

INTEGRAL GROUP RINGS WITH ALL CENTRAL UNITS TRIVIAL: SOLVABLE GROUPS

Sugandha Maheshwary¹

*Indian Institute of Science Education and Research, Mohali,
Sector 81, Mohali (Punjab) 140 306, India
e-mail: sugandha@iisermohali.ac.in*

(Received 6 January 2017; accepted 8 August 2017)

The object of this paper is to examine finite solvable groups whose integral group rings have only trivial central units.

Key words : Solvable groups; nilpotent groups; integral group rings; unit group; trivial central units; cut-property.

1. INTRODUCTION

Let G be a finite group. It is easy to see that the group $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]))$ of central units of the integral group ring $\mathbb{Z}[G]$ contains the *trivial central units* $\pm g$, $g \in \mathcal{Z}(G)$, the centre of G . Following [2], we say that G is a group with the cut-property, if $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]))$ consists of only the trivial central units. Nilpotent groups and metacyclic groups with the cut-property have been investigated and a complete classification of finite metacyclic groups with this property has been given in [2]. Continuing this investigation, and drawing on the work in [3], we provide, in Section 2, a precise characterization for a solvable group G to have the cut-property in case (i) the order of G is odd, or (ii) the order of G is even and every element of G has prime-power order (Theorems 1 & 2). We also complete the classification of finite nilpotent groups with the cut-property (Theorem 3).

2. SOLVABLE GROUPS WITH THE Cut-PROPERTY

Throughout the paper, G denotes a finite group and for elements $x, y \in G$, $x \sim y$, denotes “ x is conjugate to y .” We recall ([9] & [4], Lemma 2) that G has the cut-property, if, and only if, for every

¹Research supported by DST-SERB, India (PDF/2016/000731).

$x \in G$ and for every integer j relatively prime to the order $o(x)$ of x ,

$$x^j \sim x \text{ or } x^{-1}. \quad (1)$$

Theorem 1 — *An odd-order group G has the cut-property if, and only if, every element $x \in G$ satisfies (i) $x^5 \sim x^{-1}$, and (ii) $o(x)$ is either 7 or a power of 3.*

PROOF : Let G be an odd-order group with the cut-property, so that G satisfies (1), i.e., in the terminology of Chillag-Dolfi [3], G is an *inverse semi-rational group*. Consequently, by ([3], Theorem 3) together with an observation in its proof that the order of every element in such a group is a prime-power and (1), we have that for every element $x \in G$,

$$o(x) = 3^a \text{ or } 7^a, \quad a \geq 0 \quad (2)$$

and

$$x^5 \sim x \text{ or } x^{-1}. \quad (3)$$

We claim that $x^5 \not\sim x$ for any $1 \neq x \in G$. For, if $x^5 \sim x$, i.e., $x^5 = g^{-1}xg$, for some $g \in G$, then $x^{5^{o(g)}} = x$, which implies

$$5^{o(g)} \equiv 1 \pmod{o(x)}. \quad (4)$$

Observe that in either of the two possibilities given by (2), for the value of $o(x)$, 5 is a primitive root of $o(x)$. Therefore, (4) yields that $\varphi(o(x))$ divides $o(g)$, where φ denotes the Euler's phi function. This is clearly not possible as $o(g)$ is odd. It thus follows that

$$x^5 \sim x^{-1} \text{ for all } x \in G, \quad (5)$$

i.e., (i) holds. Arguing as above, it is easy to see that if $o(x) = 7^a$, then (5) is possible only if $a \leq 1$. Hence, (ii) holds.

Conversely, let G be a finite group satisfying (i) and (ii). Since 5 is a primitive root of 7 and of 3^a , for all $a \geq 1$, (i) implies that for every $x \in G$, $x^j \sim x$ or x^{-1} , for all j relatively prime to $o(x)$. Hence, in view of (1), G has the cut-property. \square

Let $V(\mathbb{Z}[G])$ denote the group of units of augmentation 1 in $\mathbb{Z}[G]$. In [2], a classification of metacyclic groups has been given according to the central height of $V(\mathbb{Z}[G])$, i.e., the least integer n at which the upper central series $\{\mathcal{Z}_i(V(\mathbb{Z}[G]))\}_{i \geq 0}$ of $V(\mathbb{Z}[G])$ stabilizes. In view of ([1], Theorem 3.7) and Theorem 1, we conclude the following:

Corollary 1 — Let G be an odd-order group. The central height of $V(\mathbb{Z}[G])$ is 1, except when G is a group with the cut-property such that G contains an element of order 7, and in this case, the central height of $V(\mathbb{Z}[G])$ is 0.

