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LAPLACIAN SPECTRUM OF NON-COMMUTING GRAPHS OF FINITE GROUPS

Parama Dutta, Jutirekha Dutta and Rajat Kanti Nath

Department of Mathematical Sciences, Tezpur University, Napaam 784 028,
Sonitpur, Assam, India
e-mails: parama@gonitsora.com, jutirekhadutta@yahoo.com;
rajatkantinath@yahoo.com

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In this paper, we compute the Laplacian spectrum of non-commuting graphs of some classes of finite non-abelian groups. Our computations reveal that the non-commuting graphs of all the groups considered in this paper are L-integral. We also obtain some conditions on a group so that its non-commuting graph is L-integral.

Key words : Non-commuting graph; spectrum; L-integral graph; finite group.

1. INTRODUCTION

Let G be a finite group with centre $Z(G)$. The non-commuting graph of a non-abelian group G, denoted by $\mathcal{A}_G$, is a simple undirected graph whose vertex set is $G \setminus Z(G)$ and two vertices x and y are adjacent if and only if $xy \neq yx$. Various aspects of non-commuting graphs of different finite groups can be found in [1, 4, 8, 12, 23]. In [12], Elvierayani and Abdussakir have computed the Laplacian spectrum of the non-commuting graph of dihedral groups D_{2m} where m is odd and suggested to consider the case when m is even. In this paper, we compute the Laplacian spectrum of the non-commuting graph of D_{2m} for any $m \geq 3$ using a different method. Our method also enables to compute the Laplacian spectrum of the non-commuting graphs of several well-known families of finite non-abelian groups such as the quasidihedral groups, generalized quaternion groups, some projective special linear groups, general linear groups etc. In a separate paper [11], we study the Laplacian energy of non-commuting graphs of the groups considered in this paper.

For a graph $\mathcal{G}$ we write $\overline{\mathcal{G}}$ and $V(\mathcal{G})$ to denote the complement of $\mathcal{G}$ and the set of vertices of $\mathcal{G}$ respectively. Let $A(\mathcal{G})$ and $D(\mathcal{G})$ denote the adjacency matrix and degree matrix of a graph $\mathcal{G}$

respectively. Then the Laplacian matrix of $\mathcal{G}$ is given by $L(\mathcal{G}) = D(\mathcal{G}) - A(\mathcal{G})$. We write L-spec$(\mathcal{G})$ to denote the Laplacian spectrum of $\mathcal{G}$ and L-spec$(\mathcal{G}) = \{\alpha_1^{a_1}, \alpha_2^{a_2}, \dots, \alpha_n^{a_n}\}$ where $\alpha_1 < \alpha_2 < \cdots < \alpha_n$ are the eigenvalues of $L(\mathcal{G})$ with multiplicities $a_1, a_2, \dots, a_n$ respectively. The Laplacian spectrum of graphs plays an important role in solving many Physical and Chemical problems. The reader may conf. [18, Section 5] for details. A graph $\mathcal{G}$ is called L-integral if L-spec $(\mathcal{G})$ contains only integers. As a consequence of our results, it follows that the non-commuting graphs of all the groups considered in this paper are L-integral. It is worth mentioning that L-integral graphs are studied extensively in [3, 15, 17].

2. Preliminary Results

It is well-known that L-spec$(K_n) = \{0^1, n^{n-1}\}$ where K_n denotes the complete graph on n vertices. Further, we have the following results.

***Theorem* 2.1** — *If $\mathcal{G} = l_1K_{m_1} \sqcup l_2K_{m_2} \sqcup \cdots \sqcup l_kK_{m_k}$, where $l_iK_{m_i}$ denotes the disjoint union of l_i copies of K_{m_i} for $1 \le i \le k$ and $m_1 < m_2 < \cdots < m_k$, then*

$$\text{L-spec}(\mathcal{G}) = \left\{0^{\sum_{i=1}^{k} l_i}, m_1^{l_1(m_1-1)}, m_2^{l_2(m_2-1)}, \dots, m_k^{l_k(m_k-1)}\right\}.$$

***Theorem* 2.2** — *[18, Theorem 3.6]. Let $\mathcal{G}$ be a graph such that* L-spec$(\mathcal{G}) = \{\alpha_1^{a_1}, \alpha_2^{a_2}, \dots, \alpha_n^{a_n}\}$ *then* L-spec$(\overline{\mathcal{G}})$ *is given by*

$$\{0, (|V(\mathcal{G})| - \alpha_n)^{a_n}, (|V(\mathcal{G})| - \alpha_{n-1})^{a_{n-1}}, (|V(\mathcal{G})| - \alpha_{n-2})^{a_{n-2}}, \dots, (|V(\mathcal{G})| - \alpha_1)^{a_1 - 1}\}.$$

As a corollary of the above two theorems we have the following result.

