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EFFECT OF A FLOATING ELASTIC PLATE/MEMBRANE ON THE MOTION DUE TO A RING SOURCE IN WATER WITH POROUS BED

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The velocity potentials due to the presence of a horizontal circular ring of wave sources of time-dependent strength in water of finite constant depth with a floating elastic plate or a floating membrane are determined. The uniform bottom is composed of non-dissipative porous medium. The problems are formulated as the initial value problems and the Laplace transform method is used to solve these. For time-harmonic source strength, the steady-state analysis of the potentials reveals the existence of outgoing progressive waves. Graphs for the surface profiles are presented for different values of the tension parameter for the membrane, flexural rigidity of ice and the porous-effect parameter.

Key words : Ring source potential; floating elastic plate; floating membrane; porous bottom.

1. Introduction

Investigation of water wave problems in which a part or whole of water surface is covered by a thin elastic plate has gained considerable importance since the last few decades due to their applications in two particular areas. One of these is to understand the mechanism and effects of wave propagation through Marginal Ice Zone (MIZ) in the Antarctic (cf. Squire [32]). An MIZ is formed of broken ice or continuous sheets of ice. The latter can be modeled as a thin elastic plate composed of elastic material of uniform area density. Significant ice-wave interaction problems may be found in Chakrabarti [2], Gayen and Mandal [9], Korobkin *et al*. [14] and Mohapatra and Sahoo [26]. Another application area of the hydroelasticity problems results from the construction of Very Large Floating Structures

(VLFS), where the large structure is modeled as a thin elastic plate using the Kirchhoff theory. In this connection, we may mention the research works of Kashiwagi [13], Mirafzalia *et al*. [23] and Watanabe [34].

For the temporary protection of coastal regions and construction sites, investigations involving membranes floating on the surface of water are also important. Membranes are mainly made of synthetic fiber, rubber or a polymeric material. They are easily transportable, deployable, detouchable and somewhat inexpensive. The membrane is modeled as a thin, homogeneous and inextensible sheet with uniform mass. Water wave scattering by horizontal membranes submerged in water was considered by Cho and Kim [3]. Manam [19] studied the scattering of membrane coupled gravity waves by a surface piercing plate.

All the above studies have been carried out when the fluid region has a uniform rigid bottom of finite depth or when it is of infinite depth. In coastal engineering, porosity of the bed is also a very important aspect due to the fact that the porous bottom is inherent in the natural events of water wave propagation. Its presence gives rise to many interesting phenomena like wave-energy dissipation, damping etc. Thus there exist a substantial amount of research papers which take the permeability of the bed into account in analyzing various hydrodynamic properties, when the water surface is free (cf. Jeng [12], Martha *et al*. [21], Mohapatra [27], Panda and Martha [28]). The only example of simultaneous consideration of floating elastic plate and a porous bottom is found in Maiti and Mandal [17], who examined water wave scattering by the edge(s) of a semi-infinite or a finite floating elastic plate in the presence of a porous bed.

There has been a long standing interest in studying water wave generation problems by various disturbances present either on the surface or inside the water. The velocity potentials due to a two-dimensional wave source of time-dependent strength submerged in water with an inertial surface was determined by Rhodes-Robinson [29] for deep water. Ghosh [10] investigated the cylindrical wave-maker problem in presence of surface-tension in single-layered fluid. Sherief *et al*. [31] investigated the vertical cylindrical porous wave maker problem in two-layered fluids. Lu and Dai [16] investigated the surface waves generated by unsteady concentrated disturbances in an initially quiescent fluid of infinite depth with an inertial surface. Since the last few years some remarkable research works have been published in which the wave generation problems were solved in the presence of an ice-cover modeled as a thin elastic plate. Several problems in this direction were considered by Banerjea *et al*. [1], Gayen(Chowdhury) and Mandal [5], Maiti *et al*. [18], Mohanty *et al*. [24], Savin and Savin [30] and Lu *et al*. [15].

To the best of the authors' knowledge there does not exist any research paper which considers either the effect of a floating membrane or a porous medium at the bottom on the wave radiation. Thus, the authors are motivated to consider a particular case of wave generation problem which investigates the wave motion generated due to the presence of a ring source placed in water with a porous bottom when the upper surface is covered either by an ice-cover or a floating membrane. There are several usefulness of the study of ring sources. Some of them are listed below. If an obstacle is in the form of a vertical body of revolution having a common vertical axis of symmetry with the fluid motion, one needs to consider potential due to submerged horizontal circular rings of wave sources since the problem can then be formulated in terms of suitable distribution of rings of wave sources around the body [11]. An offshore structure in the high sea for the purpose of oil prospecting may be modeled as a vertical cylinder of circular cross section. Hence consideration of velocity potentials due to submerged circular rings of wave sources is of importance. A substantial amount of research consists of finding suitable forms of the green's function due to the ring sources (see Teng and Kato [33] and the papers cited therein). Also the existence of trapped modes can be realized by placing suitably an axisymmetric ring source in the free surface (cf. McIver and McIver [22]).

Mandal and Kundu [20] used a Laplace transform technique to obtain the velocity potential due to a ring source of time-dependent strength submerged in finite depth water with an inertial surface in the presence of surface tension, the inertial surface being composed of uniformly distributed non-interacting floating material. Gayen(Chowdhury) and Mandal [4] employed a Laplace transform technique to determine the velocity potential due to a ring source of time-dependent strength submerged for both deep water and finite depth water with an ice-cover.

