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HOMOGENIZATION OF MONOTONE SYSTEMS OF NON-COERCIVE HAMILTON-JACOBI EQUATIONS[1]

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In this article, we study homogenization for a class of monotone systems of first-order time-dependent Hamilton-Jacobi equations in the case of non-coercive Hamiltonians. And we prove the uniform convergence of the solution of oscillating systems to the solution of the homogenized systems.

Key words : Viscosity solutions; non-coercive; Hamilton-Jacobi equations; homogenization.

1. INTRODUCTION

In this paper, we study the behavior as $\epsilon \to 0$ of the monotone system of non-coercive Hamilton-Jacobi equations of the form

$$\begin{cases} \frac{\partial u_i^\epsilon}{\partial t} + H_i(\epsilon^{-1}x, \epsilon^{-1}y, \epsilon^{-1}t, u^\epsilon, D_x u_i^\epsilon, D_y u_i^\epsilon) = 0 \ \text{ in } \mathbf{R}^{N+1} \times (0, +\infty); \\ u_i^\epsilon(x, y, 0) = u_{0,i}(x, y) \ \text{ in } \mathbf{R}^{N+1}, \ i = 1, 2, \ldots, M, \end{cases} \tag{1.1}$$

where $u^\epsilon = (u_1^\epsilon, u_2^\epsilon, \ldots, u_M^\epsilon)$ and u_i^ϵ is a real-valued function defined in $\mathbf{R}^{N+1} \times (0, +\infty)$. The Hamiltonians $H_i(x, y, t, r, p_x, p_y), i = 1, 2, \ldots, M$, are continuous functions on $\mathbf{R}^{N+1} \times (0, +\infty) \times \mathbf{R}^M \times \mathbf{R}^N \times \mathbf{R}$ which are Z^N-periodic in x and 1-periodic in y and t. We assume that H_i is monotone, which is a standard assumption to establish a comparison principle for the system (1.1) (we refer to [2, 6-8]) and more other conditions on H_i will be given later on. It is worth pointing out that H_i is assumed to be coercive with respect to p_x but not with respect to p_y. This is the key and new point for monotone systems of Hamilton-Jacobi equations.

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For non-coercive Hamilton-Jacobi equations, it is difficult to solve the "cell problem". We refer to [5] by Barles who studies the limit as $\epsilon \to 0$ of the solution $u^\epsilon(x, y, t)$ of (non-coercive) Hamilton-Jacobi equation

$$u_t^\epsilon + H(\epsilon^{-1}x, \epsilon^{-1}y, \epsilon^{-1}t, D_x u^\epsilon, D_y u^\epsilon) = 0 \text{ in } \mathbf{R}^{N+1} \times (0, +\infty)$$

where $H(x, y, t, p_x, p_y)$ is a continuous and non-coercive Hamiltonian, which most of our ideas used in this paper are borrowed from.

We know that the coercivity plays a central role to solve the cell problem. In [1], Imbert and Monneau study the homogenization of coercive Hamilton-Jacobi equation with a u^ϵ/ϵ-dependence, namely

$$u_t^\epsilon + H(\epsilon^{-1}x, \epsilon^{-1}u^\epsilon, Du^\epsilon) = 0 \text{ in } \mathbf{R}^N \times (0, +\infty)$$

where H is a continuous and coercive Hamiltonian. In [2], Camili *et al*. study the homogenization of monotone systems of coercive Hamilton-Jacobi equations, namely

$$\begin{cases} \frac{\partial u_i^\epsilon}{\partial t} + H_i(x, \epsilon^{-1}x, u^\epsilon, Du_i^\epsilon) = 0 \text{ in } \mathbf{R}^N \times (0, T]; \\ u_i^\epsilon(x, 0) = u_{0,i}(x) \text{ in } \mathbf{R}^N, \ i = 1, 2, \dots, M, \end{cases}$$

where the Hamiltonians $H_i(x, y, r, p)$ are continuous, time-independent and coercive.

Concerning the homogenization of Hamilton-Jacobi equations, a pioneer work is the literature [18] by Lions *et al*. who completely solve the problem

$$H(x, y, p_x, p_y) \to +\infty \text{ as } |p_x| + |p_y| \to +\infty, \text{ uniformly w.r.t } x \text{ and } y,$$

where H is periodic and coercive Hamiltonian. Whereafter it has been adapted to many different homogenization problems (see [1, 3-5, 11-14, 15, 17]). Concerning the homogenization of systems of Hamilton-Jacobi equations, we refer to [2, 10, 11].

