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A NEW INTEGRATOR FOR SPECIAL THIRD ORDER DIFFERENTIAL EQUATIONS WITH APPLICATION TO THIN FILM FLOW PROBLEM

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In recent time, Runge-Kutta methods that integrate special third order ordinary differential equations (ODEs) directly are proposed to address efficiency issues associated with classical Runge-Kutta methods. Albeit, the methods require evaluation of three set of equations to proceed with the numerical integration. In this paper, we propose a class of multistep-like Runge-Kutta methods (hybrid methods), which integrates special third order ODEs directly. The method is completely derivative-free. Algebraic order conditions of the method are derived. Using the order conditions, a four-stage method is presented. Numerical experiment is conducted on some test problems. The method is also applied to a practical problem in Physics and engineering to ascertain its validity. Results from the experiment show that the new method is more accurate and efficient than the classical Runge-Kutta methods and a class of direct Runge-Kutta methods recently designed for special third order ODEs.

Key words : Hybrid method; three-step method; Taylor series; order conditions; third order ordinary differential equations; numerical integrator.

1. INTRODUCTION

Third order ordinary differential equation (ODE) is used in modeling problems arising in a number of areas of applied science such as biology, quantum mechanics, celestial mechanics and chemical engineering [1]. For instance, the models that describe draining coating flows, thin film flow, acoustic wave propagation in relaxing media e.t.c. are all third order differential equations [1-7].

Most of these equations defy analytical solutions. Hence, the search for approximate solutions by numerical means is eminent. A lot of work has been done on the solutions of third order ODEs, especially in the area of linear multistep related methods. For instance, the P-stable linear multistep method by Awoyemi [8] and the hybrid collocation method by Awoyemi and Idowu [9]. More of these can be found in the works of Majid *et al*. [10], Olabode and Yusuph [11], Mahrkanoon [12], Guo and Wang [3], Guo *et al*. [4], Majid *et al*. [13], Ken *et al*. [14] and the references therein. Traditionally, third order ODEs can be solved by first transforming them into systems of first order equations and applying Runge-Kutta methods or linear multistep methods, but this could be computationally costlier than the direct methods.

In this paper, our main concern is with the initial value problems (IVPs) of special third order ODEs of the form

$$y'''(x) = f(x, y(x)), \;\; y(x_0) = y_0, \;\; y'(x_0) = y'_0, \;\; y''(x_0) = y''_0, \tag{1}$$

where $y \in R^d, \; f : R \times R^d \to R^d$ is a vector value function. The speciality associated with (1) is the fact that f does not depend on y', y'' explicitly. A thin film flow problem, an important engineering and Physics model that describes the draining of fluid on a solid surface, is a typical example of (1) [15].

Inspired by Runge-Kutta Nyström methods (RKN), Xiong and Zhaoxia [1] proposed a Runge-Kutta method for solving (1) directly. A class of two-step hybrid methods for solving special second order ODEs proposed by Coleman [16] out performed the RKN methods in terms of computational accuracy and efficiency. This is because the methods evaluate only one equation at each step of the integration. Hence, we propose a class of three-step hybrid methods (THMD) for solving (1) directly, which (unlike RKT methods in [1] and RKD method in [15]) evaluates only one equation for the numerical integration.

The remaining part of the paper is organized as follows: in Section 2, we present how the general form of the proposed method is formulated. Algebraic order conditions of the methods are derived in Section 3. As an example, an explicit 4-stage THMD is presented in Section 4. Convergence analysis is presented in Section 5. Numerical experiment is presented in Section 6. In Section 7, application of the new method on a thin film flow problem is discussed. And finally, conclusion is presented in Section 8.

2. FORMULATION OF THMD METHODS

As mentioned above, suppose we want to solve (1) using classical Runge-Kutta method, it must be transformed into a system of first order equations as follows

$$\begin{pmatrix} y(x) \\ u(x) \\ v(x) \end{pmatrix}' = \begin{pmatrix} u(x) \\ v(x) \\ f(x, y(x)) \end{pmatrix}, \tag{2}$$

subject to initial conditions $y(x_0) = y_0,\ u(x_0) = y_0',\ v(x_0) = y_0''$. The Runge-Kutta method is given by

$$\begin{aligned} Y_i &= y_n + h\sum_{j=1}^{s} a_{i,j} f(x_n + c_j h, Y_j), i = 1, ..., s, \\ y_{n+1} &= y_n + h\sum_{j=1}^{s} b_i f(x_n + c_i h, Y_i). \end{aligned} \tag{3}$$

