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e-INVERTIBLE MATRICES OVER COMMUTATIVE SEMIRINGS[1]

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In this article, the e-invertible matrices over commutative semirings are studied. Some properties and equivalent characterizations of the e-invertible matrices are given. Also, the interrelationships between invertible matrices and e-invertible matrices over commutative semirings are discussed. The main results obtained in this article generalize and enrich the corresponding results about invertible matrices over commutative semirings.

Key words : Commutative semiring; multiplicative idempotent element; e-invertible matrix; ε-determinant.

1. Introduction

By a *semiring* [5] we mean an algebra $(S,\ +,\ \cdot,\ 0,\ 1)$ of type (2, 2, 0, 0) such that

(1) $(S,\ +,\ 0)$ is a commutative monoid;

(2) $(S,\ \cdot,\ 1)$ is a monoid;

(3) $(a+b)\cdot c = a\cdot c + b\cdot c$ and $a\cdot(b+c) = a\cdot b + a\cdot c$ for all $a, b, c \in S$;

(4) $a\cdot 0 = 0\cdot a = 0$ for all $a \in S$;

(5) $0 \neq 1$.

We use, unless otherwise stated, the symbol S to denote both the set and the semiring structure, and usually write ab instead of $a\cdot b$ for all $a, b \in S$.

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A semiring S is said to be *commutative* if $ab = ba$ for all $a, b \in S$. An element $e \in S$ is called *multiplicatively idempotent* if $e^2 = e$, and we denote by $E(S)$ the set of all non-zero multiplicative idempotent elements in S. Thus $E(S) = \{e \in S \setminus \{0\} \mid e^2 = e\}$.

Let S be a semiring. We denote by $M_n(S)$ the set of all $n \times n$ matrices over S, where n is a positive integer. For $A \in M_n(S)$, we denote by a_{ij} the (i, j)-entry of A, and by A^T the transpose of A. For any $A, B \in M_n(S)$ and $s \in S$, we define:

$$A + B = (a_{ij} + b_{ij})_{n\times n},\ sA = (sa_{ij})_{n\times n}, \text{ and } A \cdot B = \Big(\sum_{k=1}^{n} a_{ik}b_{kj}\Big)_{n\times n}.$$

It is easy to see that $M_n(S)$ forms a semiring with respect to the matrix addition and the matrix multiplication (see, [2, 5]). Also, zero matrix O_n and identity matrix I_n over S are the zero and the identity of $M_n(S)$, respectively.

The study of matrices over general semirings has a long history. In 1964, Rutherford [13] gave a proof of the Cayley-Hamiltion theorem for a commutative semiring avoiding the use of determinants. Since then, a number of works on theory of matrices over semirings were published (see, e.g. [2, 4, 5, 10, 11, 14]). In 1999, Golan described matrices over semirings in his work [5] comprehensively. The techniques of matrices over semirings also have important applications in optimization theory, models of discrete event networks and graph theory. For further examples, see [1, 3].

Invertible matrices are an important type of matrices. It is well known that $A \in M_n(S)$ is called *invertible* if $AB = BA = I_n$ for some $B \in M_n(S)$, and such a matrix B is called an inverse of A. Furthermore, if both B_1 and B_2 are inverses of A then we have $B_1 = B_1I_n = B_1AB_2 = I_nB_2 = B_2$, which means that the inverse of A is unique and denoted by A^{-1}. Over the last decades, many authors have studied invertible matrices over some special semirings (see, e.g. [6-8, 12, 15-18]). In 2009, Sirasuntorn, Sombatboriboon and Udomsub [15] obtained some necessary and sufficient conditions for a matrix over Boolean semirings to be invertible; Mora, Wasanawichit and Kemprasit [8] gave the complete descriptions of the invertible matrices over idempotent semirings. In 2011, Sombatboriboona, Mora and Kemprasit [16] discussed the conditions of the invertibility for matrices over additively inverse commutative semirings. In 2012, Kendziorra, Zumbrägel and Schmidt [6] studied invertible matrices over finite additively idempotent semirings. Recently, Tan [17] obtained the equivalent descriptions for a matrix over a commutative semiring to be invertible. Also, Liao and Wang [7] proved that a matrix is invertible in a commutative semiring if and only if it is invertible in the extensive semiring of the commutative semiring, and gave an equivalent description of the invertible matrix over commutative semirings without dependence on any auxiliary function. It is clear that Boolean semirings, idempotent semirings and additively inverse commutative semirings are special kinds of commutative semirings. In this article, we give the definition of e-invertible matrices. $A \in M_n(S)$ is ***e-invertible*** if $AB = BA = eI_n$ for some $B \in M_n(S),\ e \in E(S)$.

