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IDEALS IN CA-AG-GROUPOIDS

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An AG-groupoid S satisfying the identity $u(vw) = w(uv)\ \forall u, v, w \in S$ is called a CA-AG-groupoid [1]. This article is devoted to the study of various characterizations of (left/right) ideals in CA-AG-groupoids and to the relationships between (left/right) connected sets and (left/right) ideals in CA-AG-groupoids.

Key words : AG-groupoid; LA-semigroup; CA-AG-groupoid; ideals; connected sets.

1. Introduction and Preliminaries

A groupoid $(G, \cdot)$ or simply G satisfy $(uv)w = (wv)u\ \forall u, v, w \in G$ (called left invertive law [2]) is called an Abel-Grassmann's groupoid (in short AG-groupoid [3]). The said structure is called upon by different names by different authors, such as left almost semigroup (LA-semigroup) [2], right modular groupoid [4], left invertive groupoid [5]. Throughout the article we will denote an AG-groupoid simply by S otherwise stated else. Also to avoid excessive parenthesization and dots, we will use uv for $u \cdot v$, $uv \cdot wt$ for $(uv)(wt)$, and $(uv \cdot w)t$ for $((uv)w)t$. The medial law: $uv \cdot wt = uw \cdot vt$ always holds in S [6, Lemma 1.1(i)]. Left identity may or may not contained in S; however, if S contains left identity then it is unique [7] and S with left identity always satisfies the paramedial law: $uv \cdot wt = tv \cdot wu$ [6, Lemma 1.2(ii)]. Now, we define some elementary aspects and quote few definitions which are essential to step up this study. An element $f \in S$ is called idempotent if $f^2 = f$ and an AG-groupoid having all elements as idempotent is called AG-2-band (in short AG-band) [8]. If S is an AG-band then $S^2 = S$. A commutative AG-band is called a semilattice. S is called T^1-AG-groupoid if $uv = wt$ implies $vu = tw\ \forall\, u, v, w, t \in S$ [9]. S is called right alternative (left alternative) if $u(ww) = (uw)w$ $((uu)w = u(uw))\ \forall\, u, w \in S$. S is called alternative if it is simultaneously right and left alternative [9]. An element $e \in S$ is called left identity (right identity) of S if $eu = u \forall u \in S$ $(ue = u \forall u \in S)$. In S left identity does not implies right identity, however a right identity is always a left identity [7, Theorem 2.3]. S is called AG* [10], if $\forall u, v, w \in S$, $(uv)w = v(uw)$ and is called cyclic associative AG-groupoid (in short CA-AG-groupoid) if $u(vw) = w(uv)$ [1]. In [1] Iqbal *et al.* enumerated CA-AG-groupoids up to order 6 and further classified it into different subclasses. Further, they introduced CA-test for verification of arbitrary AG-groupoid to be cyclic associative, and studied some fundamental properties of CA-AG-groupoids. The same authors in [11] discussed different aspect of cancellativity of an element in CA-AG-groupoid and provided a partial solution to an open problem given in [12]. For detail study of properties of CA-AG-groupoids we recommend [1, 11].

For any two subsets (by a subset in the article we means a non-empty subset) M, N of S and $u \in S, uM = \{um : m \in M\}, Mu = \{mu : m \in M\}$ and $MN = \{mn : m \in M \wedge n \in N\}$. If S is a CA-AG-groupoid and $L, M, N \subseteq S$, then for all $l \in L, m \in M$ and $n \in N,\ l(mn) = n(lm) = m(nl)$, which implies that $L(MN) = N(LM) = M(NL)$.

An AG-subgroupoid of S is a subset H of S such that $H^2 \subseteq H$. A subset K of S is said to be right (left) ideal if $KS \subseteq K$ $(SK \subseteq K)$ and K is called a two-sided ideal (in short an ideal) if K is simultaneously right and left ideal of S [13]. $L, M \subseteq S$ are called right connected (left connected) sets if $LS \subseteq M$ and $MS \subseteq L($ $SL \subseteq M$ and $SM \subseteq L)$. L and M are called connected sets if they are simultaneously right and left connected [13].

It is proved in [14] that if an AG-groupoid S contains left identity e then $SS = S$ and $S = eS = Se$, i.e. e generates the same left and right ideals. Also if S has left identity then every right ideal R is ideal, SR is left ideal and RS is right ideal of S. Also for left ideal L, right ideal R and $a \in S$, aL is left ideal, R^2 is an ideal of S. The set of ideals of a regular LA-semigroup form a semilattice, for each ideal there exists a minimal prime ideal.

In [15] Mushtaq *et al.* proved that in AG-3-band S (i) right ideal implies left ideal and vice versa, (ii) for two ideals M and N of S, (a) MN is an ideal and (b) MN and NM are connected sets. They also proved that if the set of all ideals of S is totally order then every ideal of S is prime and the set of ideals of S is a semilattice.

2. IDEAL IN CA-AG-GROUPOIDS

To begin with, we illustrate the existence of (left/right) ideals in CA-AG-groupoid by providing supporting examples, and to demonstrate by various counterexamples that left and right ideal are distinct for a CA-AG-groupoid. We also provide examples of subsets of CA-AG-groupoid which are right as well as left ideal, and verify that a subset of S may or may not be an ideal. Furthermore, by examples we demonstrate that CA-AG-groupoid may have no ideal, one ideal or more than one ideals.

Example 1 : Let $S = \{0, 1, 2, 3, 4, 5, 6, 7\}$ and the binary operation on S is defined by the following Cayley's table:

$\cdot$	0	1	2	3	4	5	6	7
0	3	3	5	3	3	3	7	3
1	4	3	3	3	3	7	3	3
2	3	6	3	3	7	3	3	3
3	3	3	3	3	3	3	3	3
4	3	3	3	3	3	3	3	3
5	3	3	3	3	3	3	3	3
6	3	3	3	3	3	3	3	3
7	3	3	3	3	3	3	3	3

Then S is a CA-AG-groupoid (can be verified by CA-test [1]). Furthermore, we have the following observations;

(i) For $L = \{0, 3, 4, 7\}$, $SL = \{3, 4, 7\} \subseteq L$, thus L is left ideal of S. However, L is not right ideal as $LS = \{3, 5, 7\} \nsubseteq L$. Similarly, $\{2, 3, 5, 7\}$ and $\{0, 2, 3, 4, 5, 7\}$ are left ideals but not right ideals.