The following set of equations are obtained when (3) is applied to (2)

$$\begin{aligned} Y_i &= y_n + h\sum_{j=1}^{s} a_{i,j}\hat{Y}_j, \\ \hat{Y}_i &= y_n' + h\sum_{j=1}^{s} a_{i,j}\breve{Y}_j, \\ \breve{Y}_i &= y_n'' + h\sum_{j=1}^{s} a_{i,j} f(x_n + c_j h, Y_j), \\ y_{n+1} &= y_n + h\sum_{i=1}^{s} b_i \hat{Y}_i, \\ y_{n+1}' &= y_n' + h\sum_{i=1}^{s} b_i \breve{Y}_i, \\ y_{n+1}'' &= y_n'' + h\sum_{i=1}^{s} b_i f(x_n + c_i h, Y_i). \end{aligned}$$

Eliminating $\hat{Y}_i$ and $\breve{Y}_i$ from the equations above, see [1], we get

$$\begin{aligned} Y_i &= y_n + h\sum_{j=1}^{s} a_{i,j} y_n' + h^2 \sum_{j,k=1}^{s} a_{i,j} a_{j,k} y'' + h^3 \sum_{j=1}^{s} a_{i,j} a_{j,k} a_{k,l} f(x_n + c_l h, Y_l), \\ &\quad i = 1, ..., s, \\ y_{n+1} &= y_n + h\sum_{i=1}^{s} b_i y_n' + h^2 \sum_{i=1}^{s} b_i a_{i,j} y_n'' + h^3 \sum_{i=1}^{s} b_i a_{i,j} a_{j,k} f(x_n + c_k h, Y_k), \end{aligned}$$

$$
\begin{aligned}
y'_{n+1} &= y'_n + h\sum_{i=1}^{s} b_i y''_n + h^2 \sum_{i=1}^{s} b_i a_{i,j} f(x_n + c_j h, Y_j),\\
y''_{n+1} &= y''_n + h\sum_{i=1}^{s} b_i f(x_n + c_i h, Y_i).
\end{aligned}
$$

We assume that

$$
\sum_{j=1}^{s} a_{i,j} = \hat{c}_i,\ \sum_{j,k=1}^{s} a_{i,j}a_{j,k} = \frac{\hat{c}_i^2 + \hat{c}_i}{2},\ \sum_{i=1}^{s} b_i = \sum_{i,j=1}^{s} b_i a_{i,j} = 1,\ i = 1, ..., s.
$$

And

$$
\sum_{j,k,l=1}^{s} a_{i,j}a_{j,k}a_{k,l} = \sum_{j=1} \hat{a}_{i,j},\ \sum_{i,j,k=1}^{s} b_i a_{i,j} a_{j,k} = \sum_{i=1}^{s} \hat{b}_i.
$$

The analysis above coupled with application of difference formula leads to the proposed method

$$
\begin{aligned}
\mathbf{Y} &= \frac{1}{2}\left\{\mathbf{C}_1 \otimes y_n - 2\mathbf{C}_2 \otimes y_{n-1} + \mathbf{C}_3 \otimes y_{n-2}\right\} + h^3 \left(\mathbf{A} \otimes \mathbf{I}\right) f(x_n + \mathbf{c}h, \mathbf{Y}),\\
y_{n+1} &= 3\left(y_n - y_{n-1}\right) + y_{n-2} + h^3 \left(\hat{\mathbf{b}}^T \otimes \mathbf{I}\right) f(x_n + \mathbf{c}h, \mathbf{Y}),
\end{aligned} \tag{4}
$$

where $\mathbf{C}_1 = \mathbf{c}^2 + 3\mathbf{c} + 2\mathbf{e}$, $\mathbf{C}_2 = \mathbf{c}^2 + 2\mathbf{c}$, $\mathbf{C}_3 = \mathbf{c}^2 + \mathbf{c}$, $\hat{\mathbf{b}} = [\hat{b}_1, ..., \hat{b}_m]^T$, $\mathbf{c} = [\hat{c}_1, ..., \hat{c}_m]^T$, $\mathbf{e} = [1, ..., 1]^T$, $\mathbf{A} = [\hat{a}_{i,j}]^T$, $\mathbf{Y} = [Y_1, ..., Y_m]^T$ and $\mathbf{I}$ is identity matrix of $m \times m$ dimension. The general coefficients of the methods are summarized in Table 1.

3. Algebraic Order Conditions of THMD Methods

In this section, we derive order conditions of the proposed class of THMD methods. According to Coleman [16], "order conditions are certain relationships between coefficients of a method that causes successive terms in a Taylor series expansion of local truncation error to vanish".