For any $A \in M_n(S)$ and $1 \neq e \in E(S)$ where S is commutative. If A is invertible, then there exists $B \in M_n(S)$ such that $AB = BA = I_n$. Consequently, $A(eB) = eAB = (eB)A = eI_n$, which means that A is e-invertible. However, the converse is not true. For instance, eI_n is obviously e-invertible, we now assume that eI_n is also invertible. Then there exists $B = (b_{ij})_{n\times n} \in M_n(S)$ such that $(eI_n)B = I_n$, it follows that $1 = eb_{ii} = e^2 b_{ii} = e \cdot eb_{ii} = e \cdot 1 = e$, a contradiction. This shows that $eI_n (e \neq 1)$ is e-invertible but not invertible.

Therefore, the definition of e-invertible matrices is a generalization of the invertible matrices. For any $A, B \in M_n(S)$ where S is commutative. If $AB = I_n$ then $BA = I_n$ (see [11]). In general, $AB = eI_n$ can not imply $BA = eI_n$ with $1 \neq e \in E(S)$, as the following example shows.

Example 1.1 : Let $S = \mathbb{R} \times D_2$, in which $\mathbb{R}$ is the real number field and D_2 is the two-element distributive lattice. Then it is easy to check that S is a commutative semiring with additive identity $(0,0)$ and multiplicative identity $(1,1)$, moreover, $(0,1) \in E(S)$. Now let

$$A = \begin{pmatrix} (1,1) & (1,0) \\ (-1,0) & (-1,1) \end{pmatrix}, \quad B = \begin{pmatrix} (1,1) & (-1,0) \\ (-1,0) & (1,1) \end{pmatrix}.$$

Then

$$AB = \begin{pmatrix} (0,1) & (0,0) \\ (0,0) & (0,1) \end{pmatrix} = (0,1)I_2, \quad BA = \begin{pmatrix} (2,1) & (2,0) \\ (-2,0) & (-2,1) \end{pmatrix} \neq (0,1)I_2.$$

This article is organized as follows. In Section 2, we introduce some definitions and lemmas about matrices over a commutative semiring. In Section 3, we give the equivalent characterizations of e-invertible matrices over a commutative semiring, at the same time, the interrelationships between invertible matrices and e-invertible matrices over a commutative semiring are discussed. The main results obtained in this article generalize and enrich the corresponding results about invertible matrices over commutative semirings.

2. Preliminaries

In this section, we will give some definitions and lemmas. For convenience, we use Z^+ to denote the set of positive integers, and $\underline{n}$ to denote the set $\{1, 2, \cdots, n\} (n \in Z^+)$.

Let S be a semiring. An element a in S is called *additively invertible* if $a + b = b + a = 0$ for some b in S. Such an element b is obviously unique and denoted by $-a$. Let $V(S)$ denote the set of all additively invertible elements in S. An element b in S is called *multiplicatively invertible* if $bb' = b'b = 1$ for some b' in S. Such an element b' is also obviously unique and denoted by b^{-1}. Let $U(S)$ denote the set of all multiplicatively invertible elements in S. Then the following properties of $V(S)$ are well known [17]:

Lemma 2.1 — [17]. Let S be a semiring. Then

(1) For any $a, b \in S$, $a + b \in V(S)$ if and only if $a, b \in V(S)$;

(2) For any $a \in V(S)$ and any $s \in S$, we have $sa, as \in V(S)$.