Corollary 2.3 — If $\mathcal{G} = l_1K_{m_1} \sqcup l_2K_{m_2} \sqcup \cdots \sqcup l_kK_{m_k}$, where $l_iK_{m_i}$ denotes the disjoint union of l_i copies of K_{m_i} for $1 \le i \le k$ and $m_1 < m_2 < \cdots < m_k$, then

$$\begin{aligned}\text{L-spec}(\overline{\mathcal{G}}) = \{0, &\left(\sum_{i=1}^{k} l_i m_i - m_k\right)^{l_k(m_k-1)}, \left(\sum_{i=1}^{k} l_i m_i - m_{k-1}\right)^{l_{k-1}(m_{k-1}-1)}, \\ &\dots, \left(\sum_{i=1}^{k} l_i m_i - m_1\right)^{l_1(m_1-1)}, \left(\sum_{i=1}^{k} l_i m_i\right)^{\sum_{i=1}^{k} l_i - 1}\}.\end{aligned}$$

A group G is called a CA-group if $C_G(x)$ is abelian for all $x \in G \setminus Z(G)$. Various aspects of CA-groups can be found in [1, 10, 21]. The following result gives the Laplacian spectrum of the non-commuting graph of a finite non-abelian CA-group.

***Theorem* 2.4** — *Let G be a finite non-abelian CA-group. Then*

$$\begin{aligned}\text{L-spec}(\mathcal{A}_G) = &\{0, (|G| - |X_n|)^{|X_n|-|Z(G)|-1}, \ldots, \\ &(|G| - |X_1|)^{|X_1|-|Z(G)|-1}, (|G| - |Z(G)|)^{n-1}\}.\end{aligned}$$

where $X_1, \ldots, X_n$ are the distinct centralizers of non-central elements of G such that $|X_1| \leq \cdots \leq |X_n|$.

PROOF : Let G be a finite non-abelian CA-group and $X_i = C_G(x_i)$ where $x_i \in G \setminus Z(G)$ and $1 \leq i \leq n$. Let $x, y \in X_i \setminus Z(G)$ for some i and $x \neq y$ then, since G is a CA-group, there is an edge between x and y in $\overline{\mathcal{A}_G}$. Suppose that $x \in (X_i \cap X_j) \setminus Z(G)$ for some $1 \leq i \neq j \leq n$. Then $[x, x_i] = 1$ and $[x, x_j] = 1$. Let $s \in C_G(x)$ then $[s, x_i] = 1$ since $x_i \in C_G(x)$ and G is a CA-group. Therefore, $s \in C_G(x_i)$ and so $C_G(x) \subseteq C_G(x_i)$. Again, let $t \in C_G(x_i)$ then $[t, x] = 1$ since $x \in C_G(x_i)$ and G is a CA-group. Therefore, $t \in C_G(x)$ and so $C_G(x_i) \subseteq C_G(x)$. Thus $C_G(x) = C_G(x_i)$. Similarly, it can be seen that $C_G(x) = C_G(x_j)$, which is a contradiction. Therefore, $X_i \cap X_j = Z(G)$ for any $1 \leq i \neq j \leq n$. This shows that

$$\overline{\mathcal{A}_G} = \bigsqcup_{i=1}^{n} K_{|X_i|-|Z(G)|}. \tag{2.1}$$

Therefore, by Corollary 2.3, we have

$$\begin{aligned}\text{L-spec}(\mathcal{A}_G) = \{0, &\left(\sum_{i=1}^{n}(|X_i| - |Z(G)|) - (|X_n| - |Z(G)|)\right)^{|X_n|-|Z(G)|-1}, \ldots, \\ &\left(\sum_{i=1}^{n}(|X_i| - |Z(G)|) - (|X_1| - |Z(G)|)\right)^{|X_1|-|Z(G)|-1}, \left(\sum_{i=1}^{n}(|X_i| - |Z(G)|)\right)^{n-1}\}.\end{aligned}$$