To derive the order conditions of THMD methods, we shall consider autonomous case of (1) and re-write eqn. (4) as follows:

$$
\begin{aligned}
y_{n+1} &= y_n + \phi(h; y_n),\\
k_i &= f(Y_i),
\end{aligned} \tag{5}
$$

where

$$
\phi(h; y_n) = 2y_n - 3y_{n-1} + y_{n-2} + h^3 \sum_{j=1}^{s} \hat{b}_i k_i,
$$

Table 1: General Coefficients of THMD

-2	$\hat{a}_{1,1}$	$\hat{a}_{1,2}$	$\hat{a}_{1,3}$	$\cdots$	$\hat{a}_{1,m}$
-1	$\hat{a}_{2,1}$	$\hat{a}_{2,2}$	$\hat{a}_{2,3}$	$\cdots$	$\hat{a}_{2,m}$
0	$\hat{a}_{3,1}$	$\hat{a}_{3,2}$	$\hat{a}_{3,3}$	$\cdots$	$\hat{a}_{3,m}$
c_4	$\hat{a}_{4,1}$	$\hat{a}_{4,2}$	$\hat{a}_{4,3}$	$\cdots$	$\hat{a}_{4,m}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
c_m	$\hat{a}_{m,1}$	$\hat{a}_{m,2}$	$\hat{a}_{m,3}$	$\cdots$	$\hat{a}_{m,m}$
	$\hat{b}_1$	$\hat{b}_2$	$\hat{b}_3$	$\cdots$	$\hat{b}_m$

and k_i is defined in terms of the components of **Y** in eqn. (4). Suppose that the exact solution $y(x)$ at point x_{n+1} is defined as

$$y(x_{n+1}) = y(x_n) + \Psi(h; y(x_n)), \tag{6}$$

then the local truncation error d_{n+1} of THMD methods can be expressed as

$$d_{n+1} = y(x_{n+1}) - y_{n+1} = \Psi - \phi, \tag{7}$$

provided that the local assumption $y(x_n) = y_n$ holds. The next task is to obtain Taylor expansion for both Ψ and ϕ. The Taylor expansion of the two quantities are given below in terms of elementary differential (F).

$$\begin{aligned}
\Psi &= hy' + \frac{h^2}{2}y'' + \frac{h^3}{6}F_1^{(3)} + \frac{h^4}{24}F_1^{(4)} + \frac{h^5}{120}F_1^{(5)} + O\left(h^6\right), \\
\phi &= hy' + \frac{h^2}{2}y'' + \frac{h^3}{6}\left(6\sum_{i=1}^{s}\hat{b}_i - 5\right)F_1^{(3)} + \\
&\quad \frac{h^4}{24}\left(24\sum_{i=1}^{s}\hat{b}_i c_i + 13\right)F_1^{(4)} + \frac{h^5}{120}\left(60\sum_{i=1}^{s}\hat{b}_i c_i^2 - 29\right)F_1^{(5)} + \\
&\quad O\left(h^6\right),
\end{aligned} \tag{8}$$

where

$$\begin{aligned}
F_1^{(3)} &= f, \\
F_1^{(4)} &= f_y\left(y'\right), \\
F_1^{(5)} &= f_{yy}\left(y', y'\right) + f_y\left(y''\right), \\
F_1^{(6)} &= f_{yyy}\left(y', y', y'\right) + 3f_y\left(y', y''\right) + f_y f, \text{e.t.c.}
\end{aligned}$$

Substituting eqns. (8) into (7) gives Taylor series expansion of the local truncation error of THMD.

$$\begin{aligned}
t_{n+1} &= \left[\frac{h^3}{6}\left(6\sum_{i=1}^{s}\hat{b}_i - 6\right)F_1^{(3)} + \frac{h^4}{24}\left(24\sum_{i=1}^{s}\hat{b}_i c_i + 12\right)F_1^{(4)}\right. \\
&\quad \left. + \frac{h^5}{120}\left(60\sum_{i=1}^{s}\hat{b}_i c_i^2 - 30\right)F_1^{(5)} + O(h^6)\right].
\end{aligned} \tag{9}$$

Hence, the algebraic order conditions of THMD up to order nine are summarized in Table 2. Where all the indices run from 1 to s.

4. Construction of Explicit 4-stage THMD Method

Presented in this section is an explicit 4-stage method of the class of THMD methods proposed. The method shall be denoted by THMD4s.

To construct a 4-stage method, equations of order conditions presented in Table 2 up to order nine are considered. This results to a system of eighteen nonlinear equations to be solved in twenty one unknown parameters, which implies there are three free parameters, $c_4,\ c_5,\ c_6$. With the help of MAPLE 13 software, the system of equations are solved. Having exhausted the whole equations of order conditions derived to obtain the THMD4s method, there is not specific strategy used in choosing values of the free parameters. The parameters c_4 and c_6 are fixed at 1 and $\frac{1}{4}$ respectively, and the value of c_6 is randomly obtained as $c_6 = \frac{127}{126}$. Table 3 shows the coefficients obtained for the proposed 4-stage THMD method.

