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REMARKS ON RAMANUJAM-KAWAMATA-VIEHWEG VANISHING THEOREM

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In this article we prove a general result on a nef vector bundle E on a projective manifold X of dimension n depending on the vector space $H^{n,n}(X,E)$. It is also shown that $H^{n,n}(X,E) = 0$ for an indecomposable nef rank 2 vector bundles E on some specific type of n dimensional projective manifold X. The same vanishing shown to hold for indecomposable nef and big rank 2 vector bundles on any variety with trivial canonical bundle.

Key words : Nef and big vector bundle; vanishing and non vanishing cohomology groups.

1. Introduction

Let X be a smooth projective complex manifold of dimension n. For any coherent sheaf E on X, we denote $H^{p,q}(X,E)$ the cohomology group $H^q(X, E \otimes \Omega^p_X)$, where Ω^p_X is the sheaf of holomorphic differential forms of degree p on X.

Akizuki-Kodaira-Nakano famous vanishing theorem says:

If L is an ample line bundle on a projective manifold X of dimension n, then

$$H^{p,q}(X,L) = 0 \ \text{ for } \ p+q-n > 0.$$

The particular case $p = n$ is the Kodaira vanishing theorem. The Kodaira vanishing theorem was extended to nef and big line bundle on a smooth surface by Ramanujam [7] and for higher dimension by Kawamata [3] and Viehweg [9].

Ramanujam has given in [7] an example showing that in general, one does not expect Akizuki-Kodaira-Nakano type vanishing result for nef and big line bundle.

Le Potier [6] generalized the Akizuki-Kodaira-Nakano type vanishing theorem to the case of ample vector bundle as follows:

If E is an ample vector bundle of rank r on a projective manifold X of dimension n, then

$$H^{p,q}(X,E) = 0 \quad \text{for} \quad p+q-n > r-1. \tag{1}$$

The vanishing results of Ramanujam-Kawamata-Viehweg and Le Potier naturally led to ask the following question:

Let E be a nef and big vector bundle of rank r on a projective manifold X of dimension n. Is

$$H^{n,q}(X,E) = 0 \quad \text{for} \quad q > r-1? \tag{2}$$

The example given by Ramanujam in [7] shows that one can not expect in general “Akizuki-Kodaira-Nakano” type of vanishing for nef and big line bundle. The same example can also be used to show that the question (2) has a negative answer (see Example 4.2).

Regarding the question (2) for a nef and big rank two vector bundle E on a smooth surface X the only group which one hope to vanish is the group $H^{2,2}(X,E)$. In trying to investigate this problem we obtained the following:

***Theorem* 1.1** — *Let E be a nef vector bundle of rank r on a projective manifold X of dimension n. Set $k(E) := \dim H^{n,n}(X,E)$. Then $k(E) \leq r$ and E admits a trivial bundle of rank $k(E)$ as quotient. In particular, $k(E) = r$ if and only if E is isomorphic to trivial vector bundle of rank r.*

Corollary 1.2 — Let E be an indecomposable nef vector bundle of rank r on a projective manifold X of dimension n. Assume that $c_r(E) \neq 0$. Then $H^{n,n}(X,E) = 0$.

For the case of rank 2 vector bundles we have the following:

***Theorem* 1.3** — *Let E be an indecomposable nef vector bundle of rank 2 on a projective manifold X of dimension n. If $H^1(X, \det(E)) = 0$, then $H^{n,n}(X,E) = 0$.*

As a consequence we obtain:

Corollary 1.4 — Let X be a Grassmannian of dimension $n \geq 2$ or a complete intersection of dimension $n \geq 3$ in a Grassmannian. If E is an indecomposable nef vector bundle of rank 2 on X, then $H^{n,n}(X,E) = 0$.

Corollary 1.5 — Let X be a projective manifold of dimension $n \geq 2$ with $K_X = \mathcal{O}_X$. If E is an indecomposable nef and big vector bundle of rank 2 on X, then $H^{n,n}(X, E) = 0$.

We recall a vanishing theorem of Schneider [8] related to nef and big vector bundle, in a slightly different version from the original one, but follows from the proof given there.

***Theorem* 1.6** — *Let E (resp. L) be a vector bundle (resp. line bundle) on a projective manifold X of dimension n. If $E \otimes L$ is nef and big then*

$$H^{n,q}(X, S^k(E) \otimes \det(E) \otimes L) = 0, \text{ for } q > 0.$$

2. Notations and Definitions

Throughout we work over the field of complex numbers.

For a vector bundle E on a projective manifold X, we will denote by $E^\vee$ the dual of E, $c_i(E) \in H^{2i}(X, \mathbb{Z})$ is the i-th chern class of E, $\mathbb{P}(E)$ is the projective bundle whose fiber over a point $x \in X$ is the projective space of $1-$dimensional quotients of the vector space E_x, and $\mathcal{O}_{\mathbb{P}(E)}(1)$ the universal quotient line bundle on $\mathbb{P}(E)$.