(ii) Each of the subsets $\{3, 4\}, \{3, 5\}, \{3, 6\}$,$\{3, 4, 5\}, \{3, 4, 6\}, \{3, 5, 6\}, \{0, 3, 5, 7\}, \{3, 4, 5, 6\}$, $\{1, 3, 4, 7\}$ is a right ideal but not a left ideal of S.

(iii) $\{3\}, \{3, 7\}, \{3, 4, 7\}, \{3, 5, 6\}, \{3, 5, 7\}, \{3, 6, 7\}$, $\{3, 4, 5, 7\}$, $\{3, 4, 6, 7\}$, $\{3, 4, 5, 6, 7\}$, $\{3, 4, 5, 7\}$, $\{3, 4, 6, 7\}$,$\{3, 5, 6, 7\}$, etc. are ideals of S.

(iv) $\{0, 1\}, \{0, 2\}, \{0, 3\}, \{1, 2\}, \{1, 4\}, \{1, 5\}, \{1, 6\}, \{1, 7\}, \{2, 5\}, \{2, 6\}, \{2, 7\}, \{4, 5\}$,$\{4, 6\}$, $\{4, 7\}, \{5, 6\}, \{5, 7\}$,$\{6, 7\}, \{0, 1, 6\}, \{0, 1, 7\}, \{1, 2, 3\}$,$\{1, 3, 4\}, \{1, 3, 5\}, \{1, 3, 6\}, \{0, 1, 4\}$, $\{0, 1, 5\}, \{0, 2, 4\}$, $\{0, 2, 5\}$, $\{2, 3, 7\}, \{4, 5, 6\}$, $\{4, 5, 7\}, \{4, 6, 7\}$,$\{5, 6, 7\}, \{0, 1, 2, 4\}$, $\{0, 1, 2, 5\}, \{4, 5, 6, 7\}$, etc are neither right nor left ideals of S.

(v) Taking $M_1 = \{3, 4, 5, 6\}$ and $N_1 = \{3, 4, 7\}$, then M_1 and N_1 are right connected sets, as: $M_1S = \{3\} \subseteq N_1$ and $N_1S = \{3\} \subseteq M_1$. However, M_1 and N_1 are not left connected, as: $SN_1 = \{3, 7\} \nsubseteq M_1$. Hence M_1 and N_1 are non-connected sets. Note that M_1 is right ideal and N_1 is left ideal, however, M_1 and N_1 are not left connected.

(vi) Taking $M_2 = \{2, 3, 7\}$ and $N_2 = \{3, 5\}$, then $SM_2 = \{3, 5\} \subseteq N_2$ and $SN_2 = \{3, 7\} \subseteq M_2$, thus M_2 and N_2 are left connected sets. However, M_2 and N_2 are not right connected, as: $M_2S = \{3, 6, 7\} \nsubseteq N_2$. Note that M_2 is not an ideal and N_2 is right ideal of S, but still M_2 and N_2 are left connected sets, however both are not right connected.

Noteworthy that if right identity contains in an AG-groupoid then it form a commutative semigroup [7], thus right and left ideals coincide in such case. Also, as in CA-AG-groupoid left identity implies (right) identity [11], thus if left identity contains in CA-AG-groupoid then right and left ideals coincide.

Example 2 : Let $\mathbb{Z}_n$ be the CA-AG-groupoid as mentioned, then under usual multiplication modulo n.

(i) $\mathbb{Z}_4 = \{0, 1, 2, 3\}$ is CA-AG-groupoid, $\{0, 2\}$ is the only ideal of $\mathbb{Z}_4$.

(ii) $\mathbb{Z}_8 = \{0, 1, 2, 3, 4, 5, 6, 7\}$ is CA-AG-groupoid. $\{0, 2, 4, 6\}$ and $\{0, 4\}$ are ideals of $\mathbb{Z}_8$. However, $\{2, 4, 6\}$, $\{0, 3\}$ and $\{0, 6\}$ are not ideals of $\mathbb{Z}_8$. Hence, a subset of CA-AG-groupoid may not be an ideal.

(iii) $\mathbb{Z}_{12} = \{0, 1, ..., 11\}$ is CA-AG-groupoid. $\{0, 6\}$, $\{0, 4, 8\}$, $\{0, 2, 4, 6, 8, 10\}$ and $\{0, 3, 6, 9\}$ are ideals of $\mathbb{Z}_{12}$.

Example 3 : $\mathbb{Z}^+ \cup \{0\}$ is CA-AG-groupoid and $k(\mathbb{Z}^+ \cup \{0\}) = \{0, k, 2k, 3k, ...\}$ is an ideal of $\mathbb{Z}^+ \cup \{0\}$ for every integer $k \in \mathbb{Z}^+$.

Example 4 :

(i) $D_2 = \left\{ \begin{bmatrix} u & 0 \\ 0 & v \end{bmatrix} \mid u, v \in \mathbb{Z} \right\}$ is an infinite CA-AG-groupoid under multiplication and $A_2 = \left\{ \begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix} \mid u \in \mathbb{Z} \right\}$, is CA-AG-subgroupoid of D_2. D_2 have no proper ideal. Similarly, $D_3 = \left\{ \begin{bmatrix} u & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & w \end{bmatrix} \mid u, v, w \in \mathbb{Z} \right\}$ is CA-AG-groupoid under multiplication having no proper ideal. Generally D_n, set of all diagonal matrices over $\mathbb{Z}$, is CA-AG-groupoid under multiplication having no proper ideal.

(ii) $G_n = \{(a_{ij}) \mid a_{ij} \in \mathbb{Z}\}$, of $n \times n$ matrices is CA-AG-groupoid under usual addition, for $n = 1, 2, 3.., n$. G_n has no proper ideal in their respective order. Note that $U_n = \{$set of all upper triangular $n \times n$ matrices with entries from $\mathbb{Z}\}$ is a CA-AG-subgroupoid of G_n.

2.1 *Characterization of CA-AG-groupoids by Ideals*

In the following, we establish relationship between (left/right) ideals and (left/right) connected sets in a CA-AG-groupoid S. We prove that for right ideal R, left ideal L and $M \subseteq S$, MR, RS, SR are left ideals, LM is left ideal if $LM = ML$. We also prove that for right ideals R_1, R_2 and left ideals L_1, L_2 of S, R_1R_2 and R_2R_1 are left ideals, furthermore L_1L_2 and L_2L_1 are connected sets. We depict these results by giving some sufficient examples. Our starting points is the following lemma.