In the present paper, we determine the velocity potential due to a horizontal circular ring of wave sources of time-harmonic strength submerged in water of uniform finite depth. The bottom is composed of uniform porous materials whereas the surface of water is either covered with a thin elastic plate or a floating membrane. The problem is formulated as an initial value problem for the velocity potential describing the motion in the fluid in Section 2. In Section 3, we employ Laplace transform to solve the problem. We assume the wave-source to be time-harmonic and determine the asymptotic expressions for the potentials in the steady-state. The analyses due to a floating membrane and a floating elastic plate are described in sections 3.1 and 3.2 respectively. The asymptotic behaviour of the potentials depends on the nature of the roots of the dispersion equations. Maiti and Mandal [17] determined the roots of the dispersion equation when upper surface is covered by a thin ice-sheet and the bottom is permeable. However, such analysis is not available in the literature in the context of fluid motion in presence of a floating membrane on the top together with a uniform porous bed. In the

Appendix we establish the nature of the roots of such dispersion equation using argument principle of complex variable theory. In Section 4, the expression for the surface depression is determined and the steady state surface profiles are presented graphically. The figures reveal that how the surface profiles are affected by the fluctuations in the flexural rigidity of ice or the tension parameter of the membrane, or the permeability of the bottom. It is evidenced that decreasing values of the surface parameters and the increasing porosity result in increase in the amplitude of the waves generated due to the ring source. Finally our conclusions are summarized in Section 5.

2. Formulation of the Problem

To formulate the problem mathematically, it is assumed that the fluid is inviscid, incompressible and homogeneous with density ρ. We also assume that the bottom is of uniform finite depth h and is composed of some specific type of porous materials which constitute a non-dissipative porous medium. Here we use a cylindrical co-ordinate system (R, θ, y) in which the y-axis is taken vertically downwards passing through the center of the horizontal ring of radius a of uniformly distributed point sources each of strength $m(t)$ submerged at a depth η below the undisturbed floating ice-sheet or membrane. Since the strength $m(t)$ does not depend on θ, the motion of the fluid is axisymmetric. The plane $y = 0$ denotes the rest position of the floating membrane (MC) or the ice-cover (IC). Since the motion in water starts from rest, it is irrotational and can be described by a potential function $\phi_j(R, y, t)$ with $j = 1, 2$. Subscripts 1 and 2 are used to denote the variables related to the membrane and the ice-sheet respectively. Then $\phi_j(R, y, t)$ satisfies the Laplace's equation

$$\frac{1}{R}(R\phi_{jR})_R + \phi_{jyy} = 0 \tag{1}$$

in the fluid region except at the points on the ring.

If $\zeta_j(R, t)$ denotes the displacement of the upper surface below its mean position, then the linearized kinematic and dynamic boundary conditions on the upper surface are given by the following equations:

$$\phi_{jy} = \zeta_{jt} \text{ on } y = 0, \tag{2}$$

$$(\phi_1 - \epsilon_1\phi_{1y})_t = (-Q\nabla_R^2 + 1)g\zeta_1 \text{ on } y = 0 \text{ (for MC)}, \tag{3}$$

$$(\phi_2 - \epsilon_2\phi_{2y})_t = (D\nabla_R^4 + 1)g\zeta_2 \text{ on } y = 0 \text{ (for IC)}. \tag{4}$$

Here $(Q, D) = \frac{(T,L)}{\rho g}$ and $\epsilon_j = \frac{\rho_j}{\rho}h_j$; T being the uniform tension acting on the membrane, $L = \frac{Eh_2^3}{12(1-\nu^2)}$ being the flexural rigidity of the ice-sheet, h_j being the very small thickness of the membrane/ice sheet, ρ_j being their density. E and ν are the Young's modulus and Poisson's ratio of the ice respectively.

Eliminating ζ_1 between (2) and (3), and ζ_2 between (2) and (4) we obtain the surface boundary conditions as

$$(\phi_1 - \epsilon_1\phi_{1y})_{tt} = (-Q\nabla_R^2 + 1)g\phi_{1y} \text{ on } y = 0 \text{ (for MC)}, \tag{5}$$

$$(\phi_2 - \epsilon_2\phi_{2y})_{tt} = (D\nabla_R^4 + 1)g\phi_{2y} \text{ on } y = 0 \text{ (for IC)}. \tag{6}$$

The potential function ϕ_j also satisfies the bottom condition

$$\phi_{jy} - G\phi_j = 0 \text{ on } y = h. \tag{7}$$

Here G is the porous-effect parameter and is related to the impedance Γ of the porous medium as $G = iK\Gamma$, $K = \frac{\sigma^2}{g}$, σ is the angular frequency of the time-harmonic wave motion and g is the acceleration due to gravity. Since for a non-dissipative medium Γ is purely imaginary, G is real and this allows the existence of progressive waves [6].

At points near the ring

$$\phi_j \to m(t)\phi_0 \text{ as } \sqrt{(R-a)^2 + (y-\eta)^2} \to 0 \tag{8}$$

where, ϕ_0 is the potential due to a ring of wave sources of constant unit strength in an unbounded fluid, given by [11]

$$\phi_0 = 2\pi a \int_0^\infty e^{-k|y-\eta|} J_0(ka) J_0(kR) dk. \tag{9}$$

The initial conditions on the upper surface of water are

$$(\phi_j - \epsilon_j\phi_{jy}) = 0, (\phi_j - \epsilon_j\phi_{jy})_t = 0 \text{ on } y = 0 \text{ at } t = 0. \tag{10}$$

In the next section we describe the motion generated by a ring source of time-harmonic strength by taking $m(t) = \sin\sigma t$. The derivation of the potential due to sources with impulsive strength $(m(t) = \delta(t))$ or constant strength $(m(t) = 1)$ is straightforward [4]. We will not discuss these cases here, because, as the sources around the ring act instantaneously at $t = 0$, its effect will not be felt anywhere in the liquid region after a long lapse of time.