Hence, the result follows noting that $\sum\limits_{i=1}^{n}(|X_i| - |Z(G)|) = |G| - |Z(G)|$. □

Corollary 2.5 — Let G be a finite non-abelian CA-group and A be any finite abelian group. Then

$$\begin{aligned}\text{L-spec}(\mathcal{A}_{G\times A}) = &\{0, (|A|(|G| - |X_n|))^{|A|(|X_n|-|Z(G)|)-1}, \ldots, \\ &(|A|(|G| - |X_1|))^{|A|(|X_1|-|Z(G)|)-1}, (|A|(|G| - |Z(G)|))^{n-1}\}.\end{aligned}$$

where $X_1, \ldots, X_n$ are the distinct centralizers of non-central elements of G such that $|X_1| \leq \cdots \leq |X_n|$.

PROOF : It is easy to see that $G \times A$ is a CA-group and $X_1 \times A, X_2 \times A, \ldots, X_n \times A$ are the distinct centralizers of non-central elements of $G \times A$. Hence, the result follows from Theorem 2.4 noting that $Z(G \times A) = Z(G) \times A$. □

3. Groups With Given Central Factors

In this section, we compute the Laplacian spectrum of the non-commuting graphs of some families of finite non-abelian groups whose central factors are some well-known finite groups. We begin with the following result.

***Theorem* 3.1** — *Let G be a finite group and $\frac{G}{Z(G)} \cong Sz(2)$, where $Sz(2)$ is the Suzuki group presented by $\langle a, b : a^5 = b^4 = 1, b^{-1}ab = a^2\rangle$. Then*

$$\text{L-spec}(\mathcal{A}_G) = \{0, (15|Z(G)|)^{4|Z(G)|-1}, (16|Z(G)|)^{15|Z(G)|-5}, (19|Z(G)|)^5\}.$$

PROOF : We have

$$\frac{G}{Z(G)} = \langle aZ(G), bZ(G) : a^5Z(G) = b^4Z(G) = Z(G), b^{-1}abZ(G) = a^2Z(G)\rangle.$$

Observe that

$$\begin{aligned}
C_G(ab) &= Z(G) \sqcup abZ(G) \sqcup a^4b^2Z(G) \sqcup a^3b^3Z(G),\\
C_G(a^2b) &= Z(G) \sqcup a^2bZ(G) \sqcup a^3b^2Z(G) \sqcup ab^3Z(G),\\
C_G(a^2b^3) &= Z(G) \sqcup a^2b^3Z(G) \sqcup ab^2Z(G) \sqcup a^4bZ(G),\\
C_G(b) &= Z(G) \sqcup bZ(G) \sqcup b^2Z(G) \sqcup b^3Z(G),\\
C_G(a^3b) &= Z(G) \sqcup a^3bZ(G) \sqcup a^2b^2Z(G) \sqcup a^4b^3Z(G) \quad \text{and}\\
C_G(a) &= Z(G) \sqcup aZ(G) \sqcup a^2Z(G) \sqcup a^3Z(G) \sqcup a^4Z(G)
\end{aligned}$$

are the only centralizers of non-central elements of G. Also note that these centralizers are abelian subgroups of G. Thus G is a CA-group.

We have $|C_G(a)| = 5|Z(G)|$ and

$$|C_G(ab)| = |C_G(a^2b)| = |C_G(a^2b^3)| = |C_G(b)| = |C_G(a^3b)| = 4|Z(G)|.$$

Therefore, by Theorem 2.4, the result follows. □

***Theorem* 3.2** — *Let G be a finite group such that $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$, where p is a prime integer. Then*

$$\text{L-spec}(\mathcal{A}_G) = \{0, ((p^2-p)|Z(G)|)^{(p^2-1)|Z(G)|-p-1}, ((p^2-1)|Z(G)|)^p\}.$$

PROOF : Let $|Z(G)| = n$. Since $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$ we have $\frac{G}{Z(G)} = \langle aZ(G), bZ(G) : a^p, b^p, aba^{-1}b^{-1} \in Z(G)\rangle$, where $a, b \in G$ with $ab \neq ba$. Then for any $z \in Z(G)$, we have

$$\begin{aligned}
C_G(a) = C_G(a^iz) &= Z(G) \sqcup aZ(G) \sqcup \cdots \sqcup a^{p-1}Z(G) \text{ for } 1 \leq i \leq p-1,\\
C_G(a^jb) = C_G(a^jbz) &= Z(G) \sqcup a^jbZ(G) \sqcup \cdots \sqcup a^{(p-1)j}b^{p-1}Z(G) \text{ for } 1 \leq j \leq p.
\end{aligned}$$

These are the only centralizers of non-central elements of G. Also note that these centralizers are abelian subgroups of G. Therefore, G is a CA-group. We have $|C_G(a)| = |C_G(a^j b)| = pn$ for $1 \leq j \leq p$. Hence, the result follows from Theorem 2.4. □

As a corollary we have the following result.