Definition 2.1 — Let X be a projective manifold of dimension n. A line bundle L on X is called nef, if for every irreducible curve C in X degree of $L|_C$ is non negative. A nef line bundle L is called big if $c_1(L)^n > 0$.

A vector bundle E on X is said to be nef if the line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ on $\mathbb{P}(E)$ is nef.

A nef vector bundle E is said to be big if $\mathcal{O}_{\mathbb{P}(E)}(1)$ on $\mathbb{P}(E)$ is big or equivalently

$$s_n(E) = p_*(c_1(\mathcal{O}_{\mathbb{P}(E)}(1))^{n+r-1} > 0,$$

where $s_n(E)$ is the n-th Segre class of E and $p : \mathbb{P}(E) \to X$ be the natural projection.

3. Proof of the Results

First we recall some results which we need.

Proposition 3.1 — [5, Proposition 6.1.18(i)]. A vector bundle E on X is nef if and only if the following condition is satisfied:

Given any morphism $f : C \to X$ finite onto its image from an irreducible smooth curve C to X, and given any quotient line bundle L of $f^*(E)$, then one has $\deg L \geq 0$.

Lemma 3.2 — [1, Proposition 1.16]. Let E be a nef vector bundle on a projective manifold X of dimension n. If σ is a non-zero section of $E^\vee$ then σ is nowhere vanishing on X.

PROOF : The proof given in [1] uses analytic methods. Here we give an algebraic proof. First we prove the lemma when X is a curve. In this case if σ vanishes at some points, we get a positive degree line sub bundle of $E^\vee$. By dualizing we see that E has a line bundle quotient of negative degree. This is a contradiction to the Proposition 3.1. Thus σ is nowhere vanishing on X.

For the general case, assume σ vanishes at some points and dimension of X is greater than one. Let Z be the subscheme of X defined by the vanishing of σ and I_Z denotes its sheaf of ideals. The section σ induces surjection

$$\sigma : E \to I_Z \to 0. \tag{3}$$

Let C be a smooth curve in X with the property $D = C \cap Z$ is a non-empty proper closed subscheme of C. Then by restricting the surjective map σ to C and going modulo torsion we get a surjection:

$$\tau : E|_C \to \mathcal{O}_C(-D) \to 0. \tag{4}$$

Since C is a smooth curve $\mathcal{O}_C(-D)$ is a line bundle of negative degree, which is a contradiction to the fact that E is nef. Hence we must have $Z = \emptyset$. □

Lemma 3.3 — [10, see, Proposition 4.8]. If E is a nef and big vector bundle on a Kähler manifold X, then the line bundle $\det(E)$ on X is big.

The Dominance theorem [Theorem 3.3] in [4] ensures that $\det(E)$ is nef.

We also need to recall the proposition [Prop. 1.15 (iii)] in [1]. We will state it in a different version, which follows immediatly from the proof given there.

Lemma 3.4 — Let

$$0 \to F \to E \to Q \to 0$$

be an exact sequence of holomorphic vector bundles and $rank(E) = r, rank(F) = f$.

If $\wedge^{r-f+1} E \otimes \det Q^{-1}$ is nef (resp. ample), then F is nef (resp. ample).

PROOF OF THEOREM 1.1 : The proof is by induction on the $rank(E) = r$. If $r = 1$ and $k(E) = 0$ then there is nothing to prove.

If $k(E) > 0$, then by Lemma 3.2 there is a non zero homomorphism

$$\sigma : \mathcal{O}_X \to E^\vee$$

which is nowhere vanishing. This implies that E is a trivial bundle of rank one. Since $k(\mathcal{O}_X) = 1$, the Theorem follows in this case.

Let $r > 1$. We assume our Theorem holds for all nef vector bundles of rank less than or equal $r-1$. Again, if $k(E) = 0$ there is nothing to prove. So we assume $k(E) > 0$. Then applying Lemma 3.2 we get an exact sequence

$$0 \to \mathcal{O}_X \to E^\vee \to F^\vee \to 0, \tag{5}$$

where $F^\vee$ is a dual of vector bundle F of rank $r-1$. Dualizing (5) we get an exact sequence

$$0 \to F \to E \to \mathcal{O}_X \to 0. \tag{6}$$

By Lemma 3.4 F is a nef vector bundle. Now since $rank(F) = r-1$, we have by induction assumption $k(F) \leq r-1$ and F admits a trivial quotient of rank $k(F)$. This implies by duality $F^\vee$ admits trivial subbundle of rank $k(F)$. Now from the long cohomology exact sequence