3. Solution to the Problems

In order to solve the boundary value problem given by equations (1), (5)-(8) and (10), we employ the Laplace transform technique. Let $\overline{\phi}_j(R, y, p)$ denote the Laplace transform of $\phi_j(R, y, t)$ defined as

$$\overline{\phi}_j(R, y, p) = \int_0^\infty \phi_j(R, y, t) e^{-pt} dt \quad (p > 0). \tag{11}$$

Then, $\overline{\phi}_j$ satisfies the Laplace equation

$$\frac{1}{R}(R\overline{\phi}_{jR})_R + \overline{\phi}_{jyy} = 0 \tag{12}$$

in the fluid region except at the points on the ring,

$$\overline{\phi}_j \to \overline{m}(p)\phi_0 \text{ as } \sqrt{(R-a)^2 + (y-\eta)^2} \to 0, \tag{13}$$

$$p^2\overline{\phi}_1 - (-Q\nabla_R^2 + 1 + \frac{\epsilon_1 p^2}{g})g\overline{\phi}_{1y} = 0 \text{ on } y = 0 \text{ (for MC)}, \tag{14}$$

$$p^2\overline{\phi}_2 - (D\nabla_R^4 + 1 + \frac{\epsilon_2 p^2}{g})g\overline{\phi}_{2y} = 0 \text{ on } y = 0 \text{ (for IC)}, \tag{15}$$

$$\overline{\phi}_{jy} - G\overline{\phi}_j = 0 \text{ on } y = h. \tag{16}$$

3.1 *Ring source potential due to a floating membrane*

A solution for $\overline{\phi}_1$ satisfying (12) and (13) can be represented as

$$\begin{aligned}\overline{\phi}_1 = \overline{m}(p)\Bigg[\phi_0 - 2\pi a\int_0^\infty e^{-k(y+\eta)}J_0(ka)J_0(kR)dk \\ + \int_0^\infty \{B_1(k)\cosh k(h-y) + C_1(k)\sinh ky\}\frac{J_0(ka)}{\cosh kh}J_0(kR)dk\Bigg]\end{aligned} \tag{17}$$

where $J_0(x)$ is the Bessel function of the first kind of order zero. Using the condition (14) on the floating membrane and the porous bed condition (16), $B_1(k)$ and $C_1(k)$ are uniquely determined as

$$B_1(k) = 4\pi a\frac{gk(Qk^2 + 1 + \frac{\epsilon_1 p^2}{g})X(h-\eta)}{M_1(k)(p^2 + \Omega_1^2)}\cosh kh, \tag{18}$$

$$C_1(k) = -4\pi a\frac{H_1(k)}{M_1(k)(p^2 + \Omega_1^2)}\cosh kh \tag{19}$$

with

$$\Omega_1^2 = \frac{N_1(k)}{M_1(k)}, \tag{20}$$

$$N_1(k) = gk(Qk^2 + 1)\{X(h)\sinh kh + G\}, \tag{21}$$

$$M_1(k) = X(h)(\cosh kh + \epsilon_1 k\sinh kh) + \epsilon_1 kG, \tag{22}$$

$$H_1(k) = gk\{Qk^2 + 1 + \frac{\epsilon_1 p^2}{g}\}\{e^{-k\eta}G + l(k)\sinh kh\} + p^2 l(k)\cosh kh, \tag{23}$$

$$X(z) = (G\sinh kz - k\cosh kz),\ l(k) = (k+G)e^{-kh}\sinh k\eta. \tag{24}$$

Substituting $B_1(k)$ and $C_1(k)$ from (18), (19) into (17), $\overline{\phi}_1$ is finally obtained as

$$\overline{\phi}_1 = 2\pi a\,\overline{m}(p)P_1(R,y) + \overline{m}(p)\int_0^\infty \frac{\Omega_1^2}{\Omega_1^2+p^2}Q_1(R,y,k)dk \tag{25}$$

$$P_1(R,y) = \int_0^\infty \left[e^{-k|y-\eta|} - e^{-k(y+\eta)} + \frac{2}{M_1(k)}S_1(y,k)\right]J_0(ka)J_0(kR)dk, \tag{26}$$

$$S_1(y,k) = \epsilon_1 k\; L(y,k) - l(k)\cosh kh\;\sinh ky, \tag{27}$$

$$Q_1(R,y,k) = 4\pi a\frac{T_1(y,k)}{M_1(k)}J_0(ka)J_0(kR), \tag{28}$$

$$T_1(y,k) = \left\{\frac{M_1(k)}{X(h)\sinh kh + G} - \epsilon_1 k\right\}L(y,k) + l(k)\cosh kh\sinh ky, \tag{29}$$

$$L(y,k) = X(h-\eta)\cosh k(h-y) - \{e^{-k\eta}G + l(k)\sinh kh\}\sinh ky. \tag{30}$$

Then, Laplace inversion of (25) yields a general representation of the velocity potential due to the ring source submerged in water with membrane-cover at its top. This representation is valid for all $t > 0$ and all values of R. Thus $\phi_1(R,y,t)$ is determined as

$$\begin{aligned}\phi_1(R,y,t) &= 2\pi a\; m(t)P_1(R,y)\\ &+ \int_0^\infty \Omega_1(k)Q_1(R,y,k)\left\{\int_0^t \sin\Omega_1(t-\tau)m(\tau)d\tau\right\}dk.\end{aligned} \tag{31}$$

To analyze the motion due to a ring source of time-harmonic strength we take $m(t) = \sin\sigma t$ and we find from (31) that

$$\begin{aligned}\phi_1(R,y,t) &= 2\pi a\;\sin\sigma t\; P_1(R,y)\\ &+ \int_0^\infty \Omega_1(k)Q_1(R,y,k)\frac{\Omega_1\sin\sigma t - \sigma\sin\Omega_1 t}{\Omega_1^2-\sigma^2}dk.\end{aligned} \tag{32}$$

