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CHEBYSHEV CENTERS THAT ARE NOT FARTHEST POINTS

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In this paper, we address the question whether in a given Banach space, a Chebyshev center of a nonempty bounded subset can be a farthest point of the set. We obtain a characterization of two-dimensional real strictly convex spaces as those ones where a Chebyshev center cannot contribute to the set of farthest points of a subset. In dimension greater than two, every non-Hilbert smooth space contains a subset whose Chebyshev center is a farthest point. We explore the scenario in uniformly convex Banach spaces and further study the roles played by centerability and M-compactness in the scheme of things to obtain a step by step characterization of strictly convex Banach spaces.

Key words : Chebyshev center; farthest point; strict convexity; uniform convexity.

1. INTRODUCTION

In this paper, letter X denotes a Banach space, $B_X = \{x \in X \ : \ \|x\| \leq 1\}$ and $S_X = \{x \in X \ : \ \|x\| = 1\}$ denote the unit ball and the unit sphere of X respectively; $B[x, r]$

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$= \{y \in X : \|x-y\| \leq r\}$ is the closed ball with center x and radius r and $S[x,r] = \{y \in X : \|x-y\| = r\}$ is the closed sphere with center x and radius r. All Banach spaces are considered over the field of real numbers. For a set A, $|A|$ denotes the cardinality of A, if A is finite then $|A|$ is the number of elements in A. We call a subset A of X is *nontrivial* if $|A| \geq 2$. For a nonempty bounded set $A \subseteq X$, its *diameter* is

$$\operatorname{diam}(A) = \sup_{a_1,a_2 \in A} \|a_1 - a_2\|.$$

The *outer radius* of $A \subseteq X$ at an element $x \in X$ is defined as

$$r(x, A) = \sup_{a \in A} \|x - a\|.$$

The supremum in the definition of $r(x, A)$ may be or may not be attained at some point of A. Let

$$F(x, A) = \{a \in A : \|x - a\| = r(x, A)\}$$

denote the collection of all elements in A which are farthest from $x \in X$. If for an element $x \in X$, $r(x, A)$ is not attained then $F(x, A) = \emptyset$. The collection of all elements in A at which $r(x, A)$ is attained for some $x \in X$ is denoted by $\operatorname{Far} A$ i.e.,

$$\operatorname{Far} A = \bigcup_{x \in X} F(x, A).$$

Recall that the most intriguing unsolved problem about farthest points [8] is whether there exists a nontrivial bounded convex closed subset A of a Hilbert space H with the property that $|F(x, A)| = 1$ for every $x \in H$ (see also [1] and [10]).

The *Chebyshev radius* $r(A)$ of A is given by $r(A) = \inf_{x \in X} r(x, A)$. If there exists a point $c \in X$ such that $r(c, A) = r(A)$, then c is called a *Chebyshev center* of A. Garkavi [4] proved that if X is 1-complemented in X^{**} (in particular, if X is reflexive) then every bounded subset A of X has a Chebyshev center, and if X is uniformly convex in every direction, then every bounded subset A of X has at most one Chebyshev center (see also [3, Ch. 2, notes and remarks]). Consequently, in uniformly convex spaces, every bounded subset A has a unique Chebyshev center [6, Part 5 §33].

It is possible to characterize inner product spaces among normed linear spaces, using the notion of Chebyshev center [2]. Let c_A denote a Chebyshev center of a nontrivial bounded subset A of a Banach space X. In [2], Baronti and Papini proved the following inequality for any nonempty subset A of a Hilbert space H:

$$r^2(x, A) \geq r^2(A) + \|x - c_A\|^2 \quad \text{for all} \quad x \in H,$$

in particular,

$$r(x, A) > \|x - c_A\| \quad \text{for all} \quad x \in H,$$

for any nontrivial bounded subset A of H. It clearly follows from the above inequality that in a Hilbert space H, $c_A \notin \operatorname{Far} A$, where c_A is the unique Chebyshev center of a nontrivial bounded subset A of H.

A Banach space X is said to be *strictly convex* if S_X does not contain nontrivial linear segment i.e., there does not exist $u, v \in S_X$ $(u \neq v)$ such that $\{tu + (1-t)v : t \in [0,1]\} \subset S_X$. Equivalently, X is strictly convex if every $x \in S_X$ is an extreme point of B_X. One more reformulation: X is strictly convex if and only if for every two points $x, y \in X \setminus \{0\}$ with $x \notin \{ty : t > 0\}$, the *strict triangle inequality* $\|x + y\| < \|x\| + \|y\|$ holds true.

It is clear that if the unit sphere of a Banach space X contains a nontrivial line segment $L = \{tu + (1-t)v : t \in [0,1]\}$ (i.e., X is not strictly convex), then all the points of L are of the same distance 1 from the origin, so $L = \operatorname{Far} L$ and in particular, the Chebyshev center $\frac{(u+v)}{2}$ belongs to $\operatorname{Far} L$. This observation motivated Debmalya Sain to ask in “Research Gate” the following question:

Can a Chebyshev center of a bounded set be a farthest point of the set from a point in a strictly convex Banach space?