A simplifying feature for the classification of odd-order groups with the cut-property is that every element of such a group necessarily has prime-power order. In contrast with the odd-order groups with the cut-property, an even-order solvable group with the cut-property does not necessarily have all its elements of prime-power order. For example, the metacyclic group

$$\langle a, b \mid a^{12} = 1, b^2 = 1, b^{-1}ab = a^5 \rangle$$

has the cut-property ([2], Theorem 5), while it has elements of mixed order. For the classification of even-order solvable groups with the cut-property, we restrict to the class of solvable groups in which every element has prime-power order [8].

For a finite group G , let $\pi(G)$ denote the set of primes dividing the order of G .

Theorem 2 — *Let G be a finite solvable group such that every element of G has prime-power order. Then, G has the cut-property if, and only if, every element $x \in G$ satisfies one of the following conditions:*

- (i) $o(x) = 2^a$, $a \geq 0$ and $x^3 \sim x$ or x^{-1} ;
- (ii) $o(x) = 7$ or 3^b , $b \geq 1$ and $x^5 \sim x^{-1}$;
- (iii) $o(x) = 5$ and $x^3 \sim x^{-1}$.

PROOF : Let G be a finite solvable group with the cut-property such that every element of G has prime-power order. Then, G satisfies (1) and therefore, by ([3], Theorem 2), we have that $\pi(G) \subseteq \{2, 3, 5, 7, 13\}$. Let $1 \neq x \in G$. If $o(x)$ is even, then clearly, x satisfies (i). If $o(x)$ is odd, then $o(x) = p^\alpha$, where $p = 3, 5, 7$ or 13 and $\alpha \geq 1$, so that $o(x)$ has a primitive root, say r . By (1), we have that $x^r \sim x^\varepsilon$, $\varepsilon = \pm 1$, i.e., $x^r = g^{-1}x^\varepsilon g$, for some $g \in G$ and hence $(r\varepsilon)^{o(g)} \equiv 1 \pmod{o(x)}$. Consequently,

$$\varphi(o(x)) \text{ divides } o(g), \text{ if } \varepsilon = 1. \tag{6}$$

and

$$\varphi(o(x)) \text{ divides } 2o(g), \text{ if } \varepsilon = -1. \tag{7}$$

Clearly, neither (6) nor (7) holds, if $p = 13$. Thus, G has no element of order 13. Moreover, if $p = 7$, then (6) fails and (7) holds, only if $o(x) = 7$. Furthermore, if $p = 5$, then either of (6) and (7) holds, only if $\alpha = 1$. Note further, that $x^r \sim x$ yields $x \sim x^{-1}$, if $o(x) = 5$. Similarly, if $p = 3$, then

(6) holds only if $\alpha = 1$ and in that case, $x^5 \sim (=)x^{-1}$. These observations put together yield that x must satisfy (ii) or (iii).

Converse follows easily from (1) by observing that 5 is a primitive root of 7 and of 3^b , for all $b \geq 1$, 3 is a primitive root of 5, and the fact that $\mathcal{U}(\mathbb{Z}/2^n\mathbb{Z}) = \langle \pm 1 \rangle \oplus \langle 3 \rangle$, $n \geq 3$. \square

Recall that an element $x \in G$ is said to be real if $x \sim x^{-1}$, and G is called a *real group* if every element of G is real. It may be of interest to note that the class of real groups with the cut-property is the same as (i) the class of *rational groups* studied by Chillag and Dolfi [3], and (ii) the class \mathcal{B} of groups studied by Golomb and Hales [5]. By arguments given in the proof of Theorem 2, we observe that

$$\pi(G) \subseteq \{2, 3, 5\}, \text{ if } G \text{ is a rational solvable group ([6], Corollary 2).}$$

3. NILPOTENT GROUPS WITH THE Cut-PROPERTY

If G is a finite nilpotent group with the cut-property, then $\pi(G) \subseteq \{2, 3\}$; further, 2-groups and 3-groups with the the cut-property have also been characterized [2]. The following theorem extends the classification of finite abelian groups with the cut-property [7] to the classification of finite nilpotent groups with the cut-property.