Lemma 1 — Let R be a right ideal of CA-AG-groupoid S and $M \subseteq S$, then MR, RS and SR are left ideals.

PROOF : By cyclic associativity

$$S(MR) = R(SM) = M(RS) \subseteq MR \Rightarrow S(MR) \subseteq MR,$$
$$\text{and} \quad S(RS) = S(SR) = R(SS) \subseteq RS \Rightarrow S(RS) \subseteq RS,$$
$$\text{also} \quad S(SR) = R(SS) = S(RS) \subseteq SR \Rightarrow S(SR) \subseteq SR.$$

Hence, the result follows. □

In the following, we provide illustrative example for Lemma 1, and also provide a counterexample to verify that if right ideal R is replaced by left ideal L in Lemma 1 then ML may not be a left ideal.

Example 5 : Recall example 1, $R = \{1, 3, 4, 7\}$ and $L = \{2, 3, 5, 7\}$ are respectively right and left ideal of S. Taking $M = \{0, 5\} \subseteq S$, then $MR = \{3\}$ and $S(MR) = \{3\} \subseteq MR$, thus MR is left ideal. Now, $ML = \{3, 5\}$ and $S(ML) = \{3, 7\} \nsubseteq ML$, thus ML is not a left ideal.

***Theorem* 1** — *(i) If R_1 and R_2 are right ideals of a CA-AG-groupoid S, then R_1R_2 and R_2R_1 are left ideals,*

(ii) If L_1 and L_2 are left ideals of CA-AG-groupoid S, then L_1L_2 and L_2L_1 are connected sets.

PROOF : (i) By cyclic associativity

$$S(R_1R_2) = R_2(SR_1) = R_1(R_2S) \subseteq R_1R_2 \Rightarrow S(R_1R_2) \subseteq R_1R_2.$$

Hence, R_1R_2 is a left ideal of S. Similarly, $S(R_2R_1) \subseteq R_2R_1$, thus R_2R_1 is also a left ideal.

(ii) By cyclic associativity

$$S(L_1L_2) = L_2(SL_1) \subseteq L_2L_1 \text{ and } S(L_2L_1) = L_1(SL_2) \subseteq L_1L_2.$$

Hence, L_1L_2 and L_2L_1 are left connected sets. Now, by left invertive law

$$(L_1L_2)S = (SL_2)L_1 \subseteq L_2L_1 \text{ and } (L_2L_1)S = (SL_1)L_2 \subseteq L_1L_2.$$

Thus, L_1L_2 and L_2L_1 are also right connected. □

We illustrate Theorem 1, by the following example.

Example 6 : Recall Example 1, S is a CA-AG-groupoid.

(i) $R_1 = \{3, 4, 5\}$ and $R_2 = \{1, 3, 4, 7\}$ are right ideals of S. Now $R_1R_2 = \{3\}$, $R_2R_1 = \{3, 7\}$, $S(R_1R_2) = \{3\} \subseteq R_1R_2$ and $S(R_2R_1) = \{3\} \subseteq R_2R_1$, thus R_1R_2 and R_2R_1 are left ideals of S.

(ii) Taking $L_1 = \{0, 2, 3, 4, 5, 7\}$ and $L_2 = \{2, 3, 5, 7\}$ then L_1 and L_2 are left ideals of S. Now $L_1L_2 = \{3, 5\}$, $L_2L_1 = \{3, 7\}$, $S(L_1L_2) = \{3, 7\} \subseteq L_2L_1$ and $S(L_2L_1) = \{3\} \subseteq L_1L_2$, thus L_1L_2 and L_2L_1 are left connected. Again, $(L_1L_2)S = \{3\} \subseteq L_2L_1$ and $(L_2L_1)S = \{3\} \subseteq L_1L_2$, thus L_1L_2 and L_2L_1 are right connected, too. Hence, L_1L_2 and L_2L_1 are connected sets.

Coupling part (i) and part (ii) of Theorem 1, we have the following corollary.

Corollary 1 — If R_1, R_2 are right ideals of CA-AG-groupoid, then $R_1R_2 \cdot R_2R_1$ and $R_2R_1 \cdot R_1R_2$ are connected sets.

Example 7 : As shown in Example 6 part (i), for right ideals $R_1 = \{3, 4, 5\}$ and $R_2 = \{1, 3, 4, 7\}$, $R_1R_2 = \{3\}$ and $R_2R_1 = \{3, 7\}$ are left ideals. Now $R_1R_2 \cdot R_2R_1 = \{3\}$, $R_2R_1 \cdot R_1R_2 = \{3\}$, also $S(R_1R_2 \cdot R_2R_1) = \{3\} \subseteq R_2R_1 \cdot R_1R_2$ and $S(R_2R_1 \cdot R_1R_2) = \{3\} \subseteq R_1R_2 \cdot R_2R_1$, thus $R_1R_2 \cdot R_2R_1$ and $R_2R_1 \cdot R_1R_2$ are left connected. Again, $(R_1R_2 \cdot R_2R_1)S = \{3\} \subseteq R_2R_1 \cdot R_1R_2$ and $(R_2R_1 \cdot R_1R_2)S = \{3\} \subseteq R_1R_2 \cdot R_2R_1$, thus $R_1R_2 \cdot R_2R_1$ and $R_2R_1 \cdot R_1R_2$ are right connected, too. Hence, $R_1R_2 \cdot R_2R_1$ and $R_2R_1 \cdot R_1R_2$ are connected sets.

Lemma 2 — If R_1 and R_2 are right ideals of CA-AG*-groupoid S, then R_1R_2 and R_2R_1 are ideals of S.

PROOF : As proved in Theorem 1 part (i) that for right ideals R_1 and R_2, R_1R_2 and R_2R_1 are left ideals. Now, we proceed to show that R_1R_2 and R_2R_1 are right ideals. By left invertive law, AG* and cyclic associativity

$$(R_1R_2)S = (SR_2)R_1 = R_2(SR_1) = R_1(R_2S) \subseteq R_1R_2 \Rightarrow (R_1R_2)S \subseteq R_1R_2.$$

Similarly, $(R_2R_1)S \subseteq R_2R_1$. Hence, the result follows. □

Lemma 3 — If R_1 and R_2 are right ideals of AG*-groupoid S, then R_1R_2 and R_2R_1 are connected sets.