This question, which we answer in positive, leaded to other natural questions and answers, and all these resulted in the article which we are presenting now. We are indebted to the “Research Gate” platform that brought the authors of this paper together.

When we posted in arXiv a preliminary version of this article (arXiv: 1608.03422 [math.FA]), Pier Luigi Papini asked us a stimulating question (which remains open), whether is it true, that in every non-Hilbert space of dimension greater than 2 there is a subset having a Chebyshev center which is a farthest point. He also kindly informed us about the Klee-Garkavi theorem [7, 5], which appeared to be of great use for this project. We deeply appreciate his advises.

As we will see in this paper, whether the Chebyshev center of a nontrivial subset of a Banach space may belong to the set, is an important factor in determining the convexity properties of the space. In view of the discussions above, let us introduce the following definitions:

Definition 1.1 — A set A in a Banach space X is said to be a *CCF set* (comes from Chebyshev center in $\operatorname{Far} A$) if there is a Chebyshev center of A that belongs to $\operatorname{Far} A$. A is said to be a *CCNF set* (comes from Chebyshev center not in $\operatorname{Far} A$) if it is not a CCF set.

Note that if a set A does not contain any of its Chebyshev center(s) then the question of Chebyshev center belonging to $\operatorname{Far} A$ does not arise and so A is trivially a CCNF set.

Definition 1.2 — A Banach space X is said to be *CCF* if it contains a nontrivial CCF set. X is said to be *CCNF* if it is not CCF, i.e., all nontrivial subsets of X are CCNF.

The main results of the paper deal with the general properties of CCF and CCNF sets and spaces. These results are collected in the next few short sections. In the first of them, called "A reformulation of CCNF", we reduce in a rather elementary way the question whether X is CCNF to the question whether for every $y \in S_X$ and every $r \in (0, 1)$, the Chebyshev radius of the set $B_X \cap B[y, r]$ is strictly smaller than r.

From our earlier discussion, it easily follows that every CCNF space must be strictly convex. In the next section, using the above reformulation and a geometric lemma, for two-dimensional spaces we prove the converse result: every two-dimensional strictly convex real Banach space is CCNF. However, the result no longer holds true if the dimension of the space is greater than two, even for L_p spaces. In the Section "CCNF spaces of dimension greater than 2" we demonstrate that every smooth non-Hilbert space of dimension greater than 2 is CCF.

In the Section "Centerable CCNF sets" we present an example of a CCF set in an infinite-dimensional strictly convex Banach space X. This example has an interesting additional property that $r(A) = \frac{1}{2}\operatorname{diam}(A)$. Recall, a set with this property is called *centerable*. Our Theorem 5.3 demonstrates impossibility of such examples in uniformly convex spaces: if A is any nontrivial centerable subset of a uniformly convex Banach space X, then A is CCNF. This result implies the following characterization of finite-dimensional strictly convex Banach spaces (Theorem 5.4): a finite-dimensional Banach space X is strictly convex if and only if every nontrivial bounded centerable subset of X is CCNF.

The notion of M-compactness also plays a vital role in the study of farthest points. A sequence $\{a_n\}$ in A is said to be *maximizing* if for some $x \in X$, $\|x - a_n\| \to r(x, A)$. A subset A of X is said to be *M-compact* if every maximizing sequence in A has a subsequence that converge to an element of A. In the last short Section "M-compact centerable CCNF sets" we prove that in a strictly convex Banach space, every nontrivial, bounded, centerable, M-compact set is CCNF. It is also easy to observe that this property characterizes the strict convexity of a Banach space.

2. A Reformulation of CCNF

In this section, we reduce the CCNF property of a Banach space to subsets of the form "intersection of the unit ball with a small ball". The following two lemmas extract the main ideas of the proof. The first lemma is too easy and so we just mention it without proof.

Lemma 2.1 — Let A be a nontrivial bounded subset of X, $x \in \operatorname{Far} A$. Then, for every $N > 0$ there is a point $y \in X$ such that x is a farthest point of A from y and $\|x - y\| > N$.

Lemma 2.2 — Let A be a nontrivial bounded subset of X, containing its Chebyshev center c_A. Suppose c_A is at the same time a farthest point of A from some $y \in X$. Let r be the Chebyshev radius of A and $R = \|c_A - y\|$. Then $r \leq R$ and the subset $U = B[c_A, r] \cap B[y, R]$ has the following properties:

(a) $A \subseteq U$.

(b) The Chebyshev radius of U equals r.

(c) c_A is a Chebyshev center of U.

(d) c_A is a farthest point of U from y.

PROOF : Inclusions $A \subseteq B[c_A, r]$ and

$$A \subseteq B[y, R] \tag{2.1}$$

follow from definitions of Chebyshev center and of farthest point respectively. Consequently, (a) is correct. Because of (2.1), the Chebyshev radius r of A cannot be greater than R. Property (a) implies $r(U) \geq r$, and inclusion

$$U \subseteq B[c_A, r] \tag{2.2}$$

implies the reverse inequality, which proves (b). Taking (b) into account, we see that (2.2) means (c). Finally, (d) follows from the fact that $c_A \in A \subseteq U$ and from the inclusion $U \subseteq B[y, R]$. □

Now we are ready to prove the following characterization of CCNF Banach spaces.