Corollary 3.3 — Let G be a non-abelian group of order p^3, for any prime p, then

$$\text{L-spec}(\mathcal{A}_G) = \{0, (p^3 - p^2)^{p^3-2p-1}, (p^3 - p)^p\}.$$

PROOF : Note that $|Z(G)| = p$ and $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$. Hence the result follows from Theorem 3.2. □

***Theorem* 3.4** — *Let G be a finite group such that $\frac{G}{Z(G)} \cong D_{2m}$, for $m \geq 2$. Then*

$$\begin{aligned}\text{L-spec}(\mathcal{A}_G) =&\{0, (m|Z(G)|)^{(m-1)|Z(G)|-1}, (2(m-1)|Z(G)|)^{m|Z(G)|-m},\\ &((2m-1)|Z(G)|)^m\}.\end{aligned}$$

PROOF : Since $\frac{G}{Z(G)} \cong D_{2m}$ we have $\frac{G}{Z(G)} = \langle xZ(G), yZ(G) : x^2, y^m, xyx^{-1}y \in Z(G)\rangle$, where $x, y \in G$ with $xy \neq yx$. It is not difficult to see that for any $z \in Z(G)$,

$$C_G(xy^j) = C_G(xy^j z) = Z(G) \sqcup xy^j Z(G), 1 \leq j \leq m$$

and

$$C_G(y) = C_G(y^i z) = Z(G) \sqcup yZ(G) \sqcup \cdots \sqcup y^{m-1}Z(G), 1 \leq i \leq m-1$$

are the only centralizers of non-central elements of G. Also note that these centralizers are abelian subgroups of G. Therefore, G is a CA-group. We have $|C_G(x^j y)| = 2n$ for $1 \leq j \leq m$ and $|C_G(y)| = mn$, where $|Z(G)| = n$. Hence, the result follows from Theorem 2.4. □

Using Theorem 3.4, we now compute the Laplacian spectrum of the non-commuting graphs of the groups M_{2mn}, D_{2m} and Q_{4n} respectively.

Corollary 3.5 — Let $M_{2mn} = \langle a, b : a^m = b^{2n} = 1, bab^{-1} = a^{-1}\rangle$ be a metacyclic group, where $m > 2$. Then L-spec$(\mathcal{A}_{M_{2mn}})$

$$= \begin{cases} \{0, (mn)^{mn-n-1}, (2mn-2n)^{mn-m}, (2mn-n)^m\} & \text{if } m \text{ is odd} \\ \{0, (mn)^{mn-2n-1}, (2mn-4n)^{mn-\frac{m}{2}}, (2mn-2n)^{\frac{m}{2}}\} & \text{if } m \text{ is even.} \end{cases}$$

PROOF : Observe that $Z(M_{2mn}) = \langle b^2\rangle$ or $\langle b^2\rangle \cup a^{\frac{m}{2}}\langle b^2\rangle$ according as m is odd or even. Also, it is easy to see that $\frac{M_{2mn}}{Z(M_{2mn})} \cong D_{2m}$ or D_m according as m is odd or even. Hence, the result follows from Theorem 3.4. □

As a corollary to the above result we have the following result.