5. Convergence of THMD Method

The update stage of the THMD method (4) can be written as follows:

$$\sum_{i=0}^{3}\gamma_i y_{n-i} - h^3\sum_{i=0}^{s} b_i f(x_i + c_i h, Y_i) = 0. \tag{10}$$

5.1 *Zero stability*

Table 2: General Order Conditions of THMD

Order	condition
3	$\sum \hat{b}_i = 1$
4	$\sum \hat{b}_i c_i = -\frac{1}{2}$
5	$\sum \hat{b}_i c_i^2 = \frac{1}{2}$
6	$\sum \hat{b}_i c_i^3 = -\frac{1}{2}$
	$\sum \hat{b}_i a_{i,j} = 0$
7	$\sum \hat{b}_i c_i^4 = \frac{3}{5}$
	$\sum \hat{b}_i c_i a_{i,j} = \frac{1}{60}$
	$\sum \hat{b}_i a_{i,j} c_j = \frac{1}{240}$
8	$\sum \hat{b}_i c_i^5 = -\frac{3}{4}$
	$\sum \hat{b}_i c_i^2 a_{i,j} = \frac{1}{120}$
	$\sum \hat{b}_i c_i a_{i,j} c_j = -\frac{1}{96}$
	$\sum \hat{b}_i a_{i,j} c_j^2 = -\frac{1}{240}$
9	$\sum \hat{b}_i c_i^6 = \frac{85}{84}$
	$\sum \hat{b}_i c_i^3 a_{i,j} = -\frac{2}{315}$
	$\sum \hat{b}_i a_{i,j} a_{i,k} = 0$
	$\sum \hat{b}_i c_i^2 a_{i,j} c_j = -\frac{79}{10080}$
	$\sum \hat{b}_i a_{i,j} c_j^3 = \frac{43}{10080}$
	$\sum \hat{b}_i a_{i,j} a_{j,k} = -\frac{1}{60480}$

Definition 1 — The THMD method is said to be zero stable if the roots ξ_j, $j = 1, 2, 3$, of the first characteristics polynomial $\chi(\xi)$, which is given by

$$\chi(\xi) = \sum_{i=0}^{3} \gamma_i \xi^{3-i} = 0, \tag{11}$$

satisfy $|\xi_j| \leq 1$, $j = 1, 2, 3$ and for the roots with $|\xi_j| = 1$, the multiplicity does not exceed 1 (see [17, 18]).

5.2 *Consistency*

The TMHD method is said to be consistent if it has order $p > 1$.

Remark : We note that the first characteristics polynomial associated with (10) is

$$\chi(\xi) = \xi^3 - 3\xi^2 + 3\xi - 1 = 0,$$

which implies that $\xi = 1$ thrice. Therefore, the THMD method is zero stable. We also note from Table 2 that

Table 3: Coefficients of THMD4s

-2	0	0	0	0	0	0
-1	0	0	0	0	0	0
0	0	0	0	0	0	0
1	0	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0
$\frac{127}{126}$	$\frac{38664035575}{576216596358144}$	$\frac{97130791604297}{192072198786048}$	$\frac{97631609597071}{192072198786048}$	$\frac{311175464897}{576216596358144}$	0	0
$\frac{1}{4}$	$-\frac{242650961}{739046555648}$	$\frac{19306494513}{246673850368}$	$\frac{10446052047}{247648845824}$	$\frac{1}{59509}$	$-\frac{8753988690531}{2968288109940736}$	0
	$\frac{947}{102330}$	$\frac{563}{1265}$	$\frac{539}{635}$	$\frac{191}{90}$	$-\frac{23818477032}{11629654795}$	$-\frac{9728}{25785}$

the minimum order p of the method is 3, which implies that it's consistent. Hence, the method is convergent, as zero stability + consistency = convergence.

6. Numerical Experiment

We present in this section numerical experiment conducted by applying the new THMD method derived in this paper alongside some other existing methods in the literature on some model problems in order to test how effective the new THMD method is. Below is definition of acronyms of the new method and those of the other methods chosen for comparison in this paper:

- **THMD4s**: the 4-stage explicit THMD method derived in section 4 of this paper;
- **RKT3s**: 3-stage explicit RKT method of order five presented in [1];
- **RKD3s**: 3-stage fifth order explicit Runge-Kutta direct method presented in [15];
- **RK4s**: 4-stage explicit Runge Kutta method of order four given in [19];
- **RK6s**: 6-stage explicit Runge Kutta method of order five given in [19].

6.1 *Test problems*

Problem 1.

$$y''' = -y,\ y(0) = 1,\ y'(0) = -1,\ y''(0) = 1,\ y(x) = \exp(-x).$$

Problem 2.

$$y''' = y + \cos(x),\ y(0) = 0,\ y'(0) = 0,\ y''(0) = 1,$$
$$y(x) = \frac{1}{2}\exp(x) - \frac{1}{2}\left(\cos(x) + \sin(x)\right).$$

Problem 3.

$$y''' - y = x,\ y(0) = 0,\ y'(0) = 0,\ y''(0) = 0,$$
$$y(x) = x + \frac{1}{3}\exp(-x) - \frac{1}{3}\exp\left(\frac{1}{2}x\right)\cos\left(\frac{1}{2}\sqrt{3x}\right) - \frac{1}{3}\sqrt{3}\exp\left(\frac{1}{2}x\right)\sin\left(\frac{1}{2}\sqrt{3x}\right).$$