***Theorem* 2.3** — *Denote $r_{t,z}$ the Chebyshev radius of the set $A_{t,z} = B_X \cap B[z, t]$. Then, for a Banach space X the following three conditions are equivalent:*

(i) *X is a CCNF space*;

(ii) *for every $z \in S_X$ and every $t \in (0, 1]$, the inequality $r_{t,z} < t$ holds true*;

(iii) *for every $\varepsilon \in (0, 1]$, there is a $t_0 \in (0, \varepsilon)$ such that for every $z \in S_X$ and every $t \in (0, t_0]$, the inequality $r_{t,z} < t$ holds true.*

PROOF : (i) $\Rightarrow$ (ii). As $A_{t,z} \subseteq B[z,t]$ we have $r_{t,z} \leq t$. If $r_{t,z} = t$, then z is a Chebyshev center of $A_{t,z}$. At the same time, z is a farthest point of $A_{t,z}$ from the origin, which contradicts our assumption (i). Consequently, $r_{t,z} < t$.

The implication (ii) $\Rightarrow$ (iii) is evident, so it remains to prove (iii) $\Rightarrow$ (i). Assume contrary that X is CCF. Then, by definition, there exists a nontrivial bounded subset A of X, containing its Chebyshev center c_A, such that $c_A \in \operatorname{Far} A$. Applying Lemma 2.1 for a given $N > 0$, we can find a $y \in X$ such that c_A is a farthest point of A from y and $R := \|c_A - y\| > N$. Denote r the Chebyshev radius of A. According to Lemma 2.2, $r \leq R$. Denote $t = \frac{r}{R} \in (0,1]$. Consider the set $U = B[c_A, r] \cap B[y, R]$ from Lemma 2.2. According to (b) of that lemma, $r(U) = r$.

For every $x \in X$, denote $f(x) = \frac{1}{R}(x-y)$. Observe that $f(y) = 0$, $\|f(c_A)\| = 1$ and f multiplies all the distances by the same coefficient $\frac{1}{R}$, i.e., $\|f(x_1) - f(x_2)\| = \frac{1}{R}\|x_1 - x_2\|$ for all $x_1, x_2 \in X$. Consequently, $r(f(U)) = \frac{r}{R} = t$. On the other hand,

$$f(U) = B[f(c_A), \frac{r}{R}] \cap B[f(y), 1] = B[f(c_A), t] \cap B[0, 1] = A_{t, f(c_A)}.$$

So, $r_{t,z} = t$ for $z = f(c_A) \in S_X$ and $t = \frac{r}{R} \leq \frac{r}{N} \to 0$ as $N \to \infty$. This contradicts our assumption (iii). □

3. TWO-DIMENSIONAL CCNF SPACES

In this section, we prove that in a two-dimensional strictly convex real Banach space X, every non-trivial bounded subset of X is CCNF. To this end, we need the following lemma:

Lemma 3.1 — Let X be a two-dimensional real Banach space, $u, v \in S_X$ and let the straight line l that connects u and v does not contain origin θ. Let S denote the part of B_X not containing θ, that is cut from B_X by l; $w = \frac{u+v}{2}$, $r = \|u - w\| = \|v - w\| = \frac{1}{2}\|u - v\|$. Then, $S \subset w + rB_X$, i.e., the distance of every point of S to w does not exceed r.

PROOF : Clearly, it is sufficient to prove that $\|s - w\| \leq r$ for all $s \in S$. Let $w_t = (1-t)u + tw, \ \ 0 \leq t \leq 1$ and $w_{t'} = (1-t')w + t'v, \ \ 0 \leq t' \leq 1$. Now,

$$\|u - w\| = \|u - w_t\| + \|w_t - w\| = r, \tag{3.1}$$

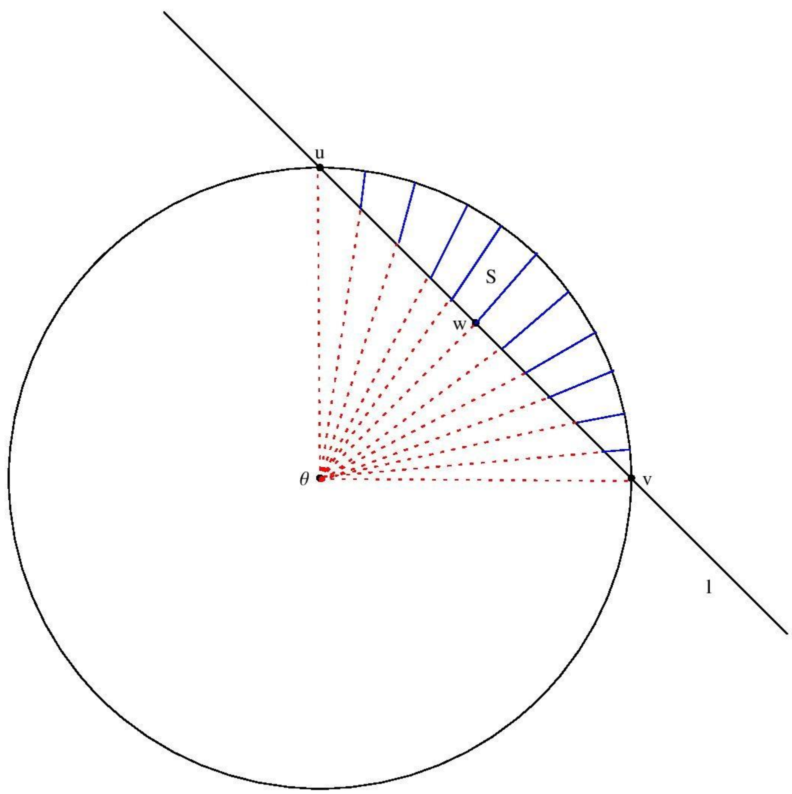
as w_t belongs to the segment $[u, w]$.