The main result of this paper is the convergence of u^ϵ, as $\epsilon \to 0$, to the function u which solves in viscosity sense the homogenized system

$$\begin{cases} \frac{\partial u_i}{\partial t} + \overline{H}_i(u, D_x u_i, D_y u_i) = 0 \text{ in } \mathbf{R}^{N+1} \times (0, +\infty); \\ u_i(x, y, 0) = u_{0,i}(x, y) \text{ in } \mathbf{R}^{N+1}, \ i = 1, 2, \dots, M, \end{cases} \tag{1.2}$$

where the Hamiltonians $\overline{H}_i$ are the so-called effective Hamiltonians which are characterized by appropriate cell problems, (see Section 3), i.e.

The cell problem : For any $i = 1, 2, \dots, M$, given (r, p_x, p_y) in $\mathbf{R}^M \times \mathbf{R}^N \times \mathbf{R}$, find $\lambda_i = \lambda_i(r, p_x, p_y)$ such that the equation

$$v_t + H_i(x, y, t, r, p_x + D_x v, p_y + D_y v) = \lambda_i \text{ in } \mathbf{R}^{N+1} \times \mathbf{R} \tag{1.3}$$

admits a space-time periodic solution in viscosity sense $v_i = v_i(r, p_x, p_y)$. The effective Hamiltonians $\overline{H}_i(r, p_x, p_y)$ associated to the Hamiltonians H_i are defined by setting $\overline{H}_i(r, p_x, p_y) = \lambda_i$.

To prove our homogenization result, we will adapt the classical perturbed test function method, which Camili *et al*. [2] use to show the homogenization result. In fact, this method was first introduced in [11] by Evans and then it has been applied to many different homogenization problems (see [2, 5, 9, 13, 16, 19]).

The paper is organized as follows. In section 2, we describe our assumptions and study the system (1) for $\epsilon > 0$. In section 3, we study the cell problem. In section 4, we prove the homogenization result.

2. Assumptions and the Problem for $\epsilon > 0$

In order to state our result, we have to impose some assumptions which ensure comparison principle and the existence and uniqueness of the viscosity solution.

$(\mathbf{A_1})$ $H_i(x, y, t, r, p_x, p_y)$ is continuous in $\mathbf{R}^N \times \mathbf{R} \times \mathbf{R} \times \mathbf{R}^M \times \mathbf{R}^N \times \mathbf{R}$, Z^N-periodic in x and 1-periodic in y and t.

$(\mathbf{A_2})$ There exist constants C_1, $C_2 > 0$ such that

$$H_i(x, y, t, r, p_x, 0) \geq C_1|p_x| - C_2 \quad \text{in } \mathbf{R}^{N+2} \times \mathbf{R}^M \times \mathbf{R}^N.$$

$(\mathbf{A_3})$ H_i is locally lipschitz continuous with respect to y, t, p_y and there exist $l \in \mathbf{R}$ and $C_3, C_4, C_5 \geq 0$ such that, for almost every $\theta = (x, y, t, r, p_x, p_y)$

$$|D_y H_i(\theta)| \leq C_3|p_y + l|,$$

$$|D_t H_i(\theta)| \leq C_4(1 + |p_y| + |H_i(\theta)|),$$

$$|D_{p_y} H_i(\theta)| \leq C_5.$$

$(\mathbf{A_4})$ H_i is monotone, that is:
if $r, s \in \mathbf{R}^M$ and $r_i - s_i = \max_{k=1,2,\dots,M} \{r_k - s_k\} \geq 0$, then
for all (x, y, t, r, p_x, p_y),

$$H_i(x, y, t, r, p_x, p_y) - H_i(x, y, t, s, p_x, p_y) \geq 0.$$

Concerning the initial datum, we assume

$(\mathbf{A_5})$ $u_{0,i}$ is bounded uniformly continuous in $\mathbf{R}^{N+1}$, for $i = 1, 2, \ldots, M$.