Corollary 3.6 — Let $D_{2m} = \langle a, b : a^m = b^2 = 1, bab^{-1} = a^{-1}\rangle$ be the dihedral group of order $2m$, where $m > 2$. Then

$$\text{L-spec}(\mathcal{A}_{D_{2m}}) = \begin{cases} \{0, m^{m-2}, (2m-1)^m\} & \text{if } m \text{ is odd} \\ \{0, m^{m-3}, (2m-4)^{\frac{m}{2}}, (2m-2)^{\frac{m}{2}}\} & \text{if } m \text{ is even.} \end{cases}$$

Corollary 3.7 — Let $Q_{4n} = \langle x, y : y^{2n} = 1, x^2 = y^n, xyx^{-1} = y^{-1}\rangle$, where $n \geq 2$, be the generalized quaternion group of order $4n$. Then

$$\text{L-spec}(\mathcal{A}_{Q_{4n}}) = \{0, (2n)^{2n-3}, (4n-4)^n, (4n-2)^n\}.$$

PROOF : The result follows from Theorem 3.4 noting that $Z(Q_{4n}) = \{1, a^n\}$ and $\frac{Q_{4n}}{Z(Q_{4n})} \cong D_{2n}$. □

4. SOME WELL-KNOWN GROUPS

In this section, we compute the Laplacian spectrum of the non-commuting graphs of some well-known families of finite groups. We begin with the family of finite groups having order pq where p and q are primes.

Proposition 4.1 — Let G be a non-abelian group of order pq, where p and q are primes with $p \mid (q-1)$. Then

$$\text{L-spec}(\mathcal{A}_G) = \{0, (pq-q)^{q-2}, (pq-p)^{pq-2q}, (pq-1)^q\}.$$

PROOF : It is easy to see that $|Z(G)| = 1$ and G is a CA-group. Also the centralizers of non-central elements of G are precisely the Sylow subgroups of G. The number of Sylow q-subgroups and Sylow p-subgroups of G are one and q respectively. Hence, the result follows from Theorem 2.4. □

Proposition 4.2 — The Laplacian spectrum of the non-commuting graph of the quasidihedral group $QD_{2^n} = \langle a, b : a^{2^{n-1}} = b^2 = 1, bab^{-1} = a^{2^{n-2}-1}\rangle$, where $n \geq 4$, is given by

$$\text{L-spec}(\mathcal{A}_{QD_{2^n}}) = \{0, (2^{n-1})^{2^{n-1}-3}, (2^n-4)^{2^{n-2}}, (2^n-2)^{2^{n-2}}\}.$$

PROOF : It is well-known that $Z(QD_{2^n}) = \{1, a^{2^{n-2}}\}$. Also

$$C_{QD_{2^n}}(a) = C_{QD_{2^n}}(a^i) = \langle a\rangle \text{ for } 1 \leq i \leq 2^{n-1}-1, i \neq 2^{n-2}$$

and

$$C_{QD_{2^n}}(a^j b) = \{1, a^{2^{n-2}}, a^j b, a^{j+2^{n-2}} b\} \text{ for } 1 \leq j \leq 2^{n-2}$$

are the only centralizers of non-central elements of QD_{2^n}. Note that these centralizers are abelian subgroups of QD_{2^n}. Therefore, QD_{2^n} is a CA-group. We have $|C_{QD_{2^n}}(a)| = 2^{n-1}$ and $|C_{QD_{2^n}}(a^j b)| = 4$ for $1 \leq j \leq 2^{n-2}$. Hence, the result follows from Theorem 2.4. □

Proposition 4.3 — The Laplacian spectrum of the non-commuting graph of the projective special linear group $PSL(2, 2^k)$, where $k \geq 2$, is given by

$$\begin{aligned} \text{L-spec}(\mathcal{A}_{PSL(2,2^k)}) = \{&0, (2^{3k} - 2^{k+1} - 1)^{2^{3k-1} - 2^{2k} + 2^{k-1}}, (2^{3k} - 2^{k+1})^{2^{2k} - 2^k - 2}, \\ &(2^{3k} - 2^{k+1} + 1)^{2^{3k-1} - 2^{2k} - 3.2^{k-1}}, (2^{3k} - 2^k - 1)^{2^{2k} + 2^k}\}. \end{aligned}$$

PROOF : We know that $PSL(2, 2^k)$ is a non-abelian group of order $2^k(2^{2k} - 1)$ with trivial center. By Proposition 3.21 of [1], the set of centralizers of non-trivial elements of $PSL(2, 2^k)$ is given by

$$\{xPx^{-1}, xAx^{-1}, xBx^{-1} : x \in PSL(2, 2^k)\}$$

where P is an elementary abelian 2-subgroup and A, B are cyclic subgroups of $PSL(2, 2^k)$ having order $2^k, 2^k - 1$ and $2^k + 1$ respectively. Also the number of conjugates of P, A and B in $PSL(2, 2^k)$ are $2^k + 1, 2^{k-1}(2^k + 1)$ and $2^{k-1}(2^k - 1)$ respectively. Note that $PSL(2, 2^k)$ is a CA-group and so, by (2.1), we have

$$\overline{\mathcal{A}_{PSL(2,2^k)}} = (2^k + 1)K_{|xPx^{-1}|-1} \sqcup 2^{k-1}(2^k + 1)K_{|xAx^{-1}|-1} \sqcup 2^{k-1}(2^k - 1)K_{|xBx^{-1}|-1}.$$