Similarly,

$$\|w - v\| = \|w - w_{t'}\| + \|w_{t'} - v\| = r. \tag{3.2}$$

Since X is a two-dimensional real Banach space, for any $s \in S$, either $s = \lambda w_t$ or $s = \lambda w_{t'}$, for some $\lambda \geq 1$. We have, $\|\lambda w_t\| \leq 1 \Rightarrow \lambda \leq \frac{1}{\|w_t\|}$ and also, $\|\lambda w_{t'}\| \leq 1 \Rightarrow \lambda \leq \frac{1}{\|w_{t'}\|}$. Now,

$$\begin{aligned} \|\lambda w_t - w_t\| &= (\lambda - 1)\|w_t\| \\ &\leq \left(\frac{1}{\|w_t\|} - 1\right)\|w_t\| \\ &= 1 - \|w_t\| \\ &= \|u\| - \|w_t\| \leq \|u - w_t\|. \end{aligned} \tag{3.3}$$



Similarly,

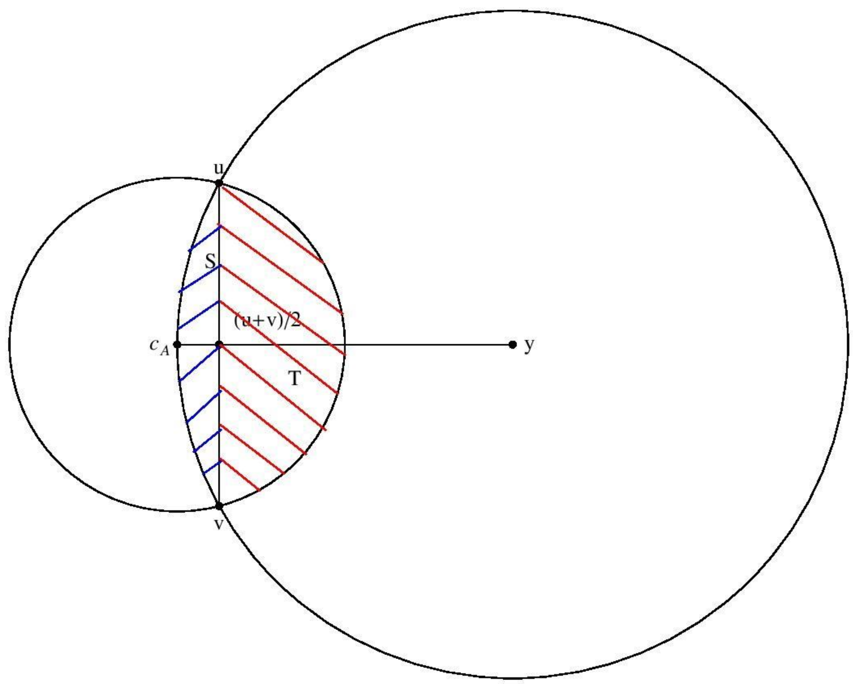
$$\|\lambda w_{t'} - w_{t'}\| \leq \|v - w_{t'}\|. \tag{3.4}$$

Now using (3.1) and (3.3), we have,

$$\begin{aligned} \|\lambda w_t - w\| &= \|\lambda w_t - w_t + w_t - w\| \\ &\leq \|\lambda w_t - w_t\| + \|w_t - w\| \\ &\leq \|u - w_t\| + \|w_t - w\| \\ &= \|u - w\| = r. \end{aligned}$$

Similarly, using (3.2) and (3.4), we can show that $\|\lambda w_{t'} - w\| \leq r$. So for all $s \in S$, $\|s - w\| \leq r$, which completes the proof. □

Now we are ready for the promised theorem. The following picture illustrates the proof.