In what follows, we always suppose that S is commutative and $e \in E(S)$, unless otherwise stated. Let $H(e) = \{a \in S \mid (\exists a' \in S)\ aa' = a'a = e\}$. Obviously, $H(1) = U(S)$. Also, we have

Lemma 2.2 — Let $a, b \in S$ and $e, f \in E(S)$. Then

(1) $ab \in H(e)$ if and only if $a, b \in H(e)$;

(2) $ef = f$ if and only if $H(e) \subseteq H(f)$;

(3) $H(e)H(f) \subseteq H(ef)$;

(4) For any $a \in H(e)$ and $ab \in V(S)$, we have $eb \in V(S)$.

PROOF : (1) Let $ab \in H(e)$. Then there exists $c \in S$ such that $abc = e$. Notice that S is commutative and so $a, b \in H(e)$.

Conversely, we assume that $a, b \in H(e)$. Then there exist $a', b' \in S$ such that $aa' = e$ and $bb' = e$. Consequently, $(ab)(b'a') = e$. This shows that $ab \in H(e)$.

(2) Let $H(e) \subseteq H(f)$. Then $a \in H(f)$ for any $a \in H(e)$, further, there exist $a', a'' \in S$ such that

$$aa' = e,\ aa'' = f.$$

It follows from (1) that $a' \in H(e)$. Since $H(e) \subseteq H(f)$, we have $a'c = f$ for some $c \in S$. Thus

$$ef = eff = (aa')(aa'')(a'c) = (aa')^2(a''c) = (aa')(a''c) = (aa'')(a'c) = ff = f.$$

Conversely, we assume that $ef = f$ holds. Then, for any $a \in H(e)$, there exists $a' \in S$ such that $aa' = e$. Thus

$$a(a'f) = (aa')f = ef = f.$$

It follows from (1) that $a \in H(f)$. This implies that $H(e) \subseteq H(f)$.

(3) From (2) we have $H(e), H(f) \subseteq H(ef)$. Clearly

$$H(e)H(f) = \{ab \mid a \in H(e), b \in H(f)\}.$$

It immediately follows from (1) that $H(e)H(f) \subseteq H(ef)$.

(4) Let $a \in H(e)$. Then there exists $a' \in S$ such that $aa' = e$, thus

$$eb = (a'a)b = a'(ab).$$

Since $ab \in V(S)$, it follows from Lemma 2.1(2) that $eb \in V(S)$. □

Obviously, $1 \in H(e)$. By Lemma 2.2(1), it yields that $H(e)$ is a submonoid of $(S, \cdot)$.

Let $A \in M_n(S)$. One can verify that A is e-invertible if and only if A^T is e-invertible. Furthermore, we have

Lemma 2.3 — Let $A \in M_n(S)$. Then A is e-invertible if and only if $AB = CA = eI_n$ for some $B, C \in M_n(S)$.

PROOF : Assume that $AB = CA = eI_n$. Then

$$eC = C(eI_n) = C(AB) = (CA)B = eB.$$

Consequently,

$$(eC)A = (eI_n)CA = (eI_n)(eI_n) = AB(eI_n) = A(eB) = A(eC).$$

Hence, A is e-invertible.

Conversely, assume that A is e-invertible. Then there exists $D \in M_n(S)$ such that $AD = DA = eI_n$, and so we just need to take $B = C = D$. □

Let $A \in M_n(S)$. The $positive\ determinant$ $|A|^+$ and the $negative\ determinant$ $|A|^-$ of A are defined as follows:

$$|A|^+ = \sum_{\sigma \in A_n} a_{1\sigma(1)}a_{2\sigma(2)} \cdots a_{n\sigma(n)},\ |A|^- = \sum_{\sigma \in S_n \setminus A_n} a_{1\sigma(1)}a_{2\sigma(2)} \cdots a_{n\sigma(n)},$$

in which S_n and A_n denote the symmetric group and the alternating group on the set $\underline{n}$, respectively [5]. In 1984, Reutenauer and Straubing [11] gave the following significant results.