PROOF : By AG* and left invertive law

$$S(R_1R_2) = (R_1S)R_2 = (R_2S)R_1 \subseteq R_2R_1.$$

Similarly, $S(R_2R_1) \subseteq R_1R_2$, thus R_1R_2 and R_2R_1 are left connected. Again

$$(R_1R_2)S = R_2(R_1S) \subseteq R_2R_1.$$

Similarly, $(R_2R_1)S \subseteq R_1R_2$, thus R_1R_2 and R_2R_1 are right connected. □

Lemma 4 — If R and L are right and left ideals of CA-AG*-groupoid S, then

(i) RL and LR are ideals of S,

(ii) RL and LR are connected sets.

PROOF : (i) By using left invertive law, AG* and cyclic associativity

$$(LR)S = (SR)L = R(SL) = L(RS) \subseteq LR,$$
$$\text{and} \quad S(LR) = R(SL) = L(RS) = (RL)S = (SL)R \subseteq LR.$$

Thus, LR is an ideal. Again

$$(RL)S = L(RS) = S(LR) = R(SL) \subseteq RL,$$
$$\text{and} \quad S(RL) = L(SR) = R(LS) = S(RL) = (RS)L \subseteq RL.$$

Hence, RL is an ideal.

(ii) By AG* and cyclic associativity

$$S(LR) = R(SL) \subseteq RS \text{ and } S(RL) = L(SR) = (SL)R \subseteq LR,$$

thus LR and RL are left connected. Again by left invertive law and AG*

$$(LR)S = (SR)L = R(SL) \subseteq RS \text{ and } (RL)S = (SL)R \subseteq LR,$$

thus LR and RL are left connected. □

In the following, we prove that for left ideal of CA-AG-groupoid S and $M \subseteq S$, LM is left ideal if $LM = ML$, we also prove that LS and SL are left ideals and connected sets, and generalize this result and prove that $((((LS)S)S)...)S$ and $S(...(S(S(SL))))$ are left ideals and connected sets. We further prove that if the condition $LM = ML$ is modify to $ML \subseteq LM$ then in such situation LM is still a left ideal.

Lemma 5 — Let L is left ideal of CA-AG-groupoid S and $M \subseteq S$. Then

(i) LM is left ideal iff $LM = ML$,

(ii) LS and SL are left ideals,

(iii) LS and SL are connected sets.

PROOF :

(i) Let $LM = ML$, by cyclic associativity

$$S(LM) = M(SL) \subseteq ML = LM \Rightarrow S(LM) \subseteq LM.$$

Hence, LM is left ideal of S. Conversely, let LM is left ideal of S. Assume contrary that $LM \neq ML$. Then $S(LM) = M(SL) \subseteq ML \neq LM$, implies LM is not a left ideal, which is contradiction. Hence, $LM = ML$.

(ii) By cyclic associativity

$$S(LS) = S(SL) = L(SS) \subseteq LS \quad \text{and} \quad S(SL) \subseteq SL.$$

Thus, LS and SL are left ideals.

(iii) Again

$$S(LS) = S(SL) \subseteq SL \ \text{ and } \ S(SL) = L(SS) \subseteq LS.$$

Hence, LS and SL are left connected. Also,

$$(SL)S \subseteq LS \quad \text{and } (LS)S = (SS)L \subseteq SL.$$

Thus, SL and LS are right connected, too. □

Lemma 5 part (i) provided a guarantee for left ideal L of CA-AG-groupoid S and $M \subseteq S$ that LM will surely be a left ideal if $LM = ML$. However, if $LM \neq ML$ then it does not mean that LM will never be a left ideal. We provide example to verify that if $LM \neq ML$ then LM may or may not be a left ideal.

Example 8 : Recall Example 1, S is CA-AG-groupoid, $L = \{2, 3, 5, 7\}$ is left ideal of S. Taking $M_1 = \{1, 2\} \subseteq S$, then $LM_1 = \{3, 6\}$, $M_1L = \{3, 7\}$, thus $LM_1 \neq M_1L$, now $S(LM_1) = \{3, 7\} \nsubseteq LM_1$, thus LM_1 is not left ideal. Again, taking $M_2 = \{2, 3, 4\}$, then $LM_2 = \{3, 7\}$ and $M_2L = \{3\}$, thus $LM_2 \neq M_2L$, now $S(LM_2) = \{3\} \subseteq LM_2$, thus LM_2 is left ideal.

The above example clarify that if $ML \subset LM$ then LM is a left ideal and if neither $ML \subset LM$ nor $LM \subset ML$, then LM is not a left ideal.

As by part (ii) of Lemma 5, for left ideal L, LS and SL are left ideals. By using this result, as LS and SL are left ideals so $(LS)S$ and $S(SL)$ are left ideals and so on. Also by part (iii) of Lemma 5, for left ideal L, LS and SL are connected sets. As for left ideal L, LS and SL are left ideals (by part (ii)) so $(LS)S$ and $S(SL)$ are connected sets. Coupling part (i) and (ii), as $(LS)S$ and $S(SL)$ are left ideals so $((LS)S)S$ and $S(S(SL))$ are connected sets, and so on. Thus the following corollary is now obvious.

Corollary 2 — Let L be a left ideal of a CA-AG-groupoid S then

(i) $(LS)S$ and $S(SL)$ are left ideals and generally $((((LS)S)S)...)S$ and $S(...(S(S(SL))))$ are left ideals,

(ii) $(LS)S$ and $S(SL)$ are connected sets and generally $((((LS)S)S)...)S$ and $S(...(S(S(SL))))$ are connected sets.

Lemma 6 — Let L be a left ideal of a CA-AG-groupoid S and $M \subseteq S$ such that $ML \subset LM$, then LM is a left ideal of S.

PROOF : By cyclic associativity and given condition $S(LM) = M(SL) \subseteq ML \subset LM$, thus LM is left ideal of S. □

Remark 1 : If M and N are left connected then $M^2 = MM \subseteq SM \subseteq N \Rightarrow M^2 \subseteq N$. Similarly, $N^2 \subseteq M$. Again, if M and N are right connected then $M^2 \subseteq N$ and $N^2 \subseteq M$.

As proved in Lemma 1 that for right ideal R of CA-AG-groupoid S and $M \subseteq S$, MR is a left ideal. However, MR may not be a right ideal, similarly RM may not necessarily be a left or right ideal. In the following lemma, we prove that in case of CA-AG*-groupoid MR and RM are ideals of S and if R (right ideal) is replace by left ideal L then LM and ML are connected sets.

***Theorem* 2** — *Let R, L are respectively right and left ideals of CA-AG*-groupoid S and $M \subseteq S$. Then*

(i) MR and RM are ideals of S,

(ii) MR and RM are connected sets,

(iii) ML and LM are connected sets.