***Theorem* 3.2** — *Let X be a two-dimensional strictly convex real Banach space. Then every nontrivial bounded subset of X is CCNF.*

PROOF : Let us argue and absurdum. Let $A \subset X$ be a nontrivial bounded CCF subset. We will use the notations of Lemma 2.2. Suppose $c_A \in A$ is a farthest point of A from some $y \in X$. Let r be the Chebyshev radius of A and $R = \|c_A - y\|$. Let u, v be the intersection points of the spheres $S[c_A, r]$ and $S[y, R]$. Then by Lemma 3.1, both S and T in the above picture are subsets of the closed ball centered at $\frac{(u+v)}{2}$ and radius $\|\frac{(u-v)}{2}\|$. Then $A \subseteq B[\frac{(u+v)}{2}, \ \|\frac{(u-v)}{2}\|]$. By the definition of Chebyshev radius, $\|\frac{(u-v)}{2}\| \geq r$ which implies that $\|u - v\| \geq 2r$. On the other hand, $u, v \in S[c_A, r]$, so $\|u - v\| \leq 2r$ and consequently $\|u - v\| = 2r$. We have $\|(u - c_A) + (c_A - v)\| = 2r$ and $\|u - c_A\| + \|c_A - v\| = 2r$. As the space is strictly convex, we must have $(u - c_A) = k(c_A - v)$, for some constant $k > 0$. Since $\|u - c_A\| = \|c_A - v\| = r$, we have $k = 1$. Therefore, we have $c_A = \frac{u+v}{2}$. Now $u, v, c_A \in S[y, R]$ and so by strict convexity we get $u = v = c_A$. Then $r = 0$ and so A consists of only one point, contradicting our assumption that A is nontrivial. This completes the proof of the theorem. □

The converse of Theorem 3.2 is also true. Indeed, as we already remarked in the introduction, if X is not strictly convex, then S_X contains a straight line segment $L = \{(1-t)u+tv : u, v \in S_X, t \in [0,1]\}$. It is easy to see that $\frac{u+v}{2}$ is a Chebyshev center of L, which is also a farthest point of L from the origin. Thus, we have the following characterization of strict convexity of a two-dimensional real Banach space:

***Theorem* 3.3** — *A two-dimensional real Banach space X is strictly convex if and only if every nontrivial bounded subset of X is CCNF.*

4. CCNF SPACES OF DIMENSION GREATER THAN TWO

In general, Theorem 3.2 is not true if the dimension of the space is strictly greater than two. As an illustrating example consider three-dimensional version of ℓ_p.

As usual, by $\ell_p^{(3)}, p > 1$, we denote the space $\mathbb{R}^3$ equipped with the norm

$$\|(x_1, x_2, x_3)\| = (|x_1|^p + |x_2|^p + |x_3|^p)^{1/p},$$

and let $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, and $e_3 = (0, 0, 1)$.

Proposition 4.1 — The Chebyshev center of the set $A_0 = \{e_1, e_2, e_3\} \subset \ell_p^{(3)}$ is the point $x_p = (s_p, s_p, s_p)$, where

$$s_p = \frac{1}{1 + 2^{1/(p-1)}}.$$

PROOF : Since $\ell_p^{(3)}$ is uniformly convex, A_0 possesses unique Chebyshev center and by symmetry, this Chebyshev center must be of the form (s, s, s). What remains to do, is to minimize the quantity

$$f(s) = \|e_k - (s, s, s)\|^p = |1 - s|^p + 2|s|^p, s \in \mathbb{R}.$$

Evidently, the minimum attains on $(0, 1)$ (otherwise $f(s) \geq 1$), where $f'(s) = 2ps^{p-1} - p(1 - s)^{p-1}$, and s_p is the unique root of equation $f'(s) = 0$. □

Following the notation of the previous proposition, denote $A_p = \{e_1, e_2, e_3, x_p\} \subset \ell_p^{(3)}$.

Example 4.2 : For $p \in (1, 2) \cup (2, \infty)$, A_p is a CCF set and consequently, $\ell_p^{(3)}$ is an example of strictly convex CCF space.

PROOF : A_p is formed by A_0 together with its Chebyshev center x_p, so x_p is also the Chebyshev center of A_p. It remains to show that $x_p \in \mathrm{Far}\, A_p$. We consider the following two cases separately:

Case 1 : $p \in (1,2)$. In this case

$$0 < s_p < \frac{1}{3}. \tag{4.1}$$

We are going to demonstrate that for $t > 1$ large enough, x_p is the farthest point of A_p from $y = (t,t,t)$. The distance from y to any of e_k equals $((t-1)^p + 2t^p)^{1/p}$, $\|y - x_p\| = 3^{1/p}(t - s_p)$, so we need to check for large t the inequality

$$(t-1)^p + 2t^p < 3\,(t - s_p)^p\,.$$

Dividing by t^p and denoting $\tau = \frac{1}{t}$, we reduce this to

$$(1-\tau)^p + 2 < 3(1 - s_p\tau)^p \tag{4.2}$$

for small positive τ. At the point $\tau = 0$, the left-hand side of (4.2) equals the right-hand side. So in order to demonstrate (4.2) for τ close to 0, it is sufficient to show for $f_1(\tau) = (1-\tau)^p + 2$, $f_2(\tau) = 3(1 - s_p\tau)^p$, the validity of the inequality $f_1'(0) < f_2'(0)$. This is the inequality

$$-p < -3ps_p,$$

which follows from (4.1).