$$0 \to H^0(X, \mathcal{O}_X) \to H^0(X, E^\vee) \to H^0(X, F^\vee) \to \cdots . \tag{7}$$

associated to the exact sequence (5), we deduce

$$k(E) - 1 \leq k(F) \leq r - 1,$$

and the image of global sections of $E^\vee$ generate a trivial subbundle V of rank $k(E)-1$ in $F^\vee$. Taking the inverse image of this V we see that $E^\vee$ admits a subbundle $S^\vee$ of rank $k(E)$. Note that $S^\vee$ is an extension of ${\mathcal{O}_X}^{k(E)-1}$ by $\mathcal{O}_X$. The dual S of $S^\vee$ is nef, since it is an extension of trivial bundle of rank $k(E)-1$ by a trivial bundle of rank 1. If $k(E) < r$ then it follows by induction S is trivial. This proves the result.

If $k(E) = r$ then again by induction $F = {\mathcal{O}_X}^{r-1}$ and all the sections of $F^\vee$ lifts to sections of $E^\vee$, hence $E^\vee$ and E are isomorphic to ${\mathcal{O}_X}^r$. □

PROOF OF THEOREM 1.3 : Assume $H^{n,n}(X, E) \neq 0$, then we get by Serre duality $H^{0,0}(X, E^\vee) \neq 0$. Let σ be a non-zero section of $E^\vee$. Since E is nef by Lemma 3.2 the section σ is nowhere vanishing, and gives an exact sequence

$$0 \to \mathcal{O}_X \to E^\vee \to \det(E)^\vee \to 0. \tag{8}$$

This extension gives a class in the cohomology group $H^1(X, \det(E))$. But by our assumption this group is zero and hence the extension splits. Thus $E^\vee$ splits and hence E splits too, this is a contadiction. □

PROOF OF COROLLARY 1.2 : If $H^{n,n}(X, E) \neq 0$, then by Theorem 1.1 we get an exact sequence

$$0 \to F \to E \to \mathcal{O}_X \to 0. \tag{9}$$

This implies $c_r(E) = c_r(F) = 0$, this is a contradiction. □

PROOF OF COROLLARY 1.4 : If X is a Grassmannian of dimension ≥ 2 or a complete intersection of dimension ≥ 3 in a Grassmannian, then for any line bundle L $H^1(X, L) = 0$. Hence if E is an indecomposable vector bundle of rank two on X, then the hypothesis of Theorem 1.1 holds for E. □

PROOF OF COROLLARY 1.5 : Assume $H^{n,n}(X, E) \neq 0$. Since E is nef and big, $\det(E)$ is nef and big by the Lemma 3.3. Hence we have an exact sequence:

$$0 \to \det(E) \to E \to \mathcal{O}_X \to 0. \tag{10}$$

But K_X is trivial implies $H^1(X, \det(E)) = 0$ by Kawamata-Ramanujam-Viehweg vanishing theorem. Hence that the exact sequence (10) splits and hence E is decomposable, which is a contradiction. □

Remark 3.5 : Corollary 1.5 applies for example to complex algebraic torus, $K3$ surfaces and Calabi-Yau manifolds. □

4. COUNTER EXAMPLES OF RAMANUJAM

Example 4.1 : The following example is due to Ramanujam [7]. Denote $\mathbb{P}^3$ blown up at a point by X and $\pi : X \to \mathbb{P}^3$ be the natural morphism and $L = \pi^*(\mathcal{O}_{\mathbb{P}^3}(1))$. Clearly the line bundle L is nef and big and hence $H^1(X, \Omega^1_X \otimes L^{-1}) \neq 0$.

Example 4.2 : Note that the variety X in the Example 4.1 can be identified with $\mathbb{P}(E)$ in such a way that $L \simeq \mathcal{O}_{\mathbb{P}(E)}(1)$, where $E = \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1)$. Clearly the bundle E on $\mathbb{P}^2$ is nef and big and $H^{2,2}(\mathbb{P}^2, E) \neq 0$. This shows that one can not expect Le Potier type vanishing result for nef and big vector bundle even for $p = n$.

More general example: if Y is a projective manifold of dimension n and H is an ample line bundle on Y, then the vector bundle $E = \mathcal{O}_Y \oplus H$ is nef and big vector bundle but $H^{n,n}(Y, E) \neq 0$.

Remark 4.3 : The non vanishing of $H^{1,1}(X, L^{-1})$ of Example 4.1 can be deduced from the non vanishing of the group $H^{2,2}(\mathbb{P}^2, E)$ in Example 4.2. Indeed:

$$H^2(X, \Omega^2_X \otimes L) \simeq H^{2,2}(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1))$$

by Le Potier isomorphism [6, Lemma 8].

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