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HANKEL DETERMINANTS OF THE GENERALIZED FACTORIALS

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Denote $\langle x|d\rangle_n = x(x+d)(x+2d)\cdots(x+(n-1)d)$ for $n = 1, 2, \cdots$, and $\langle x|d\rangle_0 = 1$, where $\langle x|d\rangle_n$ is called the generalized factorial of x with increment d. In this paper, we present the evaluation of Hankel determinants of sequence of generalized factorials. The main tool used for the evaluation is the method based on exponential Riordan arrays. Furthermore, we provide Hankel determinant evaluations of the Eulerian polynomials and exponential polynomials.

Key words : Generalized factorial; Hankel determinant; exponential Riordan array; Eulerian polynomial; exponential polynomial.

1. INTRODUCTION

The nth Hankel matrix of a sequence $a = (a_n)_{n\geq 0}$ is the $(n+1)\times(n+1)$ matrix whose (i,j) entry is a_{i+j}. The nth Hankel determinant is the determinant of the corresponding Hankel matrix, that is

$$\det(a_{i+j})_{i,j=0}^{n} = \det\begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_n \\ a_1 & a_2 & a_3 & \cdots & a_{n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_n & a_{n+1} & a_{n+2} & \cdots & a_{2n} \end{pmatrix}.$$

The evaluation of Hankel determinants has attracted attention as one of the most interesting topics in the framework of the moment theory and orthogonal polynomials. In the literature (see [1, 2, 5, 6, 12-15, 18]), several techniques for the evaluation of many classes of Hankel determinants are described.

A classical result about the Hankel determinant of the factorial numbers $n!$ is (see [9, 15])

$$\det((i+j)!)_{i,j=0}^{n} = \prod_{i=0}^{n}(i!)^2. \tag{1.1}$$

The generalized factorials $\langle x|d\rangle_n$ are defined by

$$\langle x|d\rangle_n = x(x+d)(x+2d)\cdots(x+(n-1)d) \text{ for } n \geq 1, \text{ and } \langle x|d\rangle_0 = 1,$$

where x is a real number and d is a nonzero real number. Thus, $\langle x|1\rangle_n = x(x+1)(x+2)\cdots(x+n-1) = \langle x\rangle_n$ is the classical rising factorial, and $\langle x|-1\rangle_n = x(x-1)(x-2)\cdots(x-n+1) = (x)_n$ is the classical falling factorial. For more on the properties of generalized factorials, see [10, 11].

In this paper, we give generalizations of the result (1.1) by using the Riordan array method introduced in [16], and prove the following determinant evaluation.

***Theorem* 1.1** — *The nth Hankel determinant of the generalized factorials $\langle x|d\rangle_n$ is given by*

$$\det(\langle x|d\rangle_{i+j})_{i,j=0}^{n} = d^{\binom{n+1}{2}} \prod_{i=1}^{n} i! \prod_{i=1}^{n} \langle x|d\rangle_i, \tag{1.2}$$

$$\det(\langle x|d\rangle_{i+j+1})_{i,j=0}^{n} = d^{\binom{n+1}{2}} \prod_{i=1}^{n} i! \cdot \prod_{i=1}^{n+1} \langle x|d\rangle_i, \tag{1.3}$$

$$\det(\langle x|d\rangle_{i+j+2})_{i,j=0}^{n} = d^{\binom{n+2}{2}} \prod_{i=1}^{n+1} i! \prod_{i=1}^{n+1} \langle x|d\rangle_i \sum_{i=-1}^{n} \frac{\langle x|d\rangle_{i+1}}{d^{\binom{i+2}{2}}(i+1)!}. \tag{1.4}$$

Setting $x = 1$ and $d = 1$ in the identities gives the Hankel determinants of the factorial numbers (1.1), and

$$\det((i+j+1)!)_{i,j=0}^{n} = (n+1)! \prod_{i=1}^{n}(i!)^2, \tag{1.5}$$

$$\det((i+j+2)!)_{i,j=0}^{n} = (n+2) \prod_{i=1}^{n+1}(i!)^2, \tag{1.6}$$

$$\det((i+j+3)!)_{i,j=0}^{n} = \prod_{i=1}^{n+1} i! \cdot \prod_{i=1}^{n+1}(i+1)! \cdot \binom{n+3}{2}. \tag{1.7}$$