PROOF :

(i) By cyclic associativity

$$S(MR) = R(SM) = M(RS) \subseteq MR \Rightarrow S(MR) \subseteq MR.$$

Hence, MR is a left ideal of S. Again, by left invertive law, AG* and cyclic associativity

$$(MR)S = (SR)M = R(SM) = M(RS) \subseteq MR \Rightarrow (MR)S \subseteq MR.$$

Thus, MR is a right ideal and hence an ideal. Now, by definition of AG*

$$S(RM) = (RS)M \subseteq RM \Rightarrow S(RM) \subseteq RM.$$

Thus, RM is a left ideal. Again, by left invertive law, AG* and cyclic associativity

$$(RM)S = (SM)R = M(SR) = R(MS) = S(RM) = (RS)M \subseteq RM \Rightarrow (RM)S \subseteq RM.$$

Thus, RM is right ideal, too. Hence, RM is an ideal of S.

(ii) By left invertive law, AG* and cyclic associativity

$$\begin{aligned} S(MR) &= (MS)R = (RS)M \subseteq RM, \\ \text{and} \quad S(RM) &= M(SR) = R(MS) = (MR)S \\ &= (SR)M = R(SM) = M(RS) \subseteq MR. \end{aligned}$$

Thus, MR and RM are left connected. Again

$$\begin{aligned} (MR)S &= R(MS) = S(RM) = (RS)M \subseteq RM, \\ \text{and} \quad (RM)S &= M(RS) \subseteq MR. \end{aligned}$$

Thus, MR and RM are right connected.

(iii) By cyclic associativity and AG*

$$\begin{aligned} S(ML) &= L(SM) = (SL)M \subseteq LM \Rightarrow S(ML) \subseteq LM, \\ \text{and} \quad S(LM) &= M(SL) \subseteq ML \Rightarrow S(LM) \subseteq ML. \end{aligned}$$

Hence, ML and LM are left connected. Also, by left invertive law and AG*

$$\begin{aligned} (ML)S &= (SL)M \subseteq LM \Rightarrow (ML)S \subseteq LM, \\ \text{and} \quad (LM)S &= (SM)L = M(SL) \subseteq ML \Rightarrow (LM)S \subseteq ML. \end{aligned}$$

Thus, ML and LM are right connected. □

Lemma 7 — Let S be a CA-AG-groupoid and $M \subseteq S$, then

(i) MS is left ideal,

(ii) MS and SM are ideals if S is AG*.

PROOF :

(i) By cyclic associativity

$$S(MS) = S(SM) = M(SS) \subseteq MS \Rightarrow S(MS) \subseteq MS.$$

Thus, MS is left ideal.

(ii) As by part (i) MS is left ideal. Now, we proceed to prove that MS is right ideal. By AG* and cyclic associativity

$$(MS)S = S(MS) = S(SM) = M(SS) \subseteq MS.$$

Hence, MS is also a right ideal and hence an ideal of S. Now, we prove that SM is an ideal. By AG* and cyclic associativity

$$\begin{aligned} & S(SM) = (SS)M \subseteq SM, \\ \text{and} \quad & (SM)S = M(SS) = S(MS) = S(SM) = (SS)M \subseteq SM. \end{aligned}$$

Hence, SM is an ideal. □

As in part (ii) of Lemma 5, we proved that for left ideal L of CA-AG-groupoid S, LS is left ideal of S, here in Lemma 7 part (i) we proved that for any $M \subseteq S$, MS is left ideal, indeed it is the generalization of preceding mention one.

Lemma 8 — Let S be a CA-AG*-groupoid and $M \subseteq S$, then MS and SM are connected.

PROOF : By cyclic associativity and AG*

$$S(MS) = S(SM) = (SS)M \subseteq SM \quad \text{and} \quad S(SM) = M(SS) \subseteq MS.$$

Thus, MS and SM are left connected. Again, by left invertive law and AG*

$$(MS)S = (SS)M \subseteq SM \quad \text{and} \quad (SM)S = M(SS) \subseteq MS.$$

Thus, MS and SM are right connected. □

As in CA-AG*-groupoid S, for $M \subseteq S$, MS and SM are ideals (subsets) of S, thus from part (ii) of Lemma 7, $(MS)S$ and $S(SM)$ are ideals of S and on similar way $((((MS)S)S)...)S$ and $S(...(S(S(SM))))$ are ideals of S. Similarly, by using Lemma 8, $(MS)S$ and $S(SM)$ are connected sets and generally $((((MS)S)S)...)S$ and $S(...(S(S(SM))))$ are connected sets. Thus, the following corollary is obvious.

Corollary 3 — Let S be a CA-AG*-groupoid and $M \subseteq S$, then

(i) $(MS)S$ and $S(SM)$ are ideals of S and generally $((((MS)S)S)...)S$ and $S(...(S(S(SM))))$ are ideals,

(ii) $(MS)S$ and $S(SM)$ are connected sets and generally $((((MS)S)S)...)S$ and $S(...(S(S(SM))))$ are connected sets.

Example 1 shows that in CA-AG-groupoid, right and left ideals are distinct. However, in case of CA-AG-band, right and left ideals coincide, as proved in the following.

***Theorem* 3** — *Let S be a CA-AG-band and $M \subseteq S$. Then M is right ideal iff M is left ideal.*

PROOF : Let M is right ideal.Then by left invertive law and definition of AG-band

$$SM = (SS)M = (MS)S \subseteq MS \subseteq M \Rightarrow SM \subseteq M.$$

Hence, M is left ideal. Conversely, suppose M is left ideal, then by cyclic associativity and definition of AG-band

$$MS = M(SS) = S(MS) = S(SM) \subseteq SM \subseteq M \Rightarrow MS \subseteq M.$$

Hence, M is right ideal. □

***Theorem* 4** — *Let L be a left ideal of CA-T^1-AG-groupoid S and $N \subseteq S$, then*

(i) LN is right ideal,

(ii) *NL is left ideal.*

PROOF : (i) By cyclic associativity, left invertive law and definition of T^1-AG-groupoid

$$\begin{aligned} &(LN)S = (SN)L \Rightarrow S(LN) = L(SN) = N(LS) \\ \Rightarrow\ &(LN)S = (LS)N = (NS)L \\ \Rightarrow\ &S(LN) = L(NS) = S(LN) = N(SL) \\ \Rightarrow\ &(LN)S = (SL)N \subseteq LN. \end{aligned}$$

Hence, LN is a right ideal.