For a function $u : Q \to \mathbf{R}^M$, where $Q = \mathbf{R}^{N+2}$, we say that $u = (u_1, u_2, \cdots, u_M)$ is upper semi-continuous (u.s.c.in short), (respectively lower semi-continuous (l.s.c.in short)), in Q if all the components $u_i, i = 1, 2, \cdots, M$ are u.s.c., (respectively l.s.c.), in Q. Similarly, we define bounded uniformly continuous (BUC). If $u = (u_1, u_2, \cdots, u_M)$ and $v = (v_1, v_2, \cdots, v_M)$ are two functions defined in the same set, we write $u \le v$ if $u_i \le v_i$ for all $i \in \{1, 2, \cdots, M\}$.

Prior to stating our main results, we recall the definition of viscosity solution for the system (1.1).

Definition 2.1 — An u.s.c. function u is said to be a viscosity sub-solution in $\mathbf{R}^{N+1} \times (0, +\infty)$ of (1.1) if $u_i(\cdot, \cdot, 0) \le u_{0,i}$ in $\mathbf{R}^{N+1}$ for all $i \in \{1, 2, \cdots M\}$ and if

$$\frac{\partial \varphi}{\partial t}(x,y,t) + H_i(\epsilon^{-1}x, \epsilon^{-1}y, \epsilon^{-1}t, u(x,y,t), D_x\varphi(x,y,t), D_y\varphi(x,y,t)) \le 0$$

whenever $\varphi \in C^1(\mathbf{R}^{N+1} \times (0, +\infty))$, $i = 1, 2, \cdots M$ and $u_i - \varphi$ attains a local maximum at (x, y, t) with $t > 0$.

A l.s.c. function v is said to be a viscosity super-solution in $\mathbf{R}^{N+1} \times (0, +\infty)$ of (1.1) if $v_i(\cdot, \cdot, 0) \ge u_{0,i}$ in $\mathbf{R}^{N+1}$ for all $i \in \{1, 2, \cdots M\}$ and if

$$\frac{\partial \varphi}{\partial t}(x,y,t) + H_i(\epsilon^{-1}x, \epsilon^{-1}y, \epsilon^{-1}t, v(x,y,t), D_x\varphi(x,y,t), D_y\varphi(x,y,t)) \ge 0$$

whenever $\varphi \in C^1(\mathbf{R}^{N+1} \times (0, +\infty))$, $i = 1, 2, \cdots M$ and $v_i - \varphi$ attains a local minimum at (x, y, t) with $t > 0$.

Moreover, a continuous u is said to be a viscosity solution of (1.1) if it is both a viscosity sub-solution and a viscosity super-solution of (1.1).

Theorem 2.1 — *Under assumptions $(\mathbf{A_1}) - (\mathbf{A_5})$, let $\underline{u}$ be a bounded u.s.c. sub-solution and $\overline{u}$ be a bounded l.s.c. super-solution of the initial problem*

$$\begin{cases} \frac{\partial u_i}{\partial t} + H_i(x, y, t, u, D_x u_i, D_y u_i) = 0 \ \text{ in } \mathbf{R}^{N+1} \times (0, +\infty); \\ u_i(x, y, 0) = u_{0,i}(x, y) \ \text{ in } \mathbf{R}^{N+1}, \ i = 1, 2, \ldots, M. \end{cases} \tag{2.1}$$

Then $\underline{u} \le \overline{u}$ in $\mathbf{R}^{N+1} \times [0, +\infty)$ and there exists a unique continuous viscosity solution $u \in BUC(\mathbf{R}^{N+1} \times [0, T])$ for all $T > 0$.

PROOF : First, we prove the comparison principle. Define

$$\begin{aligned}\phi_i(x_1,x_2,y_1,y_2,t,s) = {} & \underline{u}_i(x_1,y_1,t) - \overline{u}_i(x_2,y_2,s) - \frac{|x_1-x_2|^2}{2\alpha^2} - \frac{|y_1-y_2|^2}{2\gamma^2} \\ & - \frac{|t-s|^2}{2\beta^2} - \mu_1(|x_1|^2+|x_2|^2) - \mu_2(|y_1|^2+|y_2|^2) \\ & - \mu_3(|t|^2+|s|^2) - \eta t, \quad i=1,2,\dots,M,\end{aligned}$$

where $\alpha,\gamma,\beta,\eta,\mu_1,\mu_2,\mu_3$ are small positive parameters devoted to tend to 0.