Problem 4.

$$y_1''' = \frac{1}{68}\left(817y_1 + 1393y_2 + 448y_3\right),$$
$$y_2''' = -\frac{1}{68}\left(1441y_1 + 2837y_2 + 896y_3\right),$$
$$y_3''' = \frac{1}{136}\left(3059y_1 + 4319y_2 + 1592y_3\right),$$

subject to initial conditions

$$y_1(0) = 2,\ y_2(0) = -2,\ y_3(0) = -12,$$
$$y_1(0) = -12,\ y_2(0) = 28,\ y_3(0) = -33,$$
$$y_1(0) = 20,\ y_2(0) = -52,\ y_3(0) = 5.$$

The exact solutions are

$$y_1 = \exp(x) - 2\exp(2x) + 3\exp(-3x),$$
$$y_2 = 3\exp(x) + 2\exp(2x) - 7\exp(-3x),$$
$$y_3 = -11\exp(x) - 5\exp(2x) + 4\exp(-3x).$$

The test problems 1-4 are solved in the interval $[0, 5]$, with step size $h = \frac{1}{2^i}, i = 1, ..., 5$ for the 3-stage methods except problem 4 whose interval of integration is $[0, 2]$ with $h = \frac{1}{2^i}, i = 4, ..., 8$. Natural $\log_{10}$ of maximum errors of each of the methods is plotted against $\log_{10}$ of te corresponding number of function evaluations as a means of evaluating the accuracy and efficiency of the methods. Figs. 1-4 show the outcome of comparison of the methods. It could be observed from the Figs. that function evaluation of all the methods appear to be the same despite the differences in the stages of the methods, the strategy used is that when we choose any step size h, $\frac{4h}{3}$ is used for four stage methods and $2h$ is used for six stage methods.

7. Application of THMD method to a Thin Film Flow Problem

This section is devoted to the application of the proposed THMD method on a known problem in engineering and Physics, the problem of thin film flow of fluid on a surface. This problem received a lot of attention from

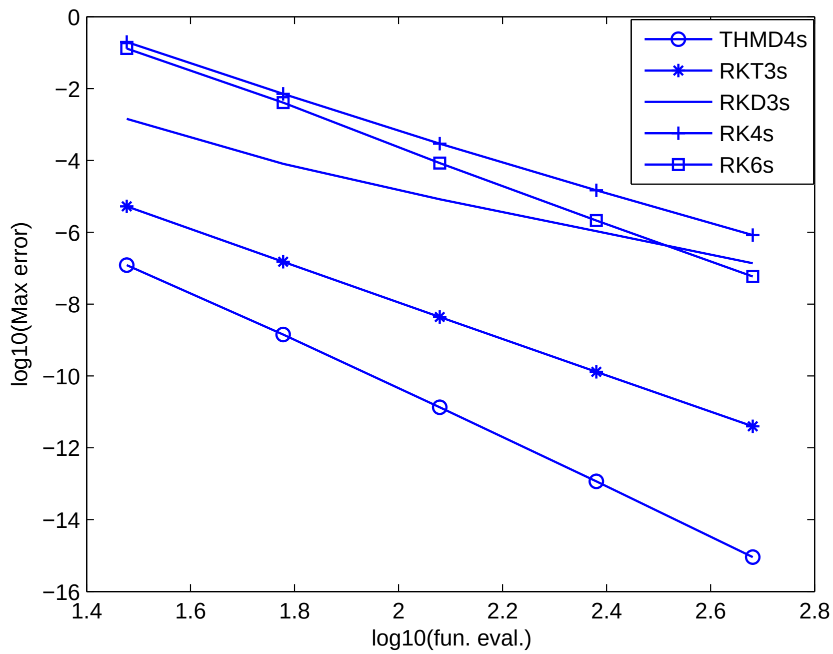

Figure 1: Efficiency curves for problem 1

authors in the last few decades. A detail discussion of this problem for a viscous fluid over a solid surface is given in [20] and the references therein, where the general form of the problem is given by

$$y''' = f(y(x)). \tag{12}$$

And $y(x)$ is the film profile in a coordinate frame moving with fluid. The form which $f(y(x))$ takes depends on the physical context under consideration. For instance,

$$f(y(x)) = \frac{1}{y^2} - 1,$$

is for a fluid draining problem on a dry wall;

$$f(y(x)) = \frac{1+\epsilon+\epsilon^2}{y^2} - \frac{\epsilon+\epsilon^2}{y^3} - 1,$$

is for a draining on a wet wall, and so on.