(ii) Again, by cyclic associativity, left invertive law and definition of T^1-AG-groupoid

$$\begin{aligned} & S(NL) = L(SN) = N(LS) \\ \Rightarrow\ & (NL)S = (LS)N = (NS)L \\ \Rightarrow\ & S(NL) = L(NS) = S(LN) = N(SL) \subseteq NL. \end{aligned}$$

Hence, NL is left ideal. □

***Theorem* 5** — *Let R be a right ideal of a CA-T^1-AG-groupoid S and $N \subseteq S$, then RN and NR are right connected.*

PROOF : By cyclic associativity and definition of T^1-AG-groupoid

$$\begin{aligned} & (MN)S = (SN)M \Rightarrow S(MN) = M(SN) = N(MS) \subseteq NM, \\ \text{and}\quad & (NM)S = (SM)N \Rightarrow S(NM) = N(SM) = M(NS) \\ \Rightarrow\ & (NM)S = (NS)M = (MS)N \subseteq MN. \end{aligned}$$

Hence, RN and NR are right connected. □

Lemma 9 — Let M and N are left connected sets of AG-groupoid S and M is left ideal of S. Then $S(M \cup N) \subseteq M$, consequently $M \cup N$ is a left ideal.

PROOF : Let $st \in S(M \cup N)$, where $s \in S$ and $t \in M \cup N$. Then $t \in M$ or $t \in N$. If $t \in M$ then $st \in SM \subseteq M$. Also if $t \in N$, then $st \in SN \subseteq M$. Hence, $S(M \cup N) \subseteq M$, as $M \subseteq M \cup N$, hence $M \cup N$ is left ideal. □

Lemma 10 — Let M and N are left connected sets of AG-groupoid S, then $M \cup N$ and $M \cap N$ are left ideals.

PROOF : Let $st \in S(M \cup N)$, where $s \in S$ and $t \in M \cup N$. Then $t \in M$ or $t \in N$. If $t \in M$ then $st \in SM \subseteq N$. Also if $t \in N$, then $st \in SN \subseteq M$. Hence, $st \in M \cup N$, thus $S(M \cup N) \subseteq M \cup N$, hence $M \cup N$ is left ideal.

Now, we prove that $M \cap N$ is left ideal. Let $sr \in S(M \cap N)$, where $s \in S$ and $r \in M \cap N$. Then $r \in M$ and $r \in N$, which implies $sr \in SM \subseteq N$ and $sr \in SN \subseteq M$. Hence, $sr \in M \cap N$, thus $S(M \cap N) \subseteq M \cap N$, hence $M \cap N$ is left ideal. □

Lemma 11 — Let M and N are right connected sets of AG-groupoid S, then $M \cup N$ and $M \cap N$ are right ideals.

PROOF : Let $ts \in (M \cup N)S$, where $t \in M \cup N$ and $s \in S$. Then $t \in M$ or $t \in N$. If $t \in M$ then $ts \in MS \subseteq N$. Again, if $t \in N$, then $ts \in NS \subseteq M$. Hence, $ts \in M \cup N$, thus $(M \cup N)S \subseteq M \cup N$, hence $M \cup N$ is right ideal.

Let $rs \in (M \cap N)S$, where $r \in M \cap N$ and $s \in S$. Then $r \in M$ and $r \in N$, which implies $rs \in MS \subseteq N$ and $rs \in NS \subseteq M$. Hence, $rs \in M \cap N$, thus $(M \cap N)S \subseteq M \cap N$, hence $M \cap N$ is right ideal. □

Corollary 4 — Let M and N are connected sets of AG-groupoid S, then $M \cup N$ and $M \cap N$ are ideals of S.

2.2 *(Left/right) Ideals associated with a fixed element of CA-AG-groupoid*

In the following, we proved that in CA-AG-groupoid S for left ideal L, right ideal R, ideal I and $u \in S$, uR is left ideal, while (i) Lu is a left ideal if $uL = Lu$ and (ii) uI is an ideal if $uI = Iu$. We also provided supporting and counter examples to illustrate these results. Further, we proved that if S is also AG* then (i) uL and uR are ideals, (ii) Lu and uL are connected sets and (iii) Ru and uR are connected sets.

Lemma 12 — Let R and L respectively be right and left ideals of a CA-AG-groupoid S and u be a fixed element of S. Then

(i) Lu is left ideal if $uL = Lu$,

(ii) uL is left ideal if $uL = Lu$,

(iii) uR is left ideal.

PROOF : (i) Let $s \in S$ and $p = lu \in Lu$, where $l \in L$. By cyclic associativity

$$\bigcup_{s \in S,\, p \in Lu} (sp) = \bigcup_{s \in S,\, l \in L} (s \cdot lu) = \bigcup_{s \in S,\, l \in L} (u \cdot sl) \subseteq uL = Lu \Rightarrow S(Lu) \subseteq Lu.$$

Hence, Lu is left ideal.

(ii) Follow by part (i).

(iii) Suppose $s \in S$ and $q = ur \in uR$, where $r \in R$. Then

$$\bigcup_{s \in S,\, q \in uR} (sq) = \bigcup_{s \in S,\, r \in R} (s \cdot ur) = \bigcup_{s \in S,\, r \in R} (r \cdot su) = \bigcup_{s \in S,\, r \in R} (u \cdot rs) \subseteq uR \Rightarrow S(uR) \subseteq uR.$$

Hence, uR is left ideal. □

The following example depicts that the condition $uL = Lu$ is necessary for uL to be a left ideal and if $uL \neq Lu$ then uL and Lu may not be left ideals.

Example 9 : Recall Example 1, S is CA-AG-groupoid.

(i) $L = \{2, 3, 5, 7\}$ is left ideal of S. For $u = 0 \in S$, $Lu = \{3\}$, $uL = \{3, 5\}$, clearly $uL \neq Lu$. Now, $S(uL) = \{3, 7\} \nsubseteq uL$, thus uL is not a left ideal of S. However, Lu is left ideal as $S(Lu) = \{3\} \subseteq Lu$.