Case 2 : $p \in (2,\infty)$. In this case

$$s_p > \frac{1}{3}. \tag{4.3}$$

we are going to demonstrate that for $t > 0$ large enough, x_p is the farthest point of A_p from $y = (-t,-t,-t)$. The distance from y to any of e_k equals $((t+1)^p + 2t^p)^{1/p}$, $\|y - x_p\| = 3^{1/p}(t + s_p)$, so we need to check for large t the inequality

$$(t+1)^p + 2t^p < 3\,(t + s_p)^p\,.$$

The same way as above, this reduces to

$$(1+\tau)^p + 2 < 3(1 + s_p\tau)^p$$

for small positive τ. Denoting $g_1(\tau) = (1+\tau)^p + 2$, $g_2(\tau) = 3(1 + s_p\tau)^p$, we have to demonstrate the inequality $g_1'(0) < g_2'(0)$, i.e., the inequality

$$p < 3ps_p,$$

which follows from (4.3). $\square$

Below, using some general Banach space theory results, we demonstrate that analogous examples exist in many Banach spaces of dimension greater than 2, including all smooth non-Hilbert spaces (a normed space is called *smooth* if its norm is Gateaux differentiable outside of the origin), and in particular in all L_p spaces, with $p \neq 2$. Nevertheless, we are not able to answer the following natural question.

Question 4.3 — (Pier Luigi Papini, private communication). Is it true, that the only CCNF Banach spaces of dimension greater than two are Hilbert spaces?

Following Phelps' terminology [9], a Banach space X has property (g), if for every bounded convex closed subset A of X and every $x \in X \setminus A$ there is a ball $B \subset X$ such that $x \notin B \supset A$.

***Theorem* 4.4** — *(Phelps, Theorem 4.4 of [9]). A finite-dimensional space X has property (g) if and only if the set of extreme points of B_{X^*} is dense in S_{X^*}.*

Remark, that the above theorem implies that every smooth finite-dimensional space X has property (g). Indeed, due to the classical Klee's duality theorem [3, Ch. 2, § 1, Theorem 2], if a reflexive (in particular, a finite-dimensional) space X is smooth, then X^* is strictly convex, so in this case the set of extreme points of B_{X^*} equals S_{X^*}. Also, if a Banach space is smooth, then all its subspaces are smooth, which makes the following theorem (the main result of this section) applicable to all smooth non-Hilbert spaces X of dimension greater than two.

***Theorem* 4.5** — *Let a non-Hilbert Banach space X, with $\dim X \geq 3$, have the following property: for every $A \subset X$ with $|A| = 3$ such that $0 \notin \operatorname{conv} A$ there is a ball $B \subset X$ such that $0 \notin B \supset A$ (in particular, this happens if every subspace $Y \subset X$ of $\dim Y = 3$ has the property (g)). Then X is a CCF space.*

PROOF : Since X is non-Hilbert and $\dim X \geq 3$, according to the Klee-Garkavi theorem [7, 5], there is a subset $A \subset X$ consisting of 3 points and having a Chebyshev center c outside of $\operatorname{conv} A$. Changing A to $A - c$, if necessary, we can assume that 0 is a Chebyshev center of A, and that $0 \notin \operatorname{conv} A$. The conditions of our theorem say that there is a ball $B \subset X$ such that $0 \notin B \supset A$. Denote z the center of the ball B, and consider $A_1 = A \cup \{0\}$. Then 0 is a Chebyshev center of A_1 and at the same time the farthest point of A_1 from z. □

Corollary 4.6 — Every smooth non-Hilbert space of dimension greater than 2 is CCF.

5. CENTERABLE CCNF SETS

In this section, we present an example of a centerable CCF set in an infinite-dimensional strictly convex Banach space X. Afterwards, it will follow from Theorem 5.3 that such an example is impossible in finite-dimensional strictly convex Banach spaces.

Firstly we recall an easy but useful way to construct equivalent strictly convex norms [3, Ch. 4 §2, Theorem 1].

Proposition 5.1 — Let X, Y be Banach spaces, Y be strictly convex and let $T\colon X \to Y$ be an injective continuous linear operator. For $x \in X$, denote $p(x) = \|x\| + \|Tx\|$. Then (X, p) is strictly convex.

Now the promised example.

Example 5.2 : Consider the space c_0 of all sequences of real numbers converging to zero, equipped with the following norm:

$$\|x\| = \max_k |x_k| + \sqrt{\sum_{k=1}^{\infty} \frac{1}{4^k} |x_k|^2} \tag{5.1}$$

where x_k $(k \in \mathbb{N})$ denote the $k-$th coordinate of $x \in c_0$. Clearly, the norm is strictly convex. Let us denote this Banach space by X. Let $\theta = (0, 0, \ldots, 0, \ldots)$ and $e_n = (0, 0, \ldots, 0, 1, 0 \ldots)$, i.e., the n-th coordinate of e_n is 1 and all other coordinates are 0. Denote

$$x_n = \frac{1}{n} e_1 + \left(1 - \frac{1}{n}\right) e_n, \quad y_n = \frac{1}{n} e_1 - \left(1 - \frac{1}{n}\right) e_n$$

and consider $A = \{\theta\} \cup \{x_n \colon n = 2, 3, ...\} \cup \{y_n \colon n = 2, 3, ...\}$.