Our proof relies on the Hankel determinant of the first column of a special exponential Riordan array. In the next section, we will give a brief introduction to the notion of exponential Riordan arrays. Then we give an addition formula for the sequence of the first column of exponential Riordan arrays with tri-diagonal production matrix. By using this addition formula, we obtain the evaluation of three Hankel determinants of the corresponding sequence. In section 3, we will prove our main result. In section 4, we will provide Hankel determinant evaluations of the Eulerian polynomials and exponential polynomials.

2. EXPONENTIAL RIORDAN ARRAY

The concepts of Riordan array and exponential Riordan array, introduced by Shapiro *et al*. [3, 4, 8, 16], are very useful in the study of combinatorial identities and Hankel determinants of combinatorial sequences. An exponential Riordan array, denoted by $[g(t), f(t)]$, is an infinite lower-triangular matrix whose exponential generating function of the kth column is $\frac{1}{k!}g(t)f(t)^k$ for $k = 0, 1, \cdots$, where $g(t) = 1 + \sum_{n=1}^{\infty} g_n \frac{t^n}{n!}$ and $f(t) = \sum_{n=1}^{\infty} f_n \frac{t^n}{n!}$ with $f_1 = 1$.

For an exponential Riordan array R, its production matrix (also called its Stieltjes matrix, see [7, 13]) is the matrix $P = R^{-1}\overline{R}$, where $\overline{R}$ is the matrix R with its first row removed. An exponential Riordan array R can also be characterized by two sequences as statement in the following proposition [7].

Proposition 2.1 — Let $R = [g(t), f(t)]$ be an exponential Riordan array and let $Z(t) = z_0 + z_1 t + z_2 t^2 + \cdots$, $A(t) = a_0 + a_1 t + a_2 t^2 + \cdots$ be two formal power series such that

$$\begin{cases} A(f(t)) = f'(t), \\ Z(f(t)) = \frac{g'(t)}{g(t)}. \end{cases} \tag{2.1}$$

Then the production matrix of R is $P = (p_{i,j})_{i,j\geq 0}$, where $p_{i,j} = \frac{i!}{j!}(z_{i-j} + ja_{i-j+1})$ $(z_{-1} = 0)$.

The sequences $(z_n)_{n\geq 0}$ and $(a_n)_{n\geq 0}$ are called the Z-sequence and the A-sequence of the exponential Riordan array $R = [g(t), f(t)]$. Let $R = (r_{n,k})_{n,k\geq 0}$ be an exponential Riordan array with Z-sequence $(p, q, 0, \cdots)$ and A-sequence $(1, a, b, 0, \cdots)$. Then the production matrix is given by

$$P = \begin{pmatrix} p & 1 & 0 & 0 & 0 & 0 & \cdots \\ q & p+a & 1 & 0 & 0 & 0 & \cdots \\ 0 & 2(q+b) & p+2a & 1 & 0 & 0 & \cdots \\ 0 & 0 & 3(q+2b) & p+3a & 1 & 0 & \cdots \\ 0 & 0 & 0 & 4(q+3b) & p+4a & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{2.2}$$

Therefore, the entries of the exponential Riordan array R satisfy the recurrence relations:

$$\begin{cases} r_{n+1,0} = pr_{n,0} + qr_{n,1},\ (n \geq 0), \\ r_{n+1,k} = r_{n,k-1} + (p+ka)r_{n,k} + (k+1)(q+kb)r_{n,k+1},\ (n \geq 0, k \geq 1). \end{cases} \tag{2.3}$$

Proposition 2.2 — Let $R = (r_{n,k})_{n,k\geq 0}$ be the exponential Riordan array with production matrix given in (2.2). Then, for nonnegative integers n and k, we have

$$r_{n+k,0} = \sum_{i=0}^{\min(n,k)} r_{n,i} r_{k,i} i! \langle q|b\rangle_i. \tag{2.4}$$