We further modify the above form of ϕ_1 to obtain its asymptotic form in the steady state. Thus we rewrite ϕ_1 as

$$\begin{aligned}\phi_1(R,y,t) \;\;=\;\; & 2\pi a\;\sin\sigma t\left[P_1(R,y) - 2\,⨍_0^\infty \frac{k(Qk^2+1)}{M_1(k)\Delta_1(k)}F(y,k)J_0(ka)J_0(kR)dk\right]\\ & -\sigma\int_0^\infty \frac{\Omega_1(k)Q_1(R,y,k)}{\Omega_1+\sigma}\frac{\sin\Omega_1 t}{\Omega_1-\sigma}dk\end{aligned} \tag{33}$$

where

$$F(y,k) = \frac{X(h)\;X(h-\eta)}{\cosh kh}\cosh k(h-y) + \frac{G\{l(k) - e^{-k\eta}X(h)\}}{\cosh kh}\sinh ky. \tag{34}$$

To obtain the form of equation (33) for large time, we now introduce a Cauchy principal value at $k = \mu_{01}$ which is the real positive zero of $\Delta_1(k)$, where

$$\Delta_1(k) \equiv K(G \tanh kh - k) + (Qk^2 + 1 - \epsilon_1 K)(k^2 \tanh kh - Gk) = 0. \tag{35}$$

The equation $\Delta_2(k)$ has three real roots $0, \pm\mu_{01}$, infinite number of imaginary roots $\pm i\alpha_{1n}$. Also, $\Delta_1(k)$ has two complex roots ν_1 and $\overline{\nu}_1$ with $Re(\nu_1) < 0$, $Im(\nu_1) > 0$ (see Appendix for proof). Hence, as $t \to \infty$ using the Riemann Lebesgue Lemma, we obtain

$$\begin{aligned} \phi_1 \quad \simeq \quad & 2\pi a \sin \sigma t \left[P_1(R, y) - 2 \,⨍_0^\infty \frac{k(Qk^2+1)}{M_1(k)\Delta_1(k)} F(y,k) J_0(ka) J_0(kR) dk \right] \\ & -2\pi^2 a g_1(y, \eta; \mu_{01}) \cos \sigma t \; J_0(\mu_{01} R) \end{aligned} \tag{36}$$

where

$$g_1(y, \eta; k) = \frac{2X(h-\eta) V_1(y,k) J_0(ka)}{M_1(k) U_1(k)} \tag{37}$$

with

$$\begin{aligned} V_1(y,k) = {} & \cosh kh \{\sinh ky + \epsilon_1 k \cosh ky\} * \\ & \left[KG \sinh kh - Gk\{Qk^2 + 1 - \epsilon_1 K\} \cosh kh \right] \\ & + k(Qk^2+1) \{G \sinh ky - G \sinh kh \cosh k(h-y)\} \\ & + k^2 (Qk^2 + 1 - \epsilon_1 K)(\cosh kh + \epsilon_1 k \sinh kh) \cosh k(h-y), \end{aligned} \tag{38}$$

$$\begin{aligned} U_1(k) = {} & K(Gh \cosh kh - \cosh kh - kh \sinh kh) \\ & + (3Qk^2 + 1 - \epsilon_1 K)(k \sinh kh - G \cosh kh) \\ & + k(Qk^2 + 1 - \epsilon_1 K)(\sinh kh + kh \cosh kh - Gh \sinh kh). \end{aligned} \tag{39}$$

To further simplify the integral in (36), we employ Jordan's lemma. For this we write $J_0(kR)$ as $J_0(kR) = \frac{H_0^{(1)}(kR) + H_0^{(2)}(kR)}{2}$ where $H_0^{(1)}(kR)$ and $H_0^{(2)}(kR)$ are the Hankel functions of the first and the second kind of order zero, respectively. We then rotate the contour along the real axis with indentations above the poles at $k = -\mu_{01}, 0$ and below the pole at $k = \mu_{01}$. This yields an alternative representation for the expression in (36) as given by

$$\begin{aligned} \phi_1 \simeq {} & 4\pi a \;\sin \sigma t \sum_{n=1}^{\infty} g_1(y, \eta; i\alpha_{1n}) K_0(\alpha_{1n} R) \\ & + 2\pi^2 i a \;\sin \sigma t \; g_1(y, \eta; \nu_1) H_0^{(1)}(\nu_1 R) \\ & - 2\pi^2 a \; g_1(y, \eta; \mu_{01}) \{\sin \sigma t \; Y_0(\mu_{01} R) + \cos \sigma t \; J_0(\mu_{01} R)\} \end{aligned} \tag{40}$$

where $K_0(x)$ is the modified Bessel function of the second kind of order zero and $Y_0(x)$ is the Bessel function of the second kind of order zero.

To analyze the behaviour of the potential at a large distance from the centre of the ring we use the following asymptotic representations of the Bessel functions: As $R \to \infty$

$$K_0(Rz) \simeq \sqrt{\frac{\pi}{2Rz}} e^{-Rz}, \; H_0^{(1)}(Rz) \simeq \sqrt{\frac{2}{\pi Rz}} e^{i(Rz-\frac{\pi}{4})}, \tag{41}$$

$$J_0(Rz) \simeq \sqrt{\frac{2}{\pi Rz}} \cos(Rz - \frac{\pi}{4}), \; Y_0(Rz) \simeq \sqrt{\frac{2}{\pi Rz}} \sin(Rz - \frac{\pi}{4}). \tag{42}$$

Using the above limiting values, the potential for large time and distance is given by

$$\phi_1 \simeq -2\pi^2 a \, g_1(y, \eta; \mu_{01}) \left(\frac{2}{\pi \mu_{01} R} \right)^{1/2} \cos\left(\mu_{01} R - \sigma t - \frac{\pi}{4} \right). \tag{43}$$

This shows that in the steady-state ϕ_1 indeed represents progressive outgoing waves at a large distance from the center of the ring.