Lemma 2.4 — [11]. Let $A, B \in M_n(S)$. Then there is an element $s \in S$ such that

$$|AB|^+ = |A|^+|B|^+ + |A|^-|B|^- + s,\ |AB|^- = |A|^+|B|^- + |A|^-|B|^+ + s.$$

Recall that a bijection ε on S is called an ε*-function* of S if $\varepsilon\big(\varepsilon(a)\big) = a$, $\varepsilon(a+b) = \varepsilon(a)+\varepsilon(b)$, $\varepsilon(ab) = \varepsilon(a)b = a\varepsilon(b)$ for all $a, b \in S$ and $\varepsilon(a) = -a$ for all $a \in V(S)$ (see [17]).

Definition 2.5 — [17]. Let ε be an ε-function of S and $A \in M_n(S)$. The ε-determinant of A, denoted by $det_\varepsilon(A)$, is defined by

$$det_\varepsilon(A) = \sum_{\sigma \in S_n} \varepsilon^{(t(\sigma))}(a_{1\sigma(1)}a_{2\sigma(2)} \cdots a_{n\sigma(n)}),$$

in which $t(\sigma)$ is the number of inversions in the permutation σ, and $\varepsilon^{(k)}$ is defined by $\varepsilon^{(k)}(a) = \varepsilon^{(k-1)}\big(\varepsilon(a)\big)$ and $\varepsilon^{(0)}(a) = a$ for any $k \in Z^+$.

Remark 2.6 : Since $\varepsilon^{(2)}(a) = a$, $det_\varepsilon(A)$ can be rewritten as follows:

$$det_\varepsilon(A) = |A|^+ + \varepsilon(|A|^-).$$

The following lemma is needed.

Lemma 2.7 [17]. Let ε be an ε-function of S and $A \in M_n(S)$. Then for any $i, j \in \underline{n}$, we have

$$\sum_{k=1}^{n} a_{ik}A_{jk} = \begin{cases} det_\varepsilon(A) & i = j \\ |A_r(i \Rightarrow j)|^+ + \varepsilon(|A_r(i \Rightarrow j)|^+) & i \neq j, \end{cases}$$

in which $A_r(i \Rightarrow j)$ denotes the matrix obtained from A by replacing row j of A by row i of A, and the A_{jk} denotes the ε-cofactor of the element a_{jk}.

3. Characterizations of e-Invertible Matrices Over a Commutative Semiring

In this section, we shall give some properties and equivalent characterizations of e-invertible matrices over a commutative semiring, at the same time, some interrelationships between invertible matrices and e-invertible matrices over a commutative semiring will be discussed. S is always supposed to be a commutative semiring with an ε-function in this section. First, by Lemma 2.4 and Remark 2.6, the following statement holds.

Proposition 3.1 — Let $A, B \in M_n(S)$ and $x_i \in S (i \in \underline{n})$. If $AB = diag(x_1, x_2, \cdots, x_n)$ then

$$det_\varepsilon(AB) = det_\varepsilon(A)det_\varepsilon(B) = \prod_{i=1}^{n} x_i.$$

PROOF : Let $AB = diag(x_1, x_2, \cdots, x_n)$. Then, by Lemma 2.4, there exists $s \in S$ such that

$$|AB|^+ = |A|^+|B|^+ + |A|^-|B|^- + s, \; |AB|^- = |A|^+|B|^- + |A|^-|B|^+ + s.$$

But

$$|AB|^+ = |diag(x_1, x_2, \cdots, x_n)|^+ = \prod_{i=1}^{n} x_i,\ |AB|^- = |diag(x_1, x_2, \cdots, x_n)|^- = 0.$$