It is easy to see that A is a subset of the unit ball and consequently, $r(A) \leq 1$. Now,

$$\lim_{n\to\infty} \|x_n - y_n\| = \lim_{n\to\infty} 2\left(1 - \frac{1}{n}\right) \|e_n\| \geq 2.$$

Consequently, $\operatorname{diam}(A) \geq 2$. Since $r(A) \geq \frac{1}{2} \operatorname{diam}(A)$, we have $r(A) = 1$. So θ is a Chebyshev center of A. Finally we prove that θ is a farthest point of A from $u = e_1$. In fact, $\|e_1 - \theta\| = \|e_1\| = \frac{3}{2}$. On the other hand, for $n \geq 2$, an easy calculation shows that $\|e_1 - x_n\| = \|e_1 - y_n\| < \frac{3}{2}$. So θ is the farthest point of A from e_1.

Next, we prove that if A is a bounded centerable subset of a uniformly convex Banach space, then A is CCNF. Before doing this, let us recall one of the standard equivalent definitions of uniform convexity: a Banach space X is said to be *uniformly convex* if for every two sequences $\{x_n\}$, $\{y_n\}$ in B_X, the condition $\lim\limits_{n\to\infty} \|x_n + y_n\| = 2$ implies $\lim\limits_{n\to\infty} \|x_n - y_n\| = 0$.

***Theorem* 5.3** — *Let X be a uniformly convex Banach space. Then every nontrivial bounded centerable subset of X is CCNF.*

PROOF : Let c_A be the Chebyshev center of a nontrivial bounded centerable set A such that $c_A \in A$ and $r > 0$ be the Chebyshev radius of A. According to the definition of a centerable set, there are $u_n, v_n \in A$, $n = 1, 2, \ldots$ such that

$$\lim_{n\to\infty} \|u_n - v_n\| = 2r. \tag{5.2}$$

Consider elements

$$x_n = \frac{1}{r}(u_n - c_A), \quad y_n = \frac{1}{r}(c_A - v_n).$$

Then $x_n, y_n \in B_X$, $\lim_{n\to\infty} \|x_n+y_n\| = 2$, so the uniform convexity of X implies $\lim_{n\to\infty} \|x_n - y_n\| = 0$. This means that $\lim_{n\to\infty} \|u_n + v_n - 2c_A\| = 0$. In other words,

$$u_n + v_n \to 2c_A.$$

Suppose c_A is a farthest point of A from some $y \in X$. Denote $R = \|c_A - y\|$. Now, denote

$$\tilde{x}_n = \frac{1}{R}(u_n - y), \quad \tilde{y}_n = \frac{1}{R}(v_n - y).$$

Then $\tilde{x}_n, \tilde{y}_n \in B_X$,

$$\lim_{n\to\infty} \|\tilde{x}_n + \tilde{y}_n\| = \frac{1}{R} \lim_{n\to\infty} \|u_n + v_n - 2y\| = \frac{1}{R}\|2c_A - 2y\| = 2.$$

Again, the uniform convexity of X implies $\lim\limits_{n\to\infty} \|\tilde{x}_n - \tilde{y}_n\| = 0$, i.e., $\|u_n - v_n\| \to 0$, which contradicts (5.2). This contradiction completes the proof of the theorem. □

Since in the finite-dimensional case, strict convexity implies uniform convexity, it is possible to obtain the following characterization of finite-dimensional strictly convex Banach spaces, simply by observing that any straight line segment in a Banach space is always a centerable set.

***Theorem* 5.4** — *A finite-dimensional Banach space X is strictly convex if and only if every nontrivial bounded centerable subset of X is CCNF.*

Remark 5.5 : Example 5.2 shows that the uniform convexity condition in Theorem 5.3 cannot be substituted by strict convexity.

6. M-COMPACT CENTERABLE CCNF SETS

In this section, we prove that if A is a bounded centerable M-compact subset of a strictly convex Banach space, then A is CCNF. Before proving the theorem, we first prove the following lemma:

Lemma 6.1 — Let X be a Banach space. Let A be any nontrivial bounded centerable M-compact subset of X, containing its Chebyshev center c_A. Then A attains its diameter.