Since $\underline{u}_i$ and $\overline{u}_i$ are bounded, $\max_i \sup_{\mathbf{R}^{2N}\times\mathbf{R}^2\times(0,+\infty)^2} \phi_i$ is finite and achieved at some point. We write that

$$\max_i \sup_{\mathbf{R}^{2N}\times\mathbf{R}^2\times(0,+\infty)^2} \phi_i(x_1,x_2,y_1,y_2,t,s) = \phi_{\bar{i}}(\overline{x}_1,\overline{x}_2,\overline{y}_1,\overline{y}_2,\overline{t},\overline{s})$$

where $(\overline{x}_1,\overline{x}_2,\overline{y}_1,\overline{y}_2,\overline{t},\overline{s})$ depends on $\alpha,\gamma,\beta,\mu_1,\mu_2,\mu_3$.

For all i and $(x,y,t)\in\mathbf{R}^{N+1}\times(0,+\infty)$, we have

$$\begin{aligned}& \underline{u}_i(x,y,t) - \overline{u}_i(x,y,t) - 2\mu_1|x|^2 - 2\mu_2|y|^2 - 2\mu_3|t|^2 - \eta t \\ &= \phi_i(x,x,y,y,t,t) \\ &\le \phi_{\bar{i}}(\overline{x}_1,\overline{x}_2,\overline{y}_1,\overline{y}_2,\overline{t},\overline{s}) \\ &\le \underline{u}_{\bar{i}}(\overline{x}_1,\overline{y}_1,\overline{t}) - \overline{u}_{\bar{i}}(\overline{x}_2,\overline{y}_2,\overline{s}).\end{aligned}$$

Obviously, if $\underline{u}_{\bar{i}}(\overline{x}_1,\overline{y}_1,\overline{t}) - \overline{u}_{\bar{i}}(\overline{x}_2,\overline{y}_2,\overline{s}) \le 0$, for all $\mu_1,\mu_2,\mu_3,\eta > 0$, then the comparison principle holds. Therefore, we may suppose that

$$\underline{u}_{\bar{i}}(\overline{x}_1,\overline{y}_1,\overline{t}) - \overline{u}_{\bar{i}}(\overline{x}_2,\overline{y}_2,\overline{s}) \ge 0 \tag{2.2}$$

for μ_1,μ_2,μ_3,η sufficiently small.

We recall the well-known facts that

$$\lim_{\mu_1\to 0}\mu_1(|\overline{x}_1|+|\overline{x}_2|)=0, \quad \lim_{\mu_2\to 0}\mu_2(|\overline{y}_1|+|\overline{y}_2|)=0, \quad \lim_{\mu_3\to 0}\mu_3(|\overline{t}|+|\overline{s}|)=0$$

and

$$\frac{|\overline{x}_1-\overline{x}_2|^2}{2\alpha^2} + \frac{|\overline{y}_1-\overline{y}_2|^2}{2\gamma^2} + \frac{|\overline{t}-\overline{s}|^2}{2\beta^2} \to 0, \text{ as } \alpha,\beta,\gamma\to 0.$$

We set

$$p_x = \frac{\overline{x}_1-\overline{x}_2}{\alpha^2}, \quad p_y = \frac{\overline{y}_1-\overline{y}_2}{\gamma^2}, \quad p_t = \frac{\overline{t}-\overline{s}}{\beta^2}.$$

First we assume for a while that it is possible to extract subsequences $\alpha, \gamma, \beta, \mu_1, \mu_2, \mu_3 \to 0$ such that

$$\bar{t} > 0 \text{ and } \bar{s} > 0. \tag{2.3}$$

The viscosity inequalities for sub-solution $\underline{u}$ and super-solution $\overline{u}$ read

$$\eta + p_t + 2\mu_3\bar{t} + H_{\bar{i}}(\overline{x}_1, \overline{y}_1, \bar{t}, \underline{u}(\overline{x}_1, \overline{y}_1, \bar{t}), p_x + 2\mu_1\overline{x}_1, p_y + 2\mu_2\overline{y}_1) \leq 0.$$

$$p_t + 2\mu_3\overline{s} + H_{\bar{i}}(\overline{x}_2, \overline{y}_2, \overline{s}, \overline{u}(\overline{x}_2, \overline{y}_2, \overline{s}), p_x - 2\mu_1\overline{x}_2, p_y - 2\mu_2\overline{y}_2) \geq 0.$$