Thus

$$0 = |AB|^- = |A|^+|B|^- + |A|^-|B|^+ + s.$$

It follows from Lemma 2.1(1) that $s \in V(S)$. Then, by the definition of ε-function, we have $s + \varepsilon(s) = 0$. Furthermore, by Remark 2.6, we have

$$det_\varepsilon(AB) = |AB|^+ + \varepsilon(|AB|^-) = \prod_{i=1}^{n} x_i.$$

Then, by Remark 2.6 and Lemma 2.4, we have

$$\begin{aligned} det_\varepsilon(AB) =& |AB|^+ + \varepsilon(|AB|^-) \\ =& |A|^+|B|^+ + |A|^-|B|^- + s + \varepsilon(|A|^+|B|^- + |A|^-|B|^+ + s) \\ =& |A|^+|B|^+ + \varepsilon(|A|^-)\varepsilon(|B|^-) + s + |A|^+\varepsilon(|B|^-) + \varepsilon(|A|^-)|B|^+ + \varepsilon(s) \\ =& |A|^+\big(|B|^+ + \varepsilon(|B|^-)\big) + \varepsilon(|A|^-)\big(|B|^+ + \varepsilon(|B|^-)\big) + \big(s + \varepsilon(s)\big) \\ =& \big(|A|^+ + \varepsilon(|A|^-)\big)\big(|B|^+ + \varepsilon(|B|^-)\big) \\ =& det_\varepsilon(A)det_\varepsilon(B). \end{aligned}$$

□

We also have

Proposition 3.2 — Let $A, B \in M_n(S)$ and $x_i \in H(e)(i \in \underline{n})$. If $AB = diag(x_1, x_2, \cdots, x_n)$ then

$$e \sum_{1 \le i \le n} a_{ij}a_{ik} \in V(S)\ (j, k \in \underline{n} \text{ with } j \ne k).$$

PROOF : Let $AB = diag(x_1, x_2, \cdots, x_n)$. Then we have

$$\sum_{k=1}^{n} a_{ik}b_{kj} = \begin{cases} x_i & i = j \\ 0 & i \ne j. \end{cases}$$

It follows from Lemma 2.1(1) that $a_{ik}b_{kj} \in V(S)$ $(i, j, k \in \underline{n}$ with $i \ne j)$.

Thus, for any $i, j, k \in \underline{n}$ with $j \neq k$, we have

$$\begin{aligned} a_{ij}a_{ik}\prod_{l=1}^{n} x_l =& a_{ij}a_{ik}\prod_{l=1}^{n}\Big(\sum_{s=1}^{n} a_{ls}b_{sl}\Big) \\ =& a_{ij}a_{ik}\Big(\sum_{1\leq s_1, s_2, \ \ldots \ , s_n \leq n} a_{1s_1}b_{s_1 1}a_{2s_2}b_{s_2 2}\cdots a_{ns_n}b_{s_n n}\Big) \\ =& \sum_{1\leq s_1, s_2, \ \ldots \ , s_n \leq n} a_{ij}a_{ik}a_{1s_1}b_{s_1 1}a_{2s_2}b_{s_2 2}\cdots a_{ns_n}b_{s_n n}. \end{aligned}$$

For any $s_1, s_2, \cdots, s_n \in \underline{n}$, consider the following two cases.

Case 1 : $s_u = s_v$ for some $u \neq v$. Then $a_{us_u}b_{s_v v} = a_{us_u}b_{s_u v} \in V(S)$.

Case 2 : $s_u \neq s_v$ for any $u \neq v$. Now let $j = s_u$ and $k = s_v$. Clearly, $i \neq u$ or $i \neq v$.

Thus $a_{ij}b_{s_u u}a_{ik}b_{s_v v} = a_{ij}b_{ju}a_{ik}b_{kv} \in V(S)$.

It follows from Lemma 2.1(1) and (2) that $a_{ij}a_{ik}\prod_{l=1}^{n} x_l \in V(S)$. Since $x_i \in H(e)$, by Lemma 2.2(4), we have $ea_{ij}a_{ik} \in V(S)$. Further, by using Lemma 2.1(1) again, we have

$$e\sum_{1\leq i\leq n} a_{ij}a_{ik} \in V(S)(j, k \in \underline{n} \text{ with } j \neq k).\square$$

Hence, from Propositions 3.1 and 3.2 we have

***Theorem* 3.3** — *Let $A \in M_n(S)$. Then the following statements are equivalent.*

(1) $AB = eI_n$ for some $B \in M_n(S)$.

(2) $det_\varepsilon(A) \in H(e)$ and $e\sum_{1\leq i\leq n} a_{ij}a_{ik} \in V(S)(j, k \in \underline{n}$ with $j \neq k)$.