The third order ODE governing the free surface of a viscous fluid whose thin film flow is affected majorly by surface tension of the fluid (see [15, 20]) is given by

$$y''' = y^{-\theta}, \quad x \geq x_0, \tag{13}$$

subject to

$$y(x_0) = \zeta_1, \quad y'(x_0) = \zeta_2, \quad y''(x_0) = \zeta_3,$$

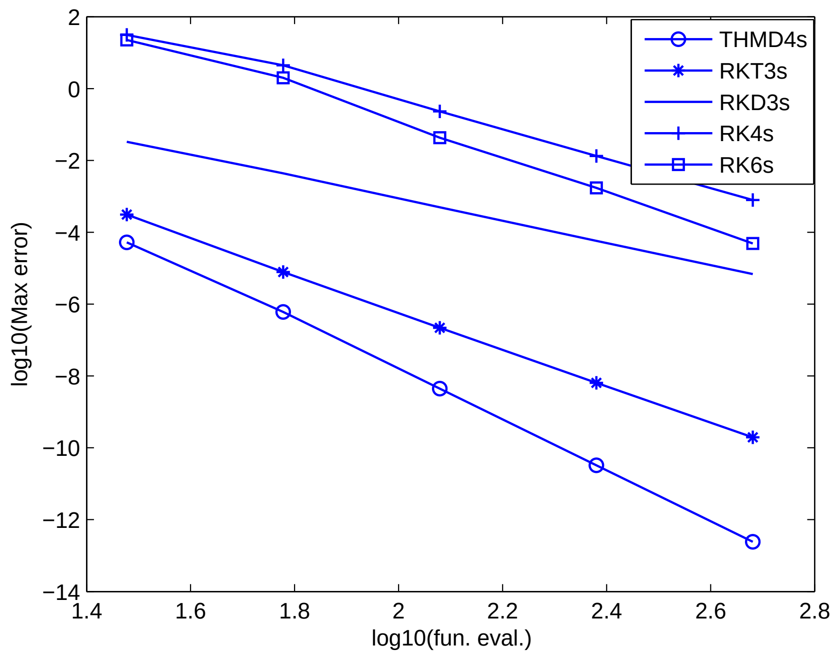

Figure 2: Efficiency curves for problem 2

where ζ_1, ζ_2 and ζ_3 are constants, and are of particular importance due to the fact that they describe the dynamic balance between surface and viscous forces in a thin fluid layer when gravity is absent or neglected [15].

Eqn. (13) defies all analytical techniques, especially for general case of θ. Hence, numerical techniques become imperative. Two cases of θ are considered here, $\theta = 2$ and $\theta = 3$.

Table 4: Numerical solutions for case $\theta = 2$ and $h = 0.01$

x	THMD4s	RK4s	RKD3s
0.0	1.0000000000	1.0000000000	1.0000000000
0.2	1.2212084858	1.2212100046	1.2212100045
0.4	1.4888467642	1.4888347800	1.4888347799
0.6	1.8073467642	1.8073613978	1.8073613977
0.8	2.1797930619	2.1798192341	2.1798192339
1.0	2.6082338883	2.6082748678	2.6082748676

Table 4-7 show numerical approximations of the film profile $y(x)$ of (13) for THMD4s, RK4s and RKD3s methods for the cases $\theta = 2$ and $\theta = 3$. It can be seen that the approximations with THMD4s method agree