Theorem 3 — *A finite nilpotent group G has the cut-property if, and only if, G is one of the following:*

- (i) *a 2-group such that for all $x \in G$, $x^3 \sim x$ or x^{-1} ;*
- (ii) *a 3-group such that for all $x \in G$, $x^2 \sim x^{-1}$;*
- (iii) *a direct sum $H \oplus K$ of a real group H satisfying (i) and a non-trivial group K satisfying (ii).*

PROOF : Let G be a finite nilpotent group with the cut-property, so that $\pi(G) \subseteq \{2, 3\}$. If $\pi(G) \neq \{2, 3\}$, then G is of type (i) or (ii) ([2], Theorems 2 & 3). In case $\pi(G) = \{2, 3\}$, then $G = H \oplus K$, where H is a non-trivial 2-group and K is a non-trivial 3-group. Since the cut-property is quotient closed, both H and K have the cut-property and hence, H satisfies (i) and K satisfies (ii). It only remains to check that H is a real group. For this, let order of H be 2^α and that of K be 3^β for some positive integers α, β . Further, let m and n be positive integers satisfying

$$m \equiv 3 \pmod{2^\alpha}; \quad m \equiv 1 \pmod{3^\beta}$$

and

$$n \equiv 3 \pmod{2^\alpha}; \quad n \equiv -1 \pmod{3^\beta}.$$

Note that neither 2 nor 3 can divide either m or n and therefore, both m and n are relatively prime to the order of (h, k) , for any $h \in H$ and $1 \neq k \in K$. Since $H \oplus K$ has the cut-property, we obtain, by (1), that $(h, k)^m = (h^3, k)$ must be conjugate to (h, k) or $(h, k)^{-1}$. Now, K being a 3-group, $k \in K$ is not real. Therefore, we must have $(h, k)^m \sim (h, k)$ and hence $h^3 \sim h$. Similarly, $(h, k)^n = (h^3, k^{-1})$ must be conjugate to $(h, k)^{-1}$, which yields that $h^3 \sim h^{-1}$. Consequently, $h \sim h^{-1}$, for any $h \in H$, i.e., H is a real group.

Conversely, if G is of type (i) or (ii), then clearly G has the cut-property ([2], Theorems 2 & 3). Let $G = H \oplus K$ be as in (iii) and let $(h, k) \in H \oplus K$. We check that for any integer i , relatively prime to the order of (h, k) , $(h, k)^i \sim (h, k)^{\pm 1}$. Since H satisfies (i) and K satisfies (ii), both H and K have the cut-property ([2], Theorems 2 & 3) and thus satisfy (1). Therefore, if $h = 1$, or $k = 1$, then $(h, k)^i \sim (h, k)^{\pm 1}$. In case neither $h = 1$ nor $k = 1$, then i is neither divisible by 2 nor by 3. Therefore, $h^i \sim h \sim h^{-1}$ and $k^i \sim k^{\pm 1}$. Hence $(h, k)^i \sim (h, k)^{\pm 1}$. It now follows from (1), that $H \oplus K$ has the cut-property. \square

Remark : It is known that the direct sum $H \oplus K$ of two 2-groups with the cut-property may not have the cut-property ([2], Remark 1). However, following the arguments of Theorem 3, we see that this is so if one of H or K is a real group. By considering the direct sum of cyclic groups of order 4, one can easily check that the above condition is sufficient, but not necessary. More precisely, the direct sum $H \oplus K$ of 2-groups H and K with the cut-property, does not have the cut-property if, and only if, there exist non-real elements $h \in H$ and $k \in K$, such that either $h^3 \sim h$ and $k^3 \sim k^{-1}$ or $h^3 \sim h^{-1}$ and $k^3 \sim k$.

In view of the proof of Theorem 3 and the above remark, we obtain the following result analogous to ([7], Theorem 7).