3.2 *Ring source potential due to a floating elastic plate*

A solution for $\overline{\phi}_2$ satisfying (12) and (13) can be represented as

$$\begin{aligned} \overline{\phi}_2 = \overline{m}(p) \Bigg[\phi_0 - 2\pi a \int_0^\infty e^{-k(y+\eta)} J_0(ka) J_0(kR) dk \\ + \int_0^\infty \{B_2(k) \cosh k(h-y) + C_2(k) \sinh ky\} \frac{J_0(ka)}{\cosh kh} J_0(kR) dk \Bigg]. \end{aligned} \tag{44}$$

Using condition (15) on the floating elastic plate and the porous bed condition (16), $B_2(k)$ and $C_2(k)$ are uniquely determined as

$$B_2(k) = 4\pi a \frac{gk(Dk^4 + 1 + \frac{\epsilon_2 p^2}{g}) X(h-\eta)}{M_2(k)(p^2 + \Omega_2^2)} \cosh kh, \tag{45}$$

$$C_2(k) = -4\pi a \frac{H_2(k)}{M_2(k)(p^2 + \Omega_2^2)} \cosh kh \tag{46}$$

with

$$\Omega_2^2 = \frac{N_2(k)}{M_2(k)}, \tag{47}$$

$$N_2(k) = gk(Dk^4 + 1)\{X(h) \sinh kh + G\}, \tag{48}$$

$$M_2(k) = X(h)(\cosh kh + \epsilon_2 k \sinh kh) + \epsilon_2 kG, \tag{49}$$

$$H_2(k) = gk\{Dk^4 + 1 + \frac{\epsilon_2 p^2}{g}\}\{e^{-k\eta}G + l(k)\sinh kh\} + p^2 l(k)\cosh kh. \tag{50}$$

Substituting $B_2(k)$ and $C_2(k)$ from (45), (46) into (44), $\overline{\phi}_2$ is found as

$$\overline{\phi}_2 = 2\pi a\, \overline{m}(p) P_2(R,y) + \overline{m}(p)\int_0^\infty \frac{\Omega_2^2}{\Omega_2^2 + p^2} Q_2(R,y,k)dk \tag{51}$$

$$P_2(R,y) = \int_0^\infty \left[e^{-k|y-\eta|} - e^{-k(y+\eta)} + \frac{2}{M_2(k)} S_2(y,k) \right] J_0(ka)J_0(kR)dk, \tag{52}$$

$$S_2(y,k) = \epsilon_2 k\, L(y,k) - l(k)\cosh kh\ \sinh ky, \tag{53}$$

$$Q_2(R,y,k) = 4\pi a \frac{T_2(y,k)}{M_2(k)} J_0(ka)J_0(kR), \tag{54}$$

$$T_2(y,k) = \left\{ \frac{M_2(k)}{X(h)\sinh kh + G} - \epsilon_2 k \right\} L(y,k) + l(k)\cosh kh \sinh ky. \tag{55}$$

The Laplace inversion of (51) produces ϕ_2 as given by

$$\begin{aligned} \phi_2(R,y,t) &= 2\pi a\, m(t) P_2(R,y) \\ &+ \int_0^\infty \Omega_2(k) Q_2(R,y,k) \left\{ \int_0^t \sin \Omega_2(t-\tau) m(\tau) d\tau \right\} dk. \end{aligned} \tag{56}$$

This is the ring-source potential due to presence of a floating elastic plate at the top and the porous bed at the bottom. If the strength of the source is prescribed by $m(t) = \sin \sigma t$, then the form of ϕ_2 is modified as

$$\begin{aligned} \phi_2(R,y,t) &= 2\pi a\ \sin \sigma t\ P_2(R,y) \\ &+ \int_0^\infty \Omega_2(k) Q_2(R,y,k) \frac{\Omega_2 \sin \sigma t - \sigma \sin \Omega_2 t}{\Omega_2^2 - \sigma^2} dk. \end{aligned} \tag{57}$$

As in 3.1, here also, we arrange ϕ_2 so as to analyze its behaviour in the steady-state. Thus ϕ_2 is rewritten as

$$\begin{aligned} \phi_2(R,y,t) \quad &= \quad 2\pi a\ \sin \sigma t \left[P_2(R,y) - 2 \fint_0^\infty \frac{k(Dk^4+1)}{M_2(k)\Delta_2(k)} F(y,k) J_0(ka) J_0(kR) dk \right] \\ & \quad -\sigma \int_0^\infty \frac{\Omega_2(k) Q_2(R,y,k)}{\Omega_2 + \sigma} \frac{\sin \Omega_2 t}{\Omega_2 - \sigma} dk. \end{aligned} \tag{58}$$

Here $\Delta_2(k)$ is a transcendental function appearing in the dispersion relation arising due to the presence of ice-cover at the surface and porous bed at the bottom. $\Delta_2(k)$ is given by

$$\Delta_2(k) \equiv K(G\tanh kh - k) + (Dk^4 + 1 - \epsilon_2 K)(k^2 \tanh kh - Gk). \tag{59}$$

The equation $\Delta_2(k) = 0$ has three real roots 0, $\pm\mu_{02}$, infinite number of imaginary roots $\pm i\alpha_{2n}$. Also, $\Delta_2(k) = 0$ has two pairs of complex conjugate roots μ_1, $\overline{\mu}_1$ and $-\mu_1$, $-\overline{\mu}_1$ with $Re(\mu_1) > 0$, $Im(\mu_1) > 0$. The existence of the roots are proved in Maiti and Mandal [17].