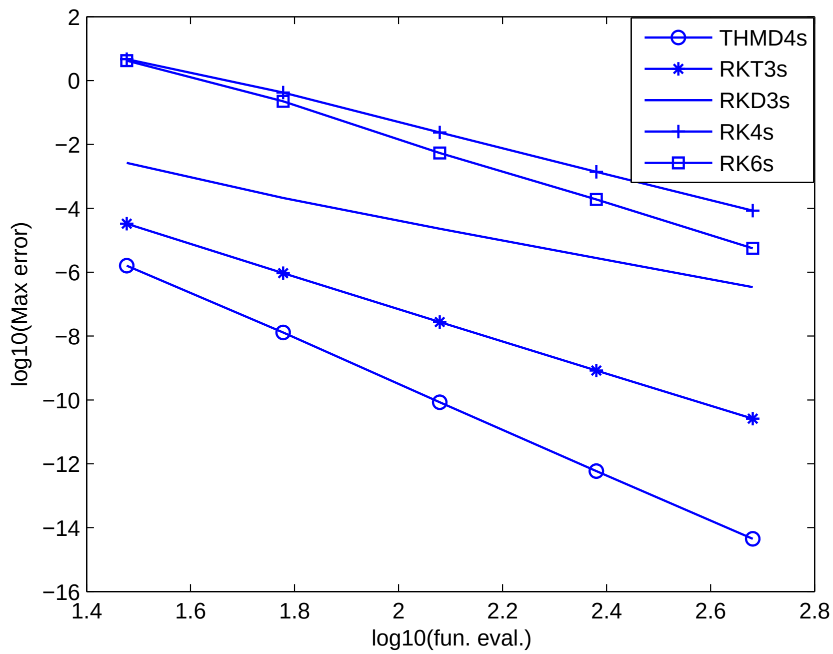

Figure 3: Efficiency curves for problem 3

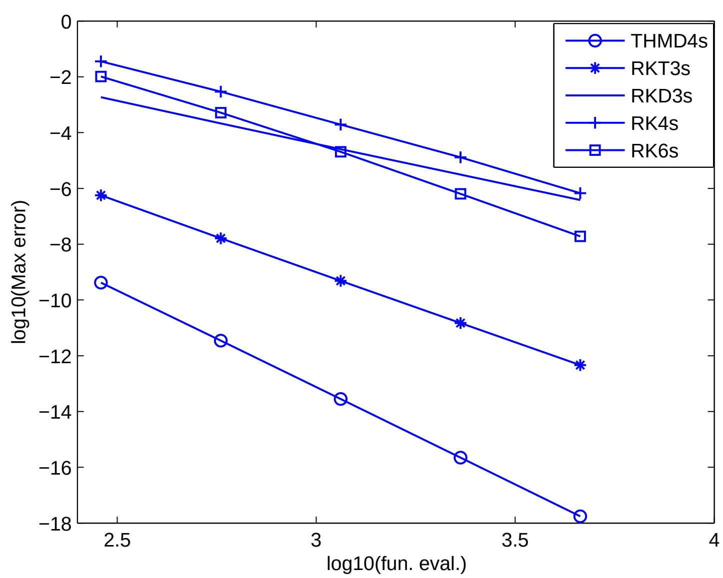

Figure 4: Efficiency curves for problem 4

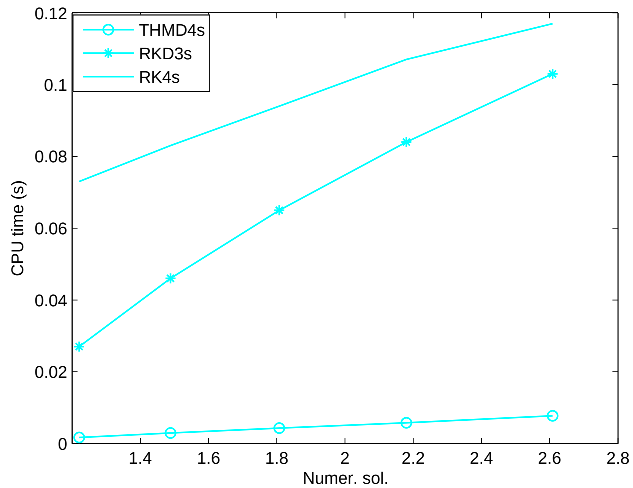

Figure 5: Efficiency curves for thin film flow problem with $\theta = 2$, $h = 0.01$

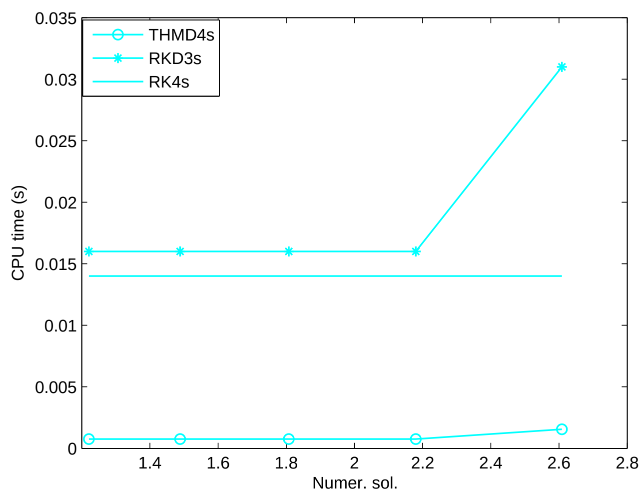

Figure 6: Efficiency curves for thin film flow problem with $\theta = 2$, $h = 0.1$

with those of RK4s and RKD3s methods up to at least five digits. This is an indication that the new method is consistent with the existing methods. Considering the better performance of the new method on the test problems 1–4, presented in Figs.1–4, one can postulate that the new method gives better approximation to the film profile $y(x)$ of (13) compared to the existing methods.