(ii) $R = \{1, 3, 4, 7\}$ is right ideal of S. For $u = 0, 1, 3, 4, 5, 6 \,\&\, 7$, $uR = \{3\}$ and $S(uR) = \{3\} \subseteq uR$, thus uR is left ideal for $u = 0, 1, 3, 4, 5, 6 \,\&\, 7$. For $u = 2$, $uR = \{3, 6, 7\}$ and $S(uR) = \{3, 7\} \subseteq uR$, thus uR is also left ideal for $u = 2$. Thus, $\forall u \in S,\ S(uR) \subseteq uR$, hence uR is left ideal of S $\forall u \in S$. Note that inspite of the fact that for

$$\begin{aligned} u = 0, &\qquad Ru = \{3, 4\} \neq \{3\} = uR, \\ u = 2, &\qquad Ru = \{3\} \neq \{6, 7\} = uR, \\ u = 5, &\qquad Ru = \{3, 7\} \neq \{3\} = uR, \end{aligned}$$

still uR is left ideal, thus condition $uR = Ru$ is not required here.

Lemma 13 — Let R and L respectively be right and left ideals of CA-AG*-groupoid S and $u \in S$ be a fixed element of S. Then

(i) uL and uR are ideals of S,

(ii) Lu and uL are connected sets,

(iii) Ru and uR are connected sets.

PROOF : (i) Let $ul \in uL$ and $s \in S$, where $l \in L$, then by AG* and cyclic associativity

$$\bigcup_{l \in L,\, s \in S} \cup\, (ul \cdot s) = \bigcup_{l \in L,\, s \in S} (l \cdot us) = \bigcup_{l \in L,\, s \in S} (s \cdot lu) = \bigcup_{l \in L,\, s \in S} (u \cdot sl) \subseteq uL \Rightarrow (uL)S \subseteq uL.$$

Hence, uL is right ideal. Again, by AG*, cyclic associativity and left invertive law

$$\begin{aligned} \bigcup_{s \in S,\, l \in L} (s \cdot ul) &= \bigcup_{s \in S,\, l \in L} (l \cdot su) = \bigcup_{s \in S,\, l \in L} (u \cdot ls) = \bigcup_{s \in S,\, l \in L} (lu \cdot s) \\ &= \bigcup_{s \in S,\, l \in L} (su \cdot l) = \bigcup_{s \in S,\, l \in L} (u \cdot sl) \subseteq uL \Rightarrow S(uL) \subseteq uL. \end{aligned}$$

Thus, uL is left ideal, too. Hence, uL is an ideal. Again, let $ur \in uR$ and $s \in S$, where $r \in R$. Then by left invertive law, AG* and cyclic associativity

$$\underset{r\in R,\, s\in S}{\cup}(ur \cdot s) = \underset{r\in R,\, s\in S}{\cup}(sr \cdot u) = \underset{r\in R,\, s\in S}{\cup}(r \cdot su)$$
$$= \underset{r\in R,\, s\in S}{\cup}(u \cdot rs) = u \cdot RS \subseteq uR \Rightarrow (uR)S \subseteq uR.$$

Hence, uR is right ideal. Again, by cyclic associativity

$$\underset{s\in S,\, r\in R}{\cup}(s \cdot ur) = \underset{s\in S,\, r\in R}{\cup}(r \cdot su) = \underset{s\in S,\, r\in R}{\cup}(u \cdot rs) \subseteq uR \Rightarrow S(uR) \subseteq uR.$$

Thus, uR is also left ideal.

(ii) For $s \in S$ and $lu \in Lu$, where $l \in L$, by left invertive law and AG*

$$\underset{l\in L,\, s\in S}{\cup}(lu \cdot s) = \underset{l\in L,\, s\in S}{\cup}(su \cdot l) = \underset{l\in L,\, s\in S}{\cup}(u \cdot sl) \subseteq uL \Rightarrow (Lu)S \subseteq uL. \tag{2.1}$$

Again, for $s \in S$ and $ul \in uL$, where $l \in L$, by left invertive law

$$\underset{l\in L,\, s\in S}{\cup}(ul \cdot s) = \underset{l\in L,\, s\in S}{\cup}(sl \cdot u) \subseteq Lu \Rightarrow (uL)S \subseteq Lu. \tag{2.2}$$

From (2.1) and (2.2), Lu and uL are right connected. Now to prove that Lu and uL are left connected, let $s \in S$ and $lu \in Lu$, for $l \in L$. By cyclic associativity

$$\underset{s\in S,\, l\in L}{\cup}(s \cdot lu) = \underset{s\in S,\, l\in L}{\cup}(u \cdot sl) \subseteq uL \Rightarrow S(Lu) \subseteq uL. \tag{2.3}$$

Again, for $ul \in uL$, where $l \in L$, by cyclic associativity and AG*

$$\underset{s\in S,\, l\in L}{\cup}(s \cdot ul) = \underset{s\in S,\, l\in L}{\cup}(l \cdot su) = \underset{s\in S,\, l\in L}{\cup}(sl \cdot u) \subseteq Lu \Rightarrow S(uL) \subseteq Lu. \tag{2.4}$$

From (2.3) and (2.4), Lu and uL are left connected. □

(iii) Suppose $s \in S$ and $ur \in uR$, where $r \in R$. Then by AG* and cyclic associativity

$$\underset{r\in R,\, s\in S}{\cup}(ur \cdot s) = \underset{r\in R,\, s\in S}{\cup}(r \cdot us) = \underset{r\in R,\, s\in S}{\cup}(s \cdot ru) = \underset{r\in R,\, s\in S}{\cup}(rs \cdot u) \subseteq Ru \Rightarrow (uR)S \subseteq Ru. \tag{2.5}$$

Again, for $ru \in Ru$, where $r \in R$, then by definition of AG*

$$\underset{r\in R,\, s\in S}{\cup}(ru \cdot s) = \underset{r\in R,\, s\in S}{\cup}(u \cdot rs) \subseteq uR \Rightarrow (Ru)S \subseteq uR. \tag{2.6}$$

From (2.5) and (2.6), uR and Ru are right connected. To prove that uR and Ru are left connected, let $s \in S$ and $ur \in uR$, for $r \in R$. Then by AG* and left invertive law

$$\underset{s\in S,\, r\in R}{\cup}(s \cdot ur) = \underset{s\in S,\, r\in R}{\cup}(us \cdot r) = \underset{s\in S,\, r\in R}{\cup}(rs \cdot u) \subseteq Ru \Rightarrow S(uR) \subseteq Ru. \tag{2.7}$$

Again, for $ru \in Ru$, where $r \in R$, by AG*, left invertive law and cyclic associativity

$$\begin{aligned} \bigcup_{s\in S,\, r\in R}(s \cdot ru) &= \bigcup_{s\in S,\, r\in R}(rs \cdot u) = \bigcup_{s\in S,\, r\in R}(us \cdot r) = \bigcup_{s\in S,\, r\in R}(s \cdot ur) \\ &= \bigcup_{s\in S,\, r\in R}(r \cdot su) = \cup(u \cdot rs) \subseteq uR \Rightarrow S(Ru) \subseteq uR. \end{aligned} \tag{2.8}$$

From (2.7) and (2.8), uR and Ru are left connected. □

Lemma 14 — Let u be a fixed element of a CA-AG-groupoid S and I be an ideal of S such that $uI = Iu$, then uI is an ideal of S.