That is, $\overline{\mathcal{A}_{PSL(2,2^k)}} = (2^k + 1)K_{2^k - 1} \sqcup 2^{k-1}(2^k + 1)K_{2^k - 2} \sqcup 2^{k-1}(2^k - 1)K_{2^k}$. Hence, the result follows from Corollary 2.3. □

Proposition 4.4 — The Laplacian spectrum of the non-commuting graph of the general linear group $GL(2, q)$, where $q = p^n > 2$ and p is a prime integer, is given by

$$\begin{aligned} \text{L-spec}(\mathcal{A}_{GL(2,q)}) = \{&0, (q^4 - q^3 - 2q^2 + q + 1)^{\frac{q^4 - 2q^3 + q}{2}}, (q^4 - q^3 - 2q^2 + 2q)^{q^3 - q^2 - 2q}, \\ &(q^4 - q^3 - 2q^2 + 3q - 1)^{\frac{q^4 - 2q^3 - 2q^2 + q}{2}}, (q^4 - q^3 - q^2 + 1)^{q^2 + q}\}. \end{aligned}$$

PROOF : We have $|GL(2, q)| = (q^2 - 1)(q^2 - q)$ and $|Z(GL(2, q))| = q - 1$. By Proposition 3.26 of [1], the set of centralizers of non-central elements of $GL(2, q)$ is given by

$$\{xDx^{-1}, xIx^{-1}, xPZ(GL(2, q))x^{-1} : x \in GL(2, q)\}$$

where D is the subgroup of $GL(2,q)$ consisting of all diagonal matrices, I is a cyclic subgroup of $GL(2,q)$ having order q^2-1 and P is the Sylow p-subgroup of $GL(2,q)$ consisting of all upper triangular matrices with 1 in the diagonal. The orders of D and $PZ(GL(2,q))$ are $(q-1)^2$ and $q(q-1)$ respectively. Also the number of conjugates of D, I and $PZ(GL(2,q))$ in $GL(2,q)$ are $\frac{q(q+1)}{2}$, $\frac{q(q-1)}{2}$ and $q+1$ respectively. Since $GL(2,q)$ is a CA-group (see Lemma 3.5 of [1]), by (2.1), we have $\overline{\mathcal{A}_{GL(2,q)}} =$

$$\frac{q(q+1)}{2}K_{|xDx^{-1}|-q+1} \sqcup \frac{q(q-1)}{2}K_{|xIx^{-1}|-q+1} \sqcup (q+1)K_{|xPZ(GL(2,q))x^{-1}|-q+1}.$$

That is, $\overline{\mathcal{A}_{GL(2,q)}} = \frac{q(q+1)}{2}K_{q^2-3q+2} \sqcup \frac{q(q-1)}{2}K_{q^2-q} \sqcup (q+1)K_{q^2-2q+1}$. Hence, the result follows from Corollary 2.3. □

Proposition 4.5 — Let $F = GF(2^n), n \geq 2$ and ϑ be the Frobenius automorphism of F, that is, $\vartheta(x) = x^2$ for all $x \in F$. Then the Laplacian spectrum of the non-commuting graph of the group

$$A(n,\vartheta) = \left\{ U(a,b) = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & \vartheta(a) & 1 \end{bmatrix} : a, b \in F \right\}$$

under matrix multiplication given by $U(a,b)U(a',b') = U(a+a', b+b'+a'\vartheta(a))$ is

$$\text{L-spec}(\mathcal{A}_{A(n,\vartheta)}) = \{0, (2^{2n}-2^{n+1})^{(2^n-1)^2}, (2^{2n}-2^n)^{2^n-2}\}.$$