PROOF : We proceed by mathematical induction on n. For $n = 0$ we have $r_{0+k,0} = r_{0,0}r_{k,0}$ since $r_{0,0} = 1$. Suppose that

$$r_{m+k,0} = \sum_{i=0}^{\min(m,k)} r_{m,i} r_{k,i} i! \langle q|b\rangle_i$$

hold for $m \leq n$ and all k. Then by (2.3) and interchanging the summation

$$\begin{aligned}
& \sum_{i=0} r_{n+1,i} r_{k,i} i! \langle q|b\rangle_i \\
= & \sum_{i=0} \left(r_{n,i-1} + (p+ia)r_{n,i} + (i+1)(q+ib)r_{n,i+1}\right) r_{k,i} i! \langle q|b\rangle_i \\
= & \sum_{i=0} r_{n,i-1} r_{k,i} i! \langle q|b\rangle_i + \sum_{i=0} (p+ia) r_{n,i} r_{k,i} i! \langle q|b\rangle_i + \sum_{i=0} r_{n,i+1} r_{k,i} (i+1)! \langle q|b\rangle_{i+1} \\
= & \sum_{i=0} r_{n,i} r_{k,i+1} (i+1)! \langle q|b\rangle_{i+1} + \sum_{i=0} (p+ia) r_{n,i} r_{k,i} i! \langle q|b\rangle_i + \sum_{i=0} r_{n,i} r_{k,i-1} i! \langle q|b\rangle_i \\
= & \sum_{i=0} \left(r_{k,i-1} + (p+ia)r_{k,i} + (i+1)(q+ib)r_{k,i+1}\right) r_{n,i} i! \langle q|b\rangle_i \\
= & \sum_{i=0} r_{k+1,i} r_{n,i} i! \langle q|b\rangle_i \\
= & \ r_{n+k+1,0},
\end{aligned}$$

which proves the induction step. □

Proposition 2.3 — Let $R = (r_{n,k})_{n,k\geq 0}$ be the exponential Riordan array with production matrix P given in (2.2). Let $H_n = (r_{i+j,0})_{i,j=0}^n$, $H_n^{(1)} = (r_{i+j+1,0})_{i,j=0}^n$, and $H_n^{(2)} = (r_{i+j+2,0})_{i,j=0}^n$. Then we have

$$\det H_n = \prod_{i=1}^n T_i, \tag{2.5}$$

$$\det H_n^{(1)} = L_n \det H_n, \tag{2.6}$$

$$\det H_n^{(2)} = \det H_{n+1} \sum_{i=-1}^{n} \frac{L_i^2}{T_{i+1}}, \tag{2.7}$$

where $L_n = \det(p_{i,j})_{i,j=0}^n$ with $L_{-1} = 1$, and $T_n = \prod_{j=1}^n j(q+(j-1)b) = n!\langle q|b\rangle_n$.

PROOF : Let $D_n = \text{diag}(1, T_1, T_2, T_3, \cdots, T_n)$, and let R_n be the $(n+1)\times(n+1)$ principal submatrix of R. Then (2.4) implies $R_n D_n R_n^t = H_n$. Thus $\det H_n = \det D_n = \prod_{i=1}^n T_i$ since $\det R_n = 1$. From (2.3), (2.4) and some computations, we obtain the matrix equation $R_n P_n D_n R_n^t = H_n^{(1)}$. Hence $\det H_n^{(1)} = \det P_n \det D_n = L_n \det H_n$. Let $G_n = R_n P_n^2 D_n R_n^t$. Then

$$G_n = \begin{pmatrix} r_{2,0} & r_{3,0} & r_{4,0} & \cdots & r_{n+1,0} & r_{n+2,0} \\ r_{3,0} & r_{4,0} & r_{5,0} & \cdots & r_{n+2,0} & r_{n+3,0} \\ r_{4,0} & r_{5,0} & r_{6,0} & \cdots & r_{n+3,0} & r_{n+4,0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ r_{n+1,0} & r_{n+2,0} & r_{n+3,0} & \cdots & r_{2n,0} & r_{2n+1,0} \\ r_{n+2,0} & r_{n+3,0} & r_{n+4,0} & \cdots & r_{2n+1,0} & r_{2n+2,0} - T_{n+1} \end{pmatrix}.$$