PROOF : $(1) \Rightarrow (2)$. Suppose that (1) holds. From Proposition 3.1 we have

$$det_\varepsilon(AB) = det_\varepsilon(A)det_\varepsilon(B) = e.$$

Since $e \in H(e)$, it follows from Lemma 2.2(1) that $det_\varepsilon(A) \in H(e)$. But $AB = eI_n$, by Proposition 3.2, we have

$$e\sum_{1\leq i\leq n} a_{ij}a_{ik} \in V(S)(j, k \in \underline{n} \text{ with } j \neq k).$$

$(2) \Rightarrow (1)$. Let $e\sum_{1\leq i\leq n} a_{ij}a_{ik} \in V(S)$ for any $j, k \in \underline{n}$ with $j \neq k$.

It follows from Lemma 2.1(1) that $ea_{ij}a_{ik} \in V(S)$. Thus,

$$e\big(a_{ij}a_{ik} + \varepsilon(a_{ij})a_{ik}\big) = ea_{ij}a_{ik} + \varepsilon(ea_{ij}a_{ik}) = 0. \qquad (*)$$

Now let $C = (ea_{ij})_{n\times n}adj_\varepsilon(A)$ where $adj_\varepsilon(A) = \big((A_{ij})_{n\times n}\big)^T$. Then, by Lemma 2.7,

$$\begin{aligned}
c_{ii} =& e\sum_{k=1}^{n} a_{ik}A_{ik} = e\big(det_\varepsilon(A)\big);\\
c_{ij} =& e\sum_{k=1}^{n} a_{ik}A_{jk} = e\big(|A_r(i\Rightarrow j)|^+ + \varepsilon(|A_r(i\Rightarrow j)|^+)\big)\\
=& e\Big(\sum_{\sigma\in A_n} a_{1\sigma(1)}\cdots a_{i\sigma(i)}\cdots a_{i\sigma(j)}\cdots a_{n\sigma(n)} + \varepsilon\big(\sum_{\sigma\in A_n} a_{1\sigma(1)}\cdots a_{i\sigma(i)}\cdots a_{i\sigma(j)}\cdots a_{n\sigma(n)}\big)\Big)\\
=& e\Big(\sum_{\sigma\in A_n} a_{1\sigma(1)}\cdots a_{i\sigma(i)}a_{i\sigma(j)}\cdots a_{n\sigma(n)} + \sum_{\sigma\in A_n} a_{1\sigma(1)}\cdots \varepsilon\big(a_{i\sigma(i)}\big)a_{i\sigma(j)}\cdots a_{n\sigma(n)}\Big)\\
=& e\sum_{\sigma\in A_n}\Big(a_{i\sigma(i)}a_{i\sigma(j)} + \varepsilon\big(a_{i\sigma(i)}\big)a_{i\sigma(j)}\Big)\prod_{k\neq i,j} a_{k\sigma(k)}.
\end{aligned}$$

Since $i \neq j$, $\sigma(i) \neq \sigma(j)$ for any $\sigma \in A_n$. By $(*)$, we have

$$e\big(a_{i\sigma(i)}a_{i\sigma(j)} + \varepsilon(a_{i\sigma(i)})a_{i\sigma(j)}\big) = 0.$$

It follows from Lemma 2.1(1) and (2) that $c_{ij} = 0$. Thus,

$$(ea_{ij})_{n\times n}ad\, j_\varepsilon(A) = e\big(det_\varepsilon(A)\big)I_n.$$

Since $det_\varepsilon(A) \in H(e)$, there exists $r \in H(e)$ such that $r\big(det_\varepsilon(A)\big) = e$.
Taking $B = (re)ad\, j_\varepsilon(A)$. Then

$$AB = (a_{ij})_{n\times n}(re)ad\, j_\varepsilon(A) = r(ea_{ij})_{n\times n}ad\, j_\varepsilon(A) = re\big(det_\varepsilon(A)\big)I_n = eI_n. \qquad \square$$

Further, we have

***Theorem* 3.4** — *Let $A \in M_n(S)$. The following statements are equivalent.*

(1) $AB = eI_n$ for some $B \in M_n(S)$.