Corollary 2 — Let G be a finite nilpotent group with the cut-property and let $G^* := G \oplus H$, where H is a real 2-group with the cut-property. Then,

$$\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G^*])) = \pm \mathcal{Z}(G^*).$$

We next give two alternative characterizations for a p -group of class 2 to have the cut-property.

For an element x in a group G , define $[x, G] := \{[x, g] := xgx^{-1}g^{-1} \mid g \in G\}$. Note that in case G is of class 2, then $[x, G]$ is a normal subgroup of G . Thus, as a consequence of Theorem 3, we have

Corollary 3 — A p -group G of class 2 has the cut-property if, and only if, one of the following holds:

- (i) $p = 2$ and, for every $x \in G$, $x^4 \in [x, G]$;

(ii) $p = 3$ and, for every $x \in G$, $x^3 \in [x, G]$.

Furthermore, the cut-property is direct sum closed for p -groups of class 2.

We also have the following:

Proposition 1 — A finite p -group G of class 2 has the cut-property if, and only if,

$$\text{for all } x \in G, \text{ both } [x, G] \text{ and } G/[x, G] \text{ have the cut-property.} \quad (8)$$

PROOF : The necessity follows from the observation that $[x, G] \subseteq \mathcal{Z}(G)$ and the fact that the cut-property is centre closed and quotient closed.

Let G be a p -group of class 2 satisfying (8). Since $[x, G]$ is a p -group which has the cut-property, $p = 2$ or 3 .

Let $p = 2$. If $x \notin [x, G]$, then $\bar{x} := x[x, G] \in G/[x, G] =: \bar{G}$, is such that $\bar{x}^4 \in [\bar{x}, \bar{G}]$, as \bar{G} has the cut-property. This yields that $x^4 \in [x, G]$ and hence G has the cut-property.

The case when $p = 3$ is similar and we omit the details. □

If a group G has the cut-property, then so do $\mathcal{Z}(G)$ and $G/\mathcal{Z}(G)$. The converse, however, is not true, even for the nilpotent groups of class 2. For example, if

$$G = \langle a, b \mid a^9 = 1, b^9 = 1, b^{-1}ab = a^4 \rangle,$$

then both $\mathcal{Z}(G)$ and $G/\mathcal{Z}(G)$, being abelian of exponent 3, have the cut-property, whereas G does not have the cut-property, since $b^2 \not\sim b^{-1}$. However, Proposition 1 yields the following:

Corollary 4 — A finite p -group G of class 2 has the cut-property if, and only if, for all normal subgroups N of G contained in $\mathcal{Z}(G)$, both N and G/N have the cut-property.

4. CONCLUDING REMARK

Our analysis leaves open the problem of characterizing a solvable group G with the cut-property in case G is a non-nilpotent solvable group of even order having an element of mixed order.

ACKNOWLEDGEMENT

The author is grateful to I. B. S. Passi for his valuable comments and suggestions.

REFERENCES

1. S. R. Arora and I. B. S. Passi, Central height of the unit group of an integral group ring, *Comm. Algebra*, **21**(10) (1993), 3673-3683.
2. G. K. Bakshi, S. Maheshwary, and I. B. S. Passi, Integral group rings with all central units trivial, *J. Pure Appl. Algebra*, **221**(8) (2017), 1955-1965.
3. D. Chillag and S. Dolfi, Semi-rational solvable groups, *J. Group Theory*, **13**(4) (2010), 535-548.
4. M. Dokuchaev, C. P. Milies and S. K. Sehgal, Integral group rings with trivial central units. II, *Comm. Algebra*, **33**(1) (2005), 37-42.
5. S. W. Golomb and A. W. Hales, On enumerative equivalence of group elements, *J. Combinatorial Theory*, **5** (1968), 308-312.
6. R. Gow, Schur indices of some groups of Lie type, *J. Algebra*, **42**(1) (1976), 102-120.
7. G. Higman, The units of group-rings, *Proc. London Math. Soc. (2)*, **46** (1940), 231-248.
8. G. Higman, Finite groups in which every element has prime-power order, *J. London Math. Soc.*, **32** (1957), 335-342.
9. J. Ritter and S. K. Sehgal, Integral group rings with trivial central units, *Proc. Amer. Math. Soc.*, **108**(2) (1990), 327-329.