PROOF : Since in our case,

$$\operatorname{diam}(A) = \sup_{a,b \in A} \|a - b\| = 2r(A),$$

there exist sequences $\{x_n\},\ \{y_n\} \subset A$ such that $\|x_n - y_n\| \to 2r(A)$. We claim that $\{x_n\}$ is a maximizing sequence in A for c_A. If not, then there exists $\varepsilon_0 > 0$ and a subsequence $\{x_{n_k}\}$ such that $\|c_A - x_{n_k}\| \leq r(A) - \varepsilon_0$. Then,

$$\begin{aligned} \|x_{n_k} - y_{n_k}\| &= \|(x_{n_k} - c_A) + (c_A - y_{n_k})\| \\ &\leq \|(x_{n_k} - c_A)\| + \|(c_A - y_{n_k})\| \\ &\leq r(A) - \varepsilon_0 + r(A) = 2r(A) - \varepsilon_0, \end{aligned}$$

which contradicts the fact that $\|x_n - y_n\| \to 2r(A)$. By the same argument, $\{y_n\}$ is a maximizing sequence in A for c_A. Consequently, as A is M-compact, there is a subsequence $\{n_k\} \subset \mathbb{N}$ and there are $\tilde{x}, \tilde{y} \in A$ such that $x_{n_k} \to \tilde{x}$ and $y_{n_k} \to \tilde{y}$. Then

$$\operatorname{diam}(A) = \lim_{n \to \infty} \|x_n - y_n\| = \lim_{k \to \infty} \|x_{n_k} - y_{n_k}\| = \|\tilde{x} - \tilde{y}\|.$$

Thus diameter of A is attained. □

We now prove the desired theorem.

Theorem 6.2 — *Let X be a strictly convex Banach space. Then every nontrivial bounded centerable M-compact subset A of X is CCNF.*

PROOF : Suppose A is CCF. Then $c_A \in \operatorname{Far} A$. By the definition, there exists $x \in X$ such that $c_A \in F(x, A)$. Denote

$$R = \|x - c_A\| = \sup_{a \in A} \|x - a\|.$$

Due to Lemma 6.1, $\operatorname{diam}(A)$ is attained and since A is centerable, $\operatorname{diam}(A) = 2r(A)$. This means that there exist $a_1, a_2 \in A$ such that

$$\|a_1 - a_2\| = \sup_{a,b \in A} \|a - b\| = 2r(A). \tag{6.1}$$

We claim that $\|c_A - a_1\| = \|c_A - a_2\| = r(A)$. Clearly $\|c_A - a_1\| \leq r(A)$ and $\|c_A - a_2\| \leq r(A)$. Moreover, the assumption that one of them is strictly smaller than $r(A)$ leads to a contradiction:

$$2r(A) = \|a_1 - a_2\| = \|a_1 - c_A + c_A - a_2\| \leq \|a_1 - c_A\| + \|a_2 - c_A\| < 2r(A).$$

So, the claim is proved. Now,

$$\left\|\frac{1}{2}\left((a_1 - c_A) + (c_A - a_2)\right)\right\| = r(A).$$

Geometrically this means that $a_1 - c_A, c_A - a_2$ and $\frac{1}{2}\left((a_1 - c_A) + (c_A - a_2)\right)$ belong to the same sphere $r(A)S_X$. By the strict convexity of X, it follows that $a_1 - c_A = c_A - a_2$, i.e., $c_A = \frac{1}{2}(a_1 + a_2)$. The following chain of inequalities

$$\begin{aligned} R &= \|x - c_A\| = \Big\|\frac{1}{2}\big((x - a_1) + (x - a_2)\big)\Big\| \\ &\leq \frac{1}{2}\big\|x - a_1\big\| + \frac{1}{2}\big\|x - a_2\big\| \leq \sup_{a \in A} \|x - a\| = R \end{aligned}$$

implies that all of them are equalities, i.e., all three vectors $x - a_1$, $x - a_2$, and $\frac{1}{2}\big((x - a_1) + (x - a_2)\big)$ belong to the same sphere RS_X. Then, the strict convexity of X implies that $x - a_1 = x - a_2$, i.e., $a_1 = a_2$. This contradiction with (6.1) completes the proof of the theorem. □

Remark 6.3 : Example 5.2 shows that the M-compactness condition in Theorem 6.2 cannot be removed.

Now, we can give a characterization of strictly convex Banach spaces, simply by observing that any closed straight line segment in a Banach space is always a centerable and M-compact set. Thus, we have the following theorem :

***Theorem* 6.4** — *A Banach space X is strictly convex if and only if every nontrivial bounded centerable and M-compact subset of X is CCNF.*

Remark 6.5 : Theorem 6.2 shows that the uniform convexity condition in Theorem 5.3 can be substituted by strict convexity if we impose an additional condition of M-compactness on the subset A of X.

We would like to add a final comment that Theorem 3.3, Theorem 5.4 and Theorem 6.4 together yield a nice step by step characterization of strict convexity of a Banach space. The characterizing properties follow an interesting trend, depending on the dimension of the space. Accordingly, we state the following theorem as the final result of this section:

***Theorem* 6.6** — *Let X be a Banach space. Then the following holds.*

(a) If X is a two-dimensional real Banach space, then X is strictly convex if and only if every nontrivial bounded subset of X is CCNF.

(b) If X is a finite-dimensional Banach space, then X is strictly convex if and only if every nontrivial bounded centerable subset of X is CCNF.

(c) If X is any Banach space, then X is strictly convex if and only if every nontrivial bounded centerable and M-compact subset of X is CCNF.

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