(2) $AB' = diag(ex_1, ex_2, \cdots, ex_n)$ for some $B' \in M_n(S)$, where $x_i \in H(e)(i \in \underline{n})$.

(3) $det_\varepsilon(A) \in H(e)$ and $e\sum_{1\le i\le n} a_{ij}a_{ik} \in V(S)(j, k \in \underline{n}\ with\ j \neq k)$.

PROOF : By Theorem 3.3, we have that the statements (1) and (3) are equivalent. Now we prove that the statements (1) and (2) are equivalent.

$(1) \Rightarrow (2)$. Assume that $AB = eI_n$ for some $B \in M_n(S)$.

We thus take $B' = B \cdot diag(x_1, x_2, \cdots, x_n)$ where $x_i \in H(e)$, hence

$$AB' = AB \cdot diag(x_1, x_2, \cdots, x_n) = diag(ex_1, ex_2, \cdots, ex_n).$$

$(2) \Rightarrow (1)$. Suppose that (2) holds. For any $x_i \in H(e)$ there exists $y_i \in H(e)$ such that $x_i y_i = y_i x_i = e$. We thus take $B = B' \cdot diag(y_1, y_2, \cdots, y_n)$, hence

$$AB = AB' \cdot diag(y_1, y_2, \cdots, y_n) = diag(ex_1y_1, ex_2y_2, \cdots, ex_ny_n) = eI_n.\Box$$

Let $A \in M_n(S)$. It is easy to verify that $|A^T|^+ = |A|^+$ and $|A^T|^- = |A|^-$ (see [9]). By Remark 2.6, We thus have $det_\varepsilon(A^T) = det_\varepsilon(A)$. The following result is obtained directly from Theorem 3.4.

Corollary 3.5 — Let $A \in M_n(S)$. The following statements are equivalent.

(1) $BA = eI_n$ for some $B \in M_n(S)$.

(2) $B'A = diag(ex_1, ex_2, \cdots, ex_n)$ for some $B' \in M_n(S)$, where $x_i \in H(e)(i \in \underline{n})$.

(3) $det_\varepsilon(A) \in H(e)$ and $e\sum_{1\leq j\leq n} a_{ij}a_{kj} \in V(S)(i, k \in \underline{n} \text{ with } i \neq k)$.

In the following we shall give the main result of this section.

***Theorem* 3.6** — *Let $A \in M_n(S)$. The following statements are equivalent.*

(1) A is e-invertible.

(2) $AB = CA = eI_n$ for some $B, C \in M_n(S)$.

(3) $AB' = C'A = diag(ex_1, ex_2, \cdots, ex_n)$ for some $B', C' \in M_n(S)$, where $x_i \in H(e)(i \in \underline{n})$.

(4) $det_\varepsilon(A) \in H(e)$, $e\sum\limits_{i=1}^{n} a_{ij}a_{ik} \in V(S)(j, k \in \underline{n} \text{ with } j \neq k)$, and $e\sum\limits_{j=1}^{n} a_{ij}a_{kj} \in V(S)(i, k \in \underline{n} \text{ with } i \neq k)$.

PROOF : It is clear from Lemma 2.3 that the statements (1) and (2) are equivalent. Since $x_i \in H(e)$, it is easy to verify that the statements (2) and (3) are equivalent. Also, by Theorem 3.4 and Corollary 3.5, the statements (3) and (4) are equivalent. $\Box$

By using Lemma 2.1(1), Lemma 2.2(2) and Theorem 3.6, we have the following Corollary.

Corollary 3.7 — Let $A \in M_n(S),\ e, f \in E(S)$. If A is e-invertible then A is ef-invertible.

For any $A \in M_n(S)$ where S is commutative, by Corollary 3.7, we know that if A is invertible then A is e-invertible. Also, If $AB = I_n$ for some $B \in M_n(S)$ then $BA = I_n$ (see [11]). Thus, by Theorem 3.4 and Corollary 3.5, we have

Corollary 3.8 — Let $A \in M_n(S)$. The following statements are equivalent.