Modifying the contour of integration for the integral in second term as done in 3.1 and using Riemann Lebesgue Lemma for the integral in the third term, we obtain

$$\begin{aligned}\phi_2 &\simeq 4\pi a\ \sin\sigma t\sum_{n=1}^{\infty} g_2(y,\eta;i\alpha_{2n})K_0(\alpha_{2n}R)\\ &+2\pi^2 ia\ \sin\sigma t\left\{g_2(y,\eta;\mu_1)H_0^{(1)}(\mu_1 R) - g_2(y,\eta;\overline{\mu}_1)H_0^{(2)}(\overline{\mu}_1 R)\right\}\\ &-2\pi^2 a\ g_2(y,\eta;\mu_{02})\left\{\sin\sigma t\ Y_0(\mu_{02}R) + \cos\sigma t\ J_0(\mu_{02}R)\right\}\end{aligned}$$

where

$$g_2(y,\eta;k) = \frac{2X(h-\eta)V_2(y,k)J_0(ka)}{M_2(k)U_2(k)} \tag{60}$$

with

$$\begin{aligned}V_2(y,k) &= \cosh kh\left\{\sinh ky + \epsilon_2 k\cosh ky\right\} *\\ &\quad\left[KG\sinh kh - Gk\{Dk^4 + 1 - \epsilon_2 K\}\cosh kh\right]\\ &\quad +k(Dk^4+1)\left\{G\sinh ky - G\sinh kh\cosh k(h-y)\right\}\\ &\quad +k^2(Dk^4 + 1 - \epsilon_2 K)(\cosh kh + \epsilon_2 k\sinh kh)\cosh k(h-y),\end{aligned} \tag{61}$$

$$\begin{aligned}U_2(k) &= K(Gh\cosh kh - \cosh kh - kh\sinh kh)\\ &\quad +(5Dk^4 + 1 - \epsilon_2 K)(k\sinh kh - G\cosh kh)\\ &\quad +k(Dk^4 + 1 - \epsilon_2 K)(\sinh kh + kh\cosh kh - Gh\sinh kh).\end{aligned} \tag{62}$$

Using the asymptotic values of $K_0(Rz), H_0^{(1),(2)}(Rz), Y_0(Rz)$, $J_0(Rz)$ as $R\to\infty$ given in (41) and (42), the potential for large time and distance is given by

$$\phi_2 \simeq -2\pi^2 a\ g_2(y,\eta;\mu_{02})\left(\frac{2}{\pi\mu_{02}R}\right)^{1/2}\cos\left(\mu_{02}R - \sigma t - \frac{\pi}{4}\right). \tag{63}$$

It is evident from the above expression that in the steady-state and at a large distance from the ring, ϕ_2 represents progressive outgoing waves.

4. Surface Profiles

Using the linearized kinematic condition on the upper surface, the dimensionless surface displacement $\frac{\zeta_j}{h}(R,t)$ can be written as

$$\frac{\zeta_j}{h}(R,t) = \frac{1}{h}Re\left\{\frac{g^2}{\sigma^3}\int_0^t \phi_{jy}(R,0,\tau)d\tau\right\}. \tag{64}$$

To display the effect of the membrane surface on the wave motion generated due to the presence of a ring source, the dimensionless surface displacement $\frac{\zeta_1}{h}$ is presented graphically against R/h. In Figure 1 the surface displacement $\frac{\zeta_1}{h}$ is plotted versus R/h for three different values of the tension-parameter $\frac{Q}{h^2} = 1, 0.5, 0.1$ for the membrane with $Gh = 0.5$, $\frac{\epsilon_1}{h} = 0.01$, $\frac{\eta}{h} = 0.8$, $\frac{a}{h} = 0.6$, $Kh = 1$ and $\sigma t = 100$. From Figure 1, it is observed that the wave amplitude increases with the decreasing values of $\frac{Q}{h^2}$. This is due to the fact that as $\frac{Q}{h^2}$ diminishes the upper surface tends to become free which makes it lighter and so, the wave height increases.

The effect of the ice-cover surface on the wave motion is illustrated in Figure 2. In this figure the surface displacement $\frac{\zeta_2}{h}$ is plotted versus R/h for three different values of the ice-cover parameter $\frac{D}{h^4} = 0.5, 0.2, 0.01$ with $Gh = 0.5$, $\frac{\epsilon_2}{h} = 0.01$, $\frac{\eta}{h} = 0.8$, $\frac{a}{h} = 0.6$, $Kh = 1$ and $\sigma t = 100$. Here also it is observed that when the waves pass over a heavier surface, its amplitude is reduced by a significant amount.

Also, to visualize the effect of the porosity on the wave motion, the dimensionless surface displacement $\frac{\zeta_1}{h}$ and $\frac{\zeta_2}{h}$ are presented graphically against R/h in Figures 3 and 4. In Figure 3 the surface displacement $\frac{\zeta_1}{h}$ is depicted for the three different values of the porous effect parameter $Gh = 0.5, 0.2, 0.01$ with tension-parameter $\frac{Q}{h^2} = 0.5$ and $\frac{\epsilon_1}{h} = 0.01$, $\frac{\eta}{h} = 0.8$, $\frac{a}{h} = 0.6$, $Kh = 1$ and $\sigma t = 100$. In Figure 4 the surface displacement $\frac{\zeta_2}{h}$ is depicted for the three different values of the porous effect parameter $Gh = 0.5, 0.2, 0.01$ with ice-cover parameter $\frac{D}{h^4} = 0.5$ and $\frac{\epsilon_2}{h} = 0.01$. Other parametric values are same as those in Figure 3. We notice from these figures that the wave amplitude steadily increases with the increase in the values of Gh. Thus the presence of porous bed amplifies the wave height irrespective of the material of the upper surface.