Let $\mu_1, \mu_2, \mu_3 \to 0$ and then we have

$$\eta + p_t + H_{\bar{i}}(\overline{x}_1, \overline{y}_1, \bar{t}, \underline{u}(\overline{x}_1, \overline{y}_1, \bar{t}), p_x, p_y) \leq 0, \tag{2.4}$$

and

$$p_t + H_{\bar{i}}(\overline{x}_2, \overline{y}_2, \overline{s}, \overline{u}(\overline{x}_2, \overline{y}_2, \overline{s}), p_x, p_y) \geq 0. \tag{2.5}$$

Since $\lim_{\beta\to 0} p_t\beta = 0$, we may as well assume that for β small enough the above viscosity inequalities also hold with $H_{\bar{i}}^{\beta} = \min(\beta^{-1}, \max(H_{\bar{i}}, -\beta^{-1}))$, which has the same properties as $H_{\bar{i}}$ except that $|D_t H_{\bar{i}}^{\beta}(\theta)| \leq C_4(1 + \beta^{-1})$ (see [5]).

Therefore, subtracting (2.5) from (2.4), we have

$$\begin{aligned}\eta &\leq H_{\bar{i}}^{\beta}(\overline{x}_2, \overline{y}_2, \overline{s}, \overline{u}(\overline{x}_2, \overline{y}_2, \overline{s}), p_x, p_y) - H_{\bar{i}}^{\beta}(\overline{x}_1, \overline{y}_1, \bar{t}, \underline{u}(\overline{x}_1, \overline{y}_1, \bar{t}), p_x, p_y) \\ &= \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4,\end{aligned}$$

where

$$\begin{aligned}\Gamma_1 &= H_{\bar{i}}^{\beta}(\overline{x}_2, \overline{y}_2, \overline{s}, \overline{u}(\overline{x}_2, \overline{y}_2, \overline{s}), p_x, p_y) - H_{\bar{i}}^{\beta}(\overline{x}_2, \overline{y}_1, \overline{s}, \overline{u}(\overline{x}_2, \overline{y}_2, \overline{s}), p_x, p_y), \\ \Gamma_2 &= H_{\bar{i}}^{\beta}(\overline{x}_2, \overline{y}_1, \overline{s}, \overline{u}(\overline{x}_2, \overline{y}_2, \overline{s}), p_x, p_y) - H_{\bar{i}}^{\beta}(\overline{x}_2, \overline{y}_1, \bar{t}, \overline{u}(\overline{x}_2, \overline{y}_2, \overline{s}), p_x, p_y), \\ \Gamma_3 &= H_{\bar{i}}^{\beta}(\overline{x}_2, \overline{y}_1, \bar{t}, \overline{u}(\overline{x}_2, \overline{y}_2, \overline{s}), p_x, p_y) - H_{\bar{i}}^{\beta}(\overline{x}_2, \overline{y}_1, \bar{t}, \underline{u}(\overline{x}_1, \overline{y}_1, \bar{t}), p_x, p_y), \\ \Gamma_4 &= H_{\bar{i}}^{\beta}(\overline{x}_2, \overline{y}_1, \bar{t}, \underline{u}(\overline{x}_1, \overline{y}_1, \bar{t}), p_x, p_y) - H_{\bar{i}}^{\beta}(\overline{x}_1, \overline{y}_1, \bar{t}, \underline{u}(\overline{x}_1, \overline{y}_1, \bar{t}), p_x, p_y).\end{aligned}$$

By using $(\mathbf{A_3})$, we have

$$\Gamma_1 \leq C_3|p_y + l||\overline{y}_1 - \overline{y}_2|$$

and

$$\Gamma_2 \leq C_4|\bar{t} - \overline{s}|(1 + |p_y| + \beta^{-1}).$$

Since $0 \le \underline{u}_{\bar{i}}(\overline{x}_1, \overline{y}_1, \bar{t}) - \overline{u}_{\bar{i}}(\overline{x}_2, \overline{y}_2, \overline{s}) = \max_i\{\underline{u}_i(\overline{x}_1, \overline{y}_1, \bar{t}) - \overline{u}_i(\overline{x}_2, \overline{y}_2, \overline{s})\}$, in view of $(\mathbf{A_4})$ and the definition of $\bar{i}$, we obtain

$$\Gamma_3 \le 0.$$

The estimate of Γ_4 may be found in [5], we write it out for reader's convenience. First fix γ, β and then let $\alpha \to 0$. According to $(\mathbf{A_2})$, (2.4) and (2.5), p_x remains bounded since p_t and p_y remain bounded. We may as well assume that $(\overline{x}_1, \overline{x}_2, \overline{y}_1, \overline{y}_2, \bar{t}, \overline{s})$ remains in a compact subset since u and v are periodic. Hence, letting $\alpha \to 0$ and using the continuity of $H^{\beta}_{\bar{i}}$ we can achieve $\Gamma_4 \to 0$.