Figs. 4-8 show the graphs of CPU time (s) required for the approximation against the approximate solutions for THMD4s, RK4s and RKD3s methods to evaluate the efficiency of the new method. The most efficient of

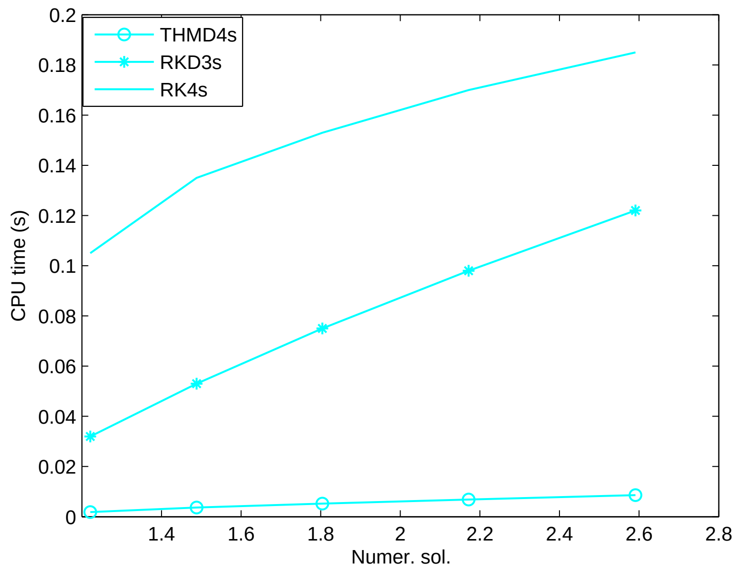

Figure 7: Efficiency curves for thin film flow problem with $\theta = 3$, $h = 0.01$

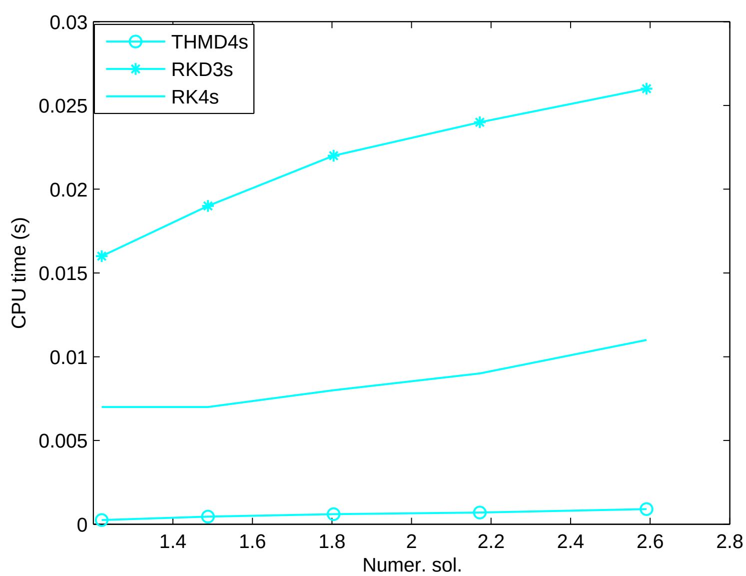

Figure 8: Efficiency curves for thin film flow problem with $\theta = 3$, $h = 0.1$

all the methods for the cases considered is the proposed THMD4s method, as its requires lesser CPU time for the approximation compared to the other methods.

Table 5: Numerical solutions for case $\theta = 2$ and $h = 0.1$

x	THMD4s	RK4s	RKD3s
0.0	1.0000000000	1.0000000000	1.0000000000
0.2	1.2212100137	1.2212105060	1.2212099592
0.4	1.4888348170	1.4888356990	1.4888344801
0.6	1.8073614815	1.8073626884	1.8073605532
0.8	2.1798193829	2.1798208831	2.1798175255
1.0	2.6082751000	2.6082768844	2.6082719667

Table 6: Numerical solutions for case $\theta = 3$ and $h = 0.01$

x	THMD4s	RK4s	RKD3s
0.0	1.0000000000	1.0000000000	1.0000000000
0.2	1.2211551425	1.2211551425	1.2211551423
0.4	1.4881052844	1.4881052844	1.4881052838
0.6	1.8042625484	1.8042625484	1.8042625471
0.8	2.1715227984	2.1715227984	2.1715227960
1.0	2.5909582594	2.5909582594	2.5909582556

Table 7: Numerical solutions for case $\theta = 3$ and $h = 0.1$

x	THMD4s	RK4s	RKD3s
0.0	1.0000000000	1.0000000000	1.0000000000
0.2	1.2211551601	1.2211559590	1.2211550887
0.4	1.4881053465	1.4881067401	1.4881049238
0.6	1.8042626817	1.8042645823	1.8042615558
0.8	2.1715230294	2.1715254210	2.1715208324
1.0	2.5909586143	2.5909615178	2.5909549758

8. Conclusion

A new family of three-step hybrid methods (THMD) for solving special third order ODEs directly is proposed. The methods are similar to the two-step hybrid methods for solving special second order ODEs directly [16].

Unlike RKT method [1] and RKD method [15], THMD method has only one equation, which is independent of first and second derivatives components - property responsible for its higher efficiency. Using Taylor approach, algebraic order conditions of the THMD methods are derived. The order conditions are used to construct a 4-stage method. Numerical results presented in section 6 and 7 reveal that the new method derived in this paper is more efficient than existing methods in the scientific literature.

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