; 0)) < q(\widetilde{\infty}_1(k_1, k_2, \ldots, k_{m-1}, k_m))$. □

Proof : By lemma 3.1, we know that $x_v = (q(\widetilde{\theta}_1(k_1, k_2, \ldots, k_m; l_1)) - 1)^{l_1+1} x_u > x_u$. It is easy to see that $\widetilde{\infty}_1(k_1, k_2, \ldots, k_{m-1}, k_m) \cong \widetilde{\theta}_1(k_1, k_2, \ldots, k_{m-1}, k_m - 1; 0) - \{(v_{1k_1}, u), (v_{2k_2}, u), \ldots, (v_{m-1k_{m-1}}, u)\} + \{(v_{1k_1}, v), (v_{2k_2}, v), \ldots, (v_{m-1k_{m-1}}, v)\}$. Thus we have $q(\widetilde{\theta}_1(k_1, k_2, \ldots, k_{m-1}, k_m - 1; 0)) < q(\widetilde{\infty}_1(k_1, k_2, \ldots, k_{m-1}, k_m))$. □

From Lemmas 4.2 and 4.3, we know that the digraph which achieves the maximum signless Laplacian spectral radius among all $\widetilde{\infty}_1$-digraphs and $\widetilde{\theta}_1$-digraphs must be in $\widetilde{\infty}$-digraphs, and the digraph which achieves the minimum signless Laplacian spectral radius among all $\widetilde{\infty}_1$-digraphs and $\widetilde{\theta}_1$-digraphs must be in $\widetilde{\theta}_1$-digraphs.

Combining Theorems 2.3 and 3.3, Lemmas 4.2 and 4.3, we can immediately get the following theorem.

***Theorem* 4.4** — *Among all* $\widetilde{\infty}_1$*-digraphs and* $\widetilde{\theta}_1$*-digraphs of order* n*, the digraph* $\widetilde{\infty}_1(1, 1, \ldots, 1, n - m)$ *is the digraph which achieves the maximum signless Laplacian spectral radius and the digraph* $\widetilde{\theta}_1(0, 1, \ldots, 1; n - m - 1)$ *the unique digraph which achieves the minimum signless Laplacian spectral radius.*

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