Then successively sending $\beta \to 0, \gamma \to 0$, the terms of right-hand side converge to 0. we obtain a contradiction since $\eta > 0$.

Secondly if (2.3) does not hold for $\mu_1, \mu_2, \mu_3, \alpha, \beta, \gamma, \eta$ small enough, without of loss generality, we may assume that $\bar{t} = 0$. Then for all i and $(x_1, y_1, t), (x_2, y_2, t) \in \mathbf{R}^{N+1} \times (0, +\infty)$, sending $\mu_1, \mu_2, \mu_3 \to 0$, we have

$$\begin{aligned} &\underline{u}_i(x_1, y_1, t) - \overline{u}_i(x_2, y_2, t) - \frac{|x_1 - x_2|^2}{2\alpha^2} - \frac{|y_1 - y_2|^2}{2\gamma^2} - \eta t \\ &\le (\underline{u}_{\bar{i}}(\overline{x}_1, \overline{y}_1, 0) - \overline{u}_{\bar{i}}(\overline{x}_2, \overline{y}_2, \overline{s}))^+ - \frac{|\overline{x}_1 - \overline{x}_2|^2}{2\alpha^2} - \frac{|\overline{y}_1 - \overline{y}_2|^2}{2\gamma^2} - \frac{|0 - \overline{s}|^2}{2\beta^2}. \end{aligned}$$

where $f^+(x) = \max\{f(x), 0\}$.

Sending $\beta \to 0$, we have

$$\begin{aligned} \underline{u}_i(x_1, y_1, t) - \overline{u}_i(x_2, y_2, t) \le\ & \eta t + \frac{|x_1 - x_2|^2}{2\alpha^2} + \frac{|y_1 - y_2|^2}{2\gamma^2} \\ & + (\underline{u}_{\bar{i}}(\overline{x}_1, \overline{y}_1, 0) - \overline{u}_{\bar{i}}(\overline{x}_2, \overline{y}_2, 0))^+ \\ & - \frac{|\overline{x}_1 - \overline{x}_2|^2}{2\alpha^2} - \frac{|\overline{y}_1 - \overline{y}_2|^2}{2\gamma^2}. \end{aligned}$$

By the definition of viscosity sub-and super-solution, we have

$$\underline{u}_{\bar{i}}(\overline{x}_1, \overline{y}_1, 0) \le u_{0,\bar{i}}(\overline{x}_1, \overline{y}_1), \quad \overline{u}_{\bar{i}}(\overline{x}_2, \overline{y}_2, 0) \ge u_{0,\bar{i}}(\overline{x}_2, \overline{y}_2).$$

It follows that

$$\underline{u}_{\bar{i}}(\overline{x}_1, \overline{y}_1, 0) - \overline{u}_{\bar{i}}(\overline{x}_2, \overline{y}_2, 0) \le u_{0,\bar{i}}(\overline{x}_1, \overline{y}_1) - u_{0,\bar{i}}(\overline{x}_2, \overline{y}_2).$$

In view of $(\mathbf{A_5})$, for any $i \in \{1, 2, \ldots, M\},\ \rho > 0$, there exists $C_{i\rho} > 0$, such that

$$u_{0,i}(x_1, y_1) - u_{0,i}(x_2, y_2) \le \rho + C_{i\rho}(|x_1 - x_2| + |y_1 - y_2|).$$

Therefore

$$u_{0,i}(x_1,y_1)-u_{0,i}(x_2,y_2)-\frac{|x_1-x_2|^2}{2\alpha^2}-\frac{|y_1-y_2|^2}{2\gamma^2}\le\rho+\frac{\alpha^2}{2}(C_{i\rho})^2+\frac{\gamma^2}{2}(C_{i\rho})^2.$$