Thus

$$\det P_n^2 \det D_n = \det H_n^{(2)} - T_{n+1} \det H_{n-1}^{(2)}.$$

Solving this last recursion we obtain $\det H_n^{(2)} = \det H_{n+1} \sum_{i=-1}^n \frac{L_i^2}{T_{i+1}}$. □

3. PROOF OF THEOREM 1.1

In this section we will prove our main result, i.e, Theorem 1.1, by using Proposition 2.3.

PROOF : We consider the exponential Riordan array $R = \left[\frac{1}{(1-dt)^{\frac{x}{d}}}, \frac{t}{1-dt}\right]$. Its general term is $r_{n,k} = \binom{n}{k} \frac{\langle x|d\rangle_n}{\langle x|d\rangle_k}$, and particularly $r_{n,0} = \langle x|d\rangle_n$. By (2.1), the generating functions of the Z- and A-Sequences are $Z(t) = x + dxt$ and $A(t) = 1 + 2dt + d^2t^2$. From Proposition 2.1, the production matrix of R turns out to

$$P = \begin{pmatrix} x & 1 & 0 & 0 & 0 & 0 & \cdots \\ dx & x+2d & 1 & 0 & 0 & 0 & \cdots \\ 0 & 2d(x+d) & x+4d & 1 & 0 & 0 & \cdots \\ 0 & 0 & 3d(x+2d) & x+6d & 1 & 0 & \cdots \\ 0 & 0 & 0 & 4d(x+3d) & x+8d & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Now, $L_n = \det(p_{i,j})_{i,j=0}^n = \langle x|d\rangle_{n+1}$, and $T_n = d^n n!\langle x|d\rangle_n$. Let $H_n = (r_{i+j,0})_{i,j=0}^n$, $H_n^{(1)} = (r_{i+j+1,0})_{i,j=0}^n$, and $H_n^{(2)} = (r_{i+j+2,0})_{i,j=0}^n$. Applying Proposition 2.3, we get

$$\begin{aligned}
\det H_n &= \det(\langle x|d\rangle_{i+j})_{i,j=0}^n = d^{\frac{n(n+1)}{2}} \prod_{i=1}^n i! \cdot \prod_{i=1}^n \langle x|d\rangle_i, \\
\det H_n^{(1)} &= \det(\langle x|d\rangle_{i+j+1})_{i,j=0}^n = d^{\frac{n(n+1)}{2}} \prod_{i=1}^n i! \cdot \prod_{i=1}^{n+1} \langle x|d\rangle_i, \\
\det H_n^{(2)} &= \det(\langle x|d\rangle_{i+j+2})_{i,j=0}^n = \det H_{n+1} \sum_{i=-1}^n \frac{\langle x|d\rangle_{i+1}}{d^{\binom{i+2}{2}}(i+1)!}.
\end{aligned}$$

This completes the proof. □

Taking $d = 1$ and $x = m$ being positive integer, we have

$$\det(\langle m\rangle_{i+j})_{i,j=0}^n = \prod_{i=1}^n i! \prod_{i=1}^n \langle m\rangle_i, \tag{3.1}$$

$$\det(\langle m\rangle_{i+j+1})_{i,j=0}^n = \prod_{i=1}^n i! \cdot \prod_{i=1}^{n+1} \langle m\rangle_i, \tag{3.2}$$

$$\det(\langle m\rangle_{i+j+2})_{i,j=0}^n = \binom{m+n+1}{m} \prod_{i=1}^{n+1} i! \prod_{i=1}^{n+1} \langle m\rangle_i. \tag{3.3}$$

4. More Examples

Example 4.1 : We consider the exponential Riordan array $R = \left[\frac{(1-x)e^t}{e^{xt}-xe^t}, \frac{e^t-e^{xt}}{e^{xt}-xe^t}\right]$. The generating functions of Z- and A-sequences are $Z(t) = 1 + xt$, $A(t) = 1 + (1 + x)t + xt^2$.