(1) A is invertible.

(2) $AB' = diag(x_1, x_2, \cdots, x_n)$ for some $B' \in M_n(S)$, where $x_i \in U(S)(i \in \underline{n})$.

(3) $det_\varepsilon(A) \in U(S)$ and $\sum_{1 \leq i \leq n} a_{ij}a_{ik} \in V(S)(j, k \in \underline{n} \text{ with } j \neq k)$.

(4) $det_\varepsilon(A) \in U(S)$ and $\sum_{1 \leq j \leq n} a_{ij}a_{kj} \in V(S)(i, k \in \underline{n} \text{ with } i \neq k)$.

Remark 3.9 : In the invertible case, from the proof of Theorem 3.3(2)⇒(1), we have

$$A^{-1} = \big(det_\varepsilon(A)\big)^{-1} ad\, j_\varepsilon(A).$$

It can be seen that Corollary 3.8(1)(3)(4) are the Theorem 3.1 in [17].

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REFERENCES

1. F. L. Baccelli and I. Mairesse, *Ergodic theorems for stochastic operators and discrete event networks*, in *Idempotency*, J. Gunawardena, ed., Cambridge University Press, Cambridge, (1998), 171-208.

2. K. R. Chowdhury, A. Sultana, N. K. Mitra and A. F. M. Khodadad Khan, On matrices over semirings, *Annals of Pure and Applied Mathematics*, **6**(1) (2014), 1-10.

3. B. Gaujal and A. Jean-Marie,Computational issues in recursive stochastic systems, in Idempotency, *J. Gunawardena*, ed., Cambridge University Press, Cambridge, (1998), 209-230.

4. S. Ghosh, Matrices over semirings, *Information Sciences*, **90** (1996), 221-230.

5. J. S. Golan, *Semirings and their applications*, Kluwer Academic Publishers, London (1999).

6. A. Kendziorra, J. Zumbrägel and E. S. Schmidt, Invertible matrices over finite additively idempotent semirings, *Semigroup Forum*, **86**(3) (2012), 525-536.

7. Yalin Liao and Xueping Wang, Note on invertible matrices over commutative semirings, *Linear and Multilinear Algebra*, **64**(3) (2016), 477-483.

8. W. Mora, A. Wasanawichit and Y. Kemprasit, Invertible matrices over Idempotent semirings, *Chamchuri Journal of Mathematics*, **1**(2) (2009), 55-61.

9. P. L. Poplin and R. E. Hartwig, Determinantal identities over commutative semirings, *Linear Algebra and its Applications*, **387** (2004), 99-132.

10. O. A. Pshenitsyna, Mappings that preserve the invertibility of matrices over semirings, *Russian Mathematical Surveys*, **64**(1) (2009), 162-164.

11. C. Reutenauer and H. Straubing, Inversion of matrices over a commutative semiring, *Journal of Algebra*, **88**(2) (1984), 350-360.

12. D. E. Rutherford, Inverses of Boolean matrices, *Proceedings of the Glasgow Mathematical Association*, **6**(1) (1963), 49-53.

13. D. E. Rutherford, The Cayley-Hamiltion theorem for semiring, *Proceedings of the Royal Society of Edinburgh*, **66A** (1964), 211-215.

14. M. K. Sen and S. K. Maity, Regular additively inverse semirings, *Acta Mathematica Universitatis Comenianae*, **75**(1) (2006), 137-146.

15. N. Sirasuntorn, S. Sombatboriboon and N. Udomsub, Inversion of matrices over Boolean semirings, *Thai Journal of Mathematics*, **7**(1) (2009), 105-113.

16. S. Sombatboriboona, W. Mora and Y. Kemprasit, Some results concerning invertible matrices over semirings, *Scienceasia*, **37**(2) (2011), 130-135.

17. Yijia Tan, On invertible matrices over commutative semirings, *Linear and Multilinear Algebra*, **61**(6) (2013), 710-724.

18. Shan Zhao and Xueping Wang, Inverible matrices and semilinear spaces over commutative semirings, *Information Sciences*, **180**(24) (2010), 5115-5124.