PROOF : Note that $Z(A(n,\vartheta)) = \{U(0,b) : b \in F\}$ and so $|Z(A(n,\vartheta))| = 2^n$. Let $U(a,b)$ be a non-central element of $A(n,\vartheta)$. It can be seen that the centralizer of $U(a,b)$ in $A(n,\vartheta)$ is $Z(A(n,\vartheta)) \sqcup U(a,0)Z(A(n,\vartheta))$. Clearly $A(n,\vartheta)$ is a CA-group and so, by (2.1), we have $\overline{\mathcal{A}_{A(n,\vartheta)}} = (2^n-1)K_{2^n}$. Hence the result follows from Corollary 2.3. □

Proposition 4.6 — Let $F = GF(p^n)$, p be a prime. Then the Laplacian spectrum of the non-commuting graph of the group

$$A(n,p) = \left\{ V(a,b,c) = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix} : a, b, c \in F \right\}$$

under matrix multiplication $V(a,b,c)V(a',b',c') = V(a+a', b+b'+ca', c+c')$ is

$$\text{L-spec}(\mathcal{A}_{A(n,p)}) = \{0, (p^{3n}-p^{2n})^{p^{3n}-2p^n-1}, (p^{3n}-p^n)^{p^n}\}.$$

PROOF : We have $Z(A(n,p)) = \{V(0,b,0) : b \in F\}$ and so $|Z(A(n,p))| = p^n$. The centralizers of non-central elements of $A(n,p)$ are given below

(i) If $b, c \in F$ and $c \neq 0$ then the centralizer of $V(0, b, c)$ in $A(n, p)$ is $\{V(0, b', c') : b', c' \in F\}$ having order p^{2n}.

(ii) If $a, b \in F$ and $a \neq 0$ then the centralizer of $V(a, b, 0)$ in $A(n, p)$ is $\{V(a', b', 0) : a', b' \in F\}$ having order p^{2n}.

(iii) If $a, b, c \in F$ and $a \neq 0, c \neq 0$ then the centralizer of $V(a, b, c)$ in $A(n, p)$ is $\{V(a', b', ca'a^{-1}) : a', b' \in F\}$ having order p^{2n}.

It can be seen that all the centralizers of non-central elements of $A(n, p)$ are abelian. Hence $A(n, p)$ is a CA-group and so, by (2.1), we have

$$\overline{\mathcal{A}_{A(n,p)}} = K_{p^{2n}-p^n} \sqcup K_{p^{2n}-p^n} \sqcup (p^n - 1)K_{p^{2n}-p^n} = (p^n + 1)K_{p^{2n}-p^n}.$$

Hence the result follows from Corollary 2.3. □

We would like to mention here that the groups considered in Proposition 4.5-4.6 are constructed by Hanaki (see [14]). These groups are also considered in [5], in order to compute their numbers of distinct centralizers.

5. Some Consequences

Note that the non-commuting graphs of all the groups considered in Section 3 and 4 are L-integral. In this section, we determine some conditions on G so that its non-commuting graph becomes L-integral.

A finite group is called an n-centralizer group if it has n numbers of distinct element centralizers. It is clear that 1-centralizer groups are precisely the abelian groups. There are no $2, 3$-centralizer finite groups. The study of these groups was initiated by Belcastro and Sherman [6] in the year 1994. We have the following results regarding n-centralizer groups.

Proposition 5.1 — If G is a finite 4-centralizer group then $\mathcal{A}_G$ is L-integral.

Proof : Let G be a finite 4-centralizer group. Then, by [6, Theorem 2], we have $\frac{G}{Z(G)} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Therefore, by Theorem 3.2, we have

$$\text{L-spec}(\mathcal{A}_G) = \{0, (2|Z(G)|)^{3|Z(G)|-3}, (3|Z(G)|)^2\}.$$

Hence, $\mathcal{A}_G$ is L-integral. □

Further, we have the following result.

Proposition 5.2 — If G is a finite $(p+2)$-centralizer p-group for any prime p, then $\mathcal{A}_G$ is L-integral.

PROOF : Let G be a finite $(p+2)$-centralizer p-group. Then, by [5, Lemma 2.7], we have $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$. Therefore, by Theorem 3.2, we have

$$\text{L-spec}(\mathcal{A}_G) = \{0, ((p^2-p)|Z(G)|)^{(p^2-1)|Z(G)|-p-1}, ((p^2-1)|Z(G)|)^{p}\}.$$

Hence, $\mathcal{A}_G$ is L-integral. □

Proposition 5.3 — If G is a finite 5-centralizer group then $\mathcal{A}_G$ is L-integral.