More examples

Therefore, from Proposition 2.1, the production matrix is

$$P = \begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 & \cdots \\
x & 2+x & 1 & 0 & 0 & 0 & \cdots \\
0 & 4x & 3+2x & 1 & 0 & 0 & \cdots \\
0 & 0 & 9x & 4+3x & 1 & 0 & \cdots \\
0 & 0 & 0 & 16x & 5+4x & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix},$$

which have general entry

$$p_{i,j} = \begin{cases} 1, & \text{if } i = j-1, \\ 1+i+ix, & \text{if } i = j, \\ i^2x, & \text{if } i = j+1, \\ 0, & \text{otherwise,} \end{cases}$$

and $\det P_n = \det (p_{i,j})_{i,j=0}^n = (n+1)!$.

The first few rows of the matrix R are

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ x+1 & x+3 & 1 & 0 \\ x^2+4x+1 & x^2+10x+7 & 3x+6 & 1 \\ x^3+11x^2+11x+1 & x^3+25x^2+55x+15 & 7x^2+40x+25 & 6x+10 \end{pmatrix}.$$

It is known that $\frac{(1-x)e^t}{e^{xt}-xe^t} = \sum_{n=0}^{\infty} EU_n(x)\frac{t^n}{n!}$, where $EU_n(x)$ are the Eulerian polynomials (see [3, 17]). Hence, by using Proposition 2.3, we have

$$\det (EU_{i+j}(x))_{i,j=0}^n = x^{\binom{n+1}{2}} \prod_{i=1}^{n} i!^2 \tag{4.1}$$

$$\det (EU_{i+j+1}(x))_{i,j=0}^n = (n+1)!x^{\binom{n+1}{2}} \prod_{i=1}^{n} i!^2, \tag{4.2}$$

$$\det (EU_{i+j+2}(x))_{i,j=0}^n = x^{\binom{n+2}{2}} \prod_{i=1}^{n+1} i!^2 \sum_{j=-1}^{n} \frac{1}{x^{j+1}}. \tag{4.3}$$

Example 4.2 : Let $R = \left[e^{x(e^t-1)}, e^t-1\right]$. Then the elements of the first column are the exponential polynomials $e_n(x) = \sum_{j=0}^{n} S(n,j)x^j$, where $S(n,j)$ are the Stirling numbers of the second kind. The case $x=1$ gives the well-known Bell numbers. The Hankel determinant of the Bell numbers and the exponential polynomials have been considered in several articles [2, 9, 15]. The first few rows of R are

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ x & 1 & 0 & 0 & 0 & \cdots \\ x^2+x & 2x+1 & 1 & 0 & 0 & \cdots \\ x^3+3x^2+x & 3x^2+6x+1 & 3x+3 & 1 & 0 & \cdots \\ x^4+6x^3+7x^2+x & 4x^3+18x^2+14x+1 & 6x^2+18x+7 & 4x+6 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

By (2.1), $Z(t) = x + xt$ and $A(t) = 1 + t$, Therefore, the production matrix is

$$P = \begin{pmatrix} x & 1 & 0 & 0 & 0 & 0 & \cdots \\ x & x+1 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 2x & x+2 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 3x & x+3 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 4x & x+4 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

for which $\det P_n = \det (p_{i,j}(x))_{i,j=0}^{n} = x^{n+1}$. An application of Proposition 2.3 gives

$$\det (e_{i+j}(x))_{i,j=0}^{n} = x^{\binom{n+1}{2}} \prod_{i=1}^{n} i!, \tag{4.4}$$

$$\det (e_{i+j+1}(x))_{i,j=0}^{n} = x^{\binom{n+2}{2}} \prod_{i=1}^{n} i!, \tag{4.5}$$

$$\det (e_{i+j+2}(x))_{i,j=0}^{n} = x^{\binom{n+2}{2}} \prod_{i=1}^{n+1} i! \sum_{j=-1}^{n} \frac{x^{j+1}}{(j+1)!}. \tag{4.6}$$

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