PROOF : By Lemma 12 part (ii), uI is a left ideal of S. To prove that uI is right ideal, suppose $ui \in uI$ and $s \in S$, where $i \in I$, then by cyclic associativity

$$\bigcup_{i\in I,\, s\in S}(ui \cdot s) = \bigcup_{i\in I,\, s\in S}(si \cdot u) \subseteq Iu = uI \Rightarrow (uI)S \subseteq uI.$$

Thus, uI is a right ideal, too. Hence, the result follows. □

Proposition 1 — [1]. Every CA-AG-groupoid is paramedial.

***Theorem* 6** — *If S is a CA-AG-groupoid and $u \in S$, then*

(i) uS *is left ideal,*

(ii) $u(Su)$ *is left ideal,*

(iii) $u(Su)$ *and* $(uS)u$ *are right connected,*

(iv) $u(Su)$ *and* $(uS)u$ *are connected, if* S *is right alternative.*

PROOF : (i) By cyclic associativity

$$S(uS) \;=\; \bigcup_{w,t\in S} w(ut) = \bigcup_{w,t\in S} t(wu) = \bigcup_{w,t\in S} u(tw) \subseteq uS \Rightarrow S(uS) \subseteq uS.$$

Hence, uS is left ideal.

(ii) By cyclic associativity and medial law

$$\begin{aligned} S(u(Su)) &= \bigcup_{w,t\in S} w(u \cdot tu) = \bigcup_{w,t\in S} tu \cdot wu = \bigcup_{w,t\in S} tw \cdot uu \\ &= \bigcup_{w,t\in S} u(tw \cdot u) \subseteq u(Su) \Rightarrow S(u(Su)) \subseteq u(Su). \end{aligned}$$

Hence, $u(Su)$ is left ideal.

(iii) By left invertive law and cyclic associativity

$$\begin{aligned}(u(Su))S &= \underset{w,t\in S}{\cup}(u\cdot wu)t = \underset{w,t\in S}{\cup}(t\cdot wu)u = \underset{w,t\in S}{\cup}(u\cdot tw)u \subseteq (uS)u \\ \Rightarrow (u(Su))S &\subseteq (uS)u. \end{aligned} \tag{2.9}$$

Again, by left invertive, proposition 1 and cyclic associativity

$$\begin{aligned}((uS)u)S &= \underset{w,t\in S}{\cup}(uw\cdot u)t = \underset{w,t\in S}{\cup}tu\cdot uw = \underset{w,t\in S}{\cup}w(tu\cdot u) \\ &= \underset{w,t\in S}{\cup}w(uu\cdot t) = \underset{w,t\in S}{\cup}t(w\cdot uu) = \underset{w,t\in S}{\cup}uu\cdot tw \\ &= \underset{w,t\in S}{\cup}wu\cdot tu = \underset{w,t\in S}{\cup}u(wu\cdot t) = \underset{w,t\in S}{\cup}t(u\cdot wu) \\ &= \underset{w,t\in S}{\cup}t(u\cdot uw) = \underset{w,t\in S}{\cup}uw\cdot tu = \underset{w,t\in S}{\cup}u(uw\cdot t) \\ &= \underset{w,t\in S}{\cup}u(tw\cdot u) \subseteq u(Su) \\ \Rightarrow ((uS)u)S &\subseteq u(Su). \end{aligned} \tag{2.10}$$

From (2.9) and (2.10), $(uS)u$ and $u(Su)$ are right connected.

(iv) By part (iii), $(uS)u$ and $u(Su)$ are right connected. We only show that $(uS)u$ and $u(Su)$ are left connected. By cyclic associativity

$$\begin{aligned}S((uS)u) &= \underset{w,t\in S}{\cup}w(ut\cdot u) = \underset{w,t\in S}{\cup}u(w\cdot ut) = \underset{w,t\in S}{\cup}u(t\cdot wu) \\ &= \underset{w,t\in S}{\cup}u(u\cdot tw) = \underset{w,t\in S}{\cup}tw\cdot uu = \underset{w,t\in S}{\cup}u(tw\cdot u) \subseteq u(Su) \\ \Rightarrow S((uS)u) &\subseteq u(Su). \end{aligned} \tag{2.11}$$

Again, by cyclic associativity, left invertive law, proposition 1 and by right alternativity

$$\begin{aligned}S(u(Su)) &= \underset{w,t\in S}{\cup}w(u\cdot tu) = \underset{w,t\in S}{\cup}w(u\cdot ut) = \underset{w,t\in S}{\cup}w(t\cdot uu) \\ &= \underset{w,t\in S}{\cup}uu\cdot wt = \underset{w,t\in S}{\cup}tu\cdot wu = \underset{w,t\in S}{\cup}(wu\cdot u)t \\ &= \underset{w,t\in S}{\cup}(w\cdot uu)t = \underset{w,t\in S}{\cup}(u\cdot wu)t = \underset{w,t\in S}{\cup}(u\cdot uw)t \\ &= \underset{w,t\in S}{\cup}(t\cdot uw)u = \underset{w,t\in S}{\cup}(w\cdot tu)u = \underset{w,t\in S}{\cup}(u\cdot wt)u \subseteq (uS)u \\ \Rightarrow S(u(Su)) &\subseteq (uS)u. \end{aligned} \tag{2.12}$$

From (2.11) and (2.12), $u(Su)$ and $(uS)u$ are left connected. □

3. Conclusion

We demonstrated that (left/right) ideals and (left/right) connected sets exist in CA-AG-groupoids. We precisely discussed some fundamental characteristics of (left/right) ideals and (left/right) connected sets in CA-AG-groupoids and established relations of (left/right) ideals among them and with (left/right) connected sets. Furthermore, we established relations of (left/right) ideals associated with fixed elements among each other and with (left/right) connected sets and investigated different conditions under which these results hold. We used the modern techniques of Prover-9, Mace-4 and GAP to produce illustrative examples, counterexamples and provide several other examples to improve the standard of this research work.

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