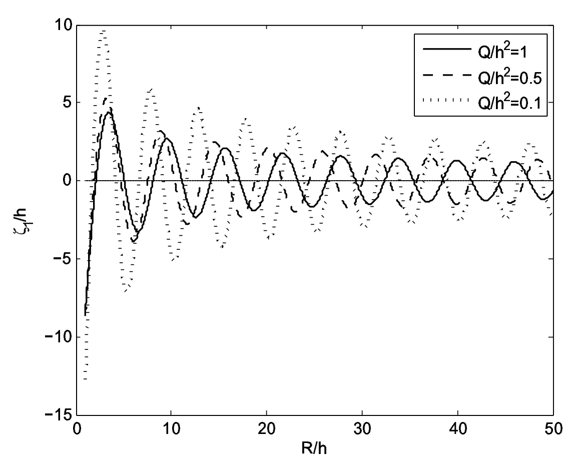

Figure 1 : Surface displacement for $Gh = 0.5$, $\frac{\epsilon_1}{h} = 0.01$, $\frac{\eta}{h} = 0.8$, $\frac{a}{h} = 0.6$, $Kh = 1$, $\sigma t = 100$.

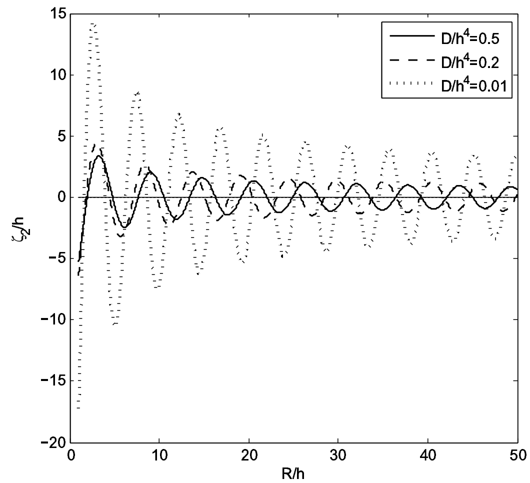

Figure 2 : Surface displacement for $Gh = 0.5$, $\frac{\epsilon_2}{h} = 0.01$, $\frac{\eta}{h} = 0.8$, $\frac{a}{h} = 0.6$, $Kh = 1$, $\sigma t = 100$.

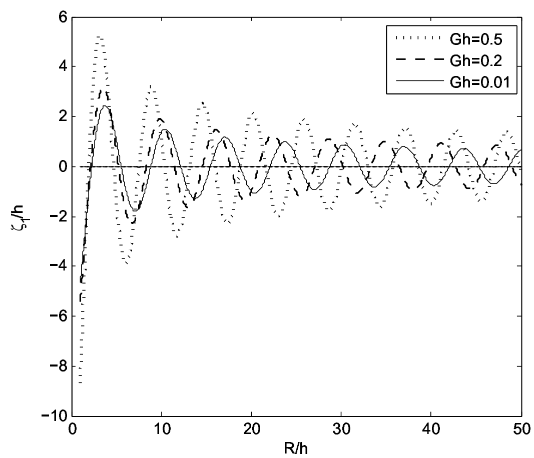

Figure 3 : Surface displacement for $\frac{\epsilon_1}{h} = 0.01$, $\frac{\eta}{h} = 0.8$, $\frac{a}{h} = 0.6$, $Kh = 1$, $\sigma t = 100$, $\frac{Q}{h^2} = 0.5$

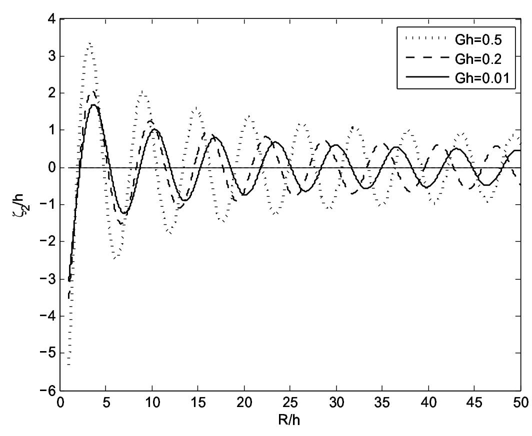

Figure 4 : Surface displacement for $\frac{\epsilon_2}{h} = 0.01$, $\frac{\eta}{h} = 0.8$, $\frac{a}{h} = 0.6$, $Kh = 1$, $\sigma t = 100$, $\frac{D}{h^4} = 0.5$

5. Conclusion

The velocity potential due to the presence of a horizontal circular ring of wave sources of time-harmonic strength in water of finite constant depth with a floating elastic plate or a floating membrane is obtained when the bottom of the fluid region is composed of non-dissipative porous medium. The steady-state analysis of the potentials reveals the existence of outgoing progressive waves of any frequency under the ice or the membrane cover. If the elastic parameter D or the tension parameter Q is put equal to zero, then the ice-covered or the membrane covered surface becomes an inertial surface. In this case outgoing time-harmonic progressive waves exist only when the angular frequency is less than a certain constant which depends on the surface density of the inertial surface. The dimensionless form of the ice-covered or the membrane covered surface profile is plotted in a number of figures for different values of the tension parameter for the membrane, flexural rigidity of ice and the porous-effect parameter. It has been observed that the wave amplitude increases with the decreasing values of the tension parameter $\frac{Q}{h^2}$ and the ice-cover parameter $\frac{D}{h^4}$, whereas it increases with the increasing values of the porous-effect parameter Gh. The results developed here are expected to be helpful in the areas of coastal and marine engineering. The problem may be further extended to a two-fluid medium (cf. Mohapatra *et al.* [25]) where the upper fluid is bounded above by a membrane or an ice cover and the lower fluid has a porous bed. Also, similar analysis may be done when the ring source is present in a fluid medium with a viscoelastic and poroelastic beds (cf. Das *et al.* [7] and Das and Sahoo [8]).