PROOF : Let G be a finite 5-centralizer group. Then by [6, Theorem 4] we have $\frac{G}{Z(G)} \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ or D_6. Now, if $\frac{G}{Z(G)} \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ then by Theorem 3.2 we have $\text{L-spec}(\mathcal{A}_G) = \{0, (6|Z(G)|)^{8|Z(G)|-4}, (8|Z(G)|)^3\}$ and hence $\mathcal{A}_G$ is L-integral. If $\frac{G}{Z(G)} \cong D_6$ then, by Theorem 3.4, we have

$$\text{L-spec}(\mathcal{A}_G) = \{0, (3|Z(G)|)^{2|Z(G)|-1}, (4|Z(G)|)^{3|Z(G)|-3}, (5|Z(G)|)^3\}$$

and hence $\mathcal{A}_G$ is L-integral. Therefore, the result follows. □

We also have the following corollary.

Corollary 5.4 — Let G be a finite non-abelian group and $\{x_1, x_2, \ldots, x_r\}$ be a set of pairwise non-commuting elements of G having maximal size. Then $\mathcal{A}_G$ is L-integral if $r = 3, 4$.

PROOF : By Lemma 2.4 in [2], we have that G is a 4-centralizer or a 5-centralizer group according as $r = 3$ or 4. Hence the result follows from Proposition 5.1 and Proposition 5.3. □

The commuting probability of a finite group G denoted by $\Pr(G)$ is the probability that any two randomly chosen elements of G commute. Clearly, $\Pr(G) = 1$ if and only if G is abelian. The study of $\Pr(G)$ is originated from a paper of Erdös and Turán [13]. Various results on $\Pr(G)$ can be found in [7, 9, 19]. The following results show that $\mathcal{A}_G$ is L-integral if $\Pr(G)$ has some particular values.

Proposition 5.5 — If $\Pr(G) \in \{\frac{5}{14}, \frac{2}{5}, \frac{11}{27}, \frac{1}{2}, \frac{5}{8}\}$ then $\mathcal{A}_G$ is L-integral.

PROOF : If $\Pr(G) \in \{\frac{5}{14}, \frac{2}{5}, \frac{11}{27}, \frac{1}{2}, \frac{5}{8}\}$ then as shown in [22, pp. 246] and [20, pp. 451], we have $\frac{G}{Z(G)}$ is isomorphic to one of the groups in $\{D_{14}, D_{10}, D_8, D_6, \mathbb{Z}_2 \times \mathbb{Z}_2\}$. If $\frac{G}{Z(G)}$ is isomorphic to D_{14}, D_{10}, D_8 or D_6 then, by Theorem 3.4, it follows that $\mathcal{A}_G$ is L-integral. If $\frac{G}{Z(G)}$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ then, by Theorem 3.2, it follows that $\mathcal{A}_G$ is L-integral. Hence, the result follows. □

Proposition 5.6 — Let G be a finite group and p the smallest prime divisor of $|G|$. If $\Pr(G) = \frac{p^2+p-1}{p^3}$ then $\mathcal{A}_G$ is L-integral.

PROOF : If $\Pr(G) = \frac{p^2+p-1}{p^3}$ then by [16, Theorem 3] we have $\frac{G}{Z(G)}$ is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$. Now, by Theorem 3.2, it follows that $\mathcal{A}_G$ is L-integral. □

Proposition 5.7 — If G is a non-solvable group with $\Pr(G) = \frac{1}{12}$ then $\mathcal{A}_G$ is L-integral.

PROOF : By [7, Proposition 3.3.7], we have that G is isomorphic to $A_5 \times B$ for some abelian group B. Since A_5 is a CA-group, by Corollary 2.5, it follows that $\mathcal{A}_G$ is L-integral. □

A graph is called planar if it can be embedded in the plane so that no two edges intersect geometrically except at a vertex to which both are adjacent. We conclude this paper with the following result.

Proposition 5.8 — Let G be a finite group then $\mathcal{A}_G$ is L-integral if $\mathcal{A}_G$ is planar.

PROOF : It was shown in Proposition 2.3 of [1] that $\mathcal{A}_G$ is planar if and only if G is isomorphic to D_6, D_8 or Q_8. Therefore, by Corollary 3.6 and Corollary 3.7, the result follows. □

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