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Appendix A: Roots of the dispersion equation (35)

In order to discuss the roots of the dispersion equation (35), we first rewrite the equation as

$$\tanh kh = \frac{kG(Qk^2 - \epsilon_1 K + 1) + kK}{k^2(Qk^2 - \epsilon_1 K + 1) + KG}. \tag{A.1}$$

If we plot $y = \tanh kh$ and $y = \frac{kG(Qk^2-\epsilon_1 K+1)+kK}{k^2(Qk^2-\epsilon_1 K+1)+KG}$ against kh, they intersect at $kh = 0$ and exactly once for $kh > 0$. But these two functions are odd in k, so there are always exactly two real roots occurring as plus and minus of some positive quantity which we denote as μ_{01}. When $k = ip$ for some real p, equation $(A.1)$ becomes

$$\tan ph = \frac{pG(-Qp^2 - \epsilon_1 K + 1) + pK}{-p^2(-Qp^2 - \epsilon_1 K + 1) + KG}. \tag{A.2}$$

The above equation has infinite number of real roots which are given by $\pm\alpha_{1n}(n = 1, 2, ...)$. Thus equation $(A.1)$ has infinite number of purely imaginary roots $\pm i\alpha_{1n}(n = 1, 2, ...)$. To find out other possible roots, we rearrange the equation $(A.1)$ as

$$\gamma_1(k) \equiv e^{2kh} - \frac{(k+G)\{k(Qk^2+1-\epsilon_1 K)+K\}}{(k-G)\{k(Qk^2+1-\epsilon_1 K)-K\}} = 0. \qquad (A.3)$$

In order to find the possible existence of complex roots, we consider the contour C formed by the boundary of the square with vertices at $(\pm(N+\frac{1}{4})\frac{\pi}{h}, \pm(N+\frac{1}{4})\frac{\pi}{h})$ in the complex k-plane where k is taken anti-clockwise on the square. For a large integer N, it is easy to see that the change in the argument of $\gamma_1(k)$ along C is $(2N+1)2\pi$ and so from the argument principle, the number of zeros of $\gamma_1(k)$ minus the number of poles of $\gamma_1(k)$ enclosed in the square is $(2N+1)$. Now, one can observe that $\gamma_1(k)$ has four poles inside the contour given by the zeros of the denominator of the fraction appearing in the equation $(A.3)$. Therefore, the number of zeros of $\gamma_1(k)$ that lie within the square is $2N+5$. But already we have found that $\gamma_1(k)$ has three real zeros and $2N$ purely imaginary zeros inside the square. So there must be two complex zeros inside C. Since the equation $(A.1)$ has real coefficient it follows that these must be conjugate to each other which we denote as ν_1 and $\overline{\nu}_1$.

Now we consider the change in argument of the function $\gamma_2(p)$ where

$$\gamma_2(p) = \tan ph\{-p^2(-Qp^2-\epsilon_1 K+1)+KG\} - \{pG(-Qp^2-\epsilon_1 K+1)+pK\},$$

while traversing the triangular contour with vertices (0,0) and $((N+\frac{1}{4})\frac{\pi}{h}, \pm(N+\frac{1}{4})\frac{\pi}{h})$. We find that the number of zeros of $\gamma_2(p)$ minus the number of poles of $\gamma_2(p)$ enclosed in the triangle is zero. Since the tan function has N real poles inside the contour, so the number of zeros of $\gamma_2(p)$ that lie within the triangle is N. But already we have found that $\gamma_2(p)$ has N real zeros inside the triangle, so there are no zeros other than the real ones.

Similarly, if we consider the change in argument of the function $\gamma_2(p)$ in traversing the triangular contour with vertices (0, 0) and $(-(N+\frac{1}{4})\frac{\pi}{h}, \pm(N+\frac{1}{4})\frac{\pi}{h})$, we find that the number of zeros of $\gamma_2(p)$ minus the number of poles of $\gamma_2(p)$ enclosed in the triangle is two. Since the tan function has N real poles inside the contour, so the number of zeros of $\gamma_2(p)$ that lie within the triangle is $N+2$. But already we have found that $\gamma_2(p)$ has N real zeros inside the triangle, so there are two zeros other than the real ones. Since zeros of $\gamma_2(p)$ are i times the roots of equation $(A.1)$, we find that Im (ν_1) $>$ Re (ν_1) where $\nu_1 = \alpha + i\beta$ with $\alpha < 0, \beta > 0$. A schematic sketch of the roots are depicted in Figure 5.

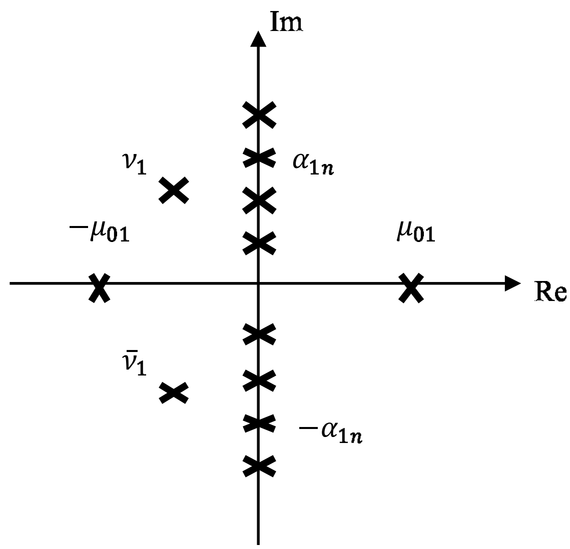

Figure 5 : Zeros of membrane cover region dispersion equation.

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