Fix $\rho>0$ and set $C_\rho=\max_i\{\frac{C_{i\rho}}{\sqrt{2}}\}$. Then

$$\underline{u}_i(x_1,y_1,t)-\overline{u}_i(x_2,y_2,t)\le\rho+\eta t+(\alpha C_\rho)^2+(\gamma C_\rho)^2+\frac{|x_1-x_2|^2}{2\alpha^2}+\frac{|y_1-y_2|^2}{2\gamma^2}$$

Sending successively $\alpha\to 0,\gamma\to 0,\eta\to 0,\rho\to 0$, we conclude that the comparison principle holds. Moreover, by Perron's method (see [8]), there exists a continuous viscosity solution u to (2.1) and applying the comparison result, we can obtain the uniqueness of the solution.

About the proof of bounded uniform continuity of u, we refer the reader to ([2], Appendix).

Therefore, it is easy to prove the following theorem.

***Theorem* 2.2** — *Under assumptions* $(\mathbf{A_1})-(\mathbf{A_5})$*, for any* $\epsilon>0$*, there exists a unique continuous viscosity solution* $u^\epsilon\in BUC(\mathbf{R}^{N+1}\times[0,T])$ *of (1.1); furthermore, the* L^∞ *bound for* u^ϵ *does not depend on* ϵ*.*

PROOF : For fixed $\varepsilon>0$, the existence and uniqueness of the solution u^ϵ to (1.1) follow immediately from Theorem 2.1. We need only to prove the L^∞ boundness for u^ϵ. Set

$$C:=\sup\{H_i(\epsilon^{-1}x,\epsilon^{-1}y,\epsilon^{-1}t,r,0,0):x\in\mathbf{R}^N,y\in\mathbf{R},t\in(0,+\infty),$$
$$|r|\le|u_0|_\infty,1\le i\le M\}$$

which is finite and independent of ϵ by periodicity of H_i in x,y and t.

Let $u^\pm(x,y,t)=(\pm|u_0|_\infty\pm Ct,\ldots,\pm|u_0|_\infty\pm Ct)$. If we can prove that u^+ is a super-solution and u^- a sub-solution of (1.1), then by comparison principle, we have

$$u^-\le u^\epsilon\le u^+\quad\text{in }\mathbf{R}^{N+1}\times(0,+\infty)$$

and we obtain the global L^∞ bound for u_ϵ,

$$|u_\epsilon|_\infty\le|u_0|_\infty+Ct.$$

In the following, we only prove that u^+ is a super-solution, the proof for u^- being analogous. First u^+ satisfies the initial condition. Since u^+ is smooth, for all i and (x,y,t),

$$\frac{\partial u_i^+}{\partial t} + H_i(\epsilon^{-1}x, \epsilon^{-1}y, \epsilon^{-1}t, u^+, D_x u_i^+, D_y u_i^+)$$
$$= C + H_i(\epsilon^{-1}x, \epsilon^{-1}y, \epsilon^{-1}t, u^+, 0, 0).$$

But $\max_j\{u_j^+(x,y,t) - |u_0|_\infty\} = Ct \geq 0$ is achieved for every index $1 \leq j \leq M$, therefore, for all i, by $(\mathbf{A_4})$,

$$H_i(\epsilon^{-1}x, \epsilon^{-1}y, \epsilon^{-1}t, u^+, 0, 0)$$
$$\geq H_i(\epsilon^{-1}x, \epsilon^{-1}y, \epsilon^{-1}t, (|u_0|_\infty, |u_0|_\infty, \ldots, |u_0|_\infty), 0, 0)$$
$$\geq -C$$

which proves the result.

3. The Cell Problem

In this section, we will prove the existence of the effective Hamiltonians, the Hamiltonians for the limit system (1.1). In order to solve the problem like (1.3) in a general way and once for all (r, p_x, p_y), we consider the ergodic problem: For any $i = 1, 2, \ldots, M$, given $r \in \mathbf{R}^M$, find $\lambda_i = \lambda_{i,r}$, such that the equation

$$w_t + F_i(x, y, t, r, D_x w, D_y w) = \lambda_i \text{ in } \mathbf{R}^{N+1} \times \mathbf{R},\ i = 1, 2, \ldots, M \tag{3.1}$$

has a periodic solution $w_i = w_{i,r}$.

Proposition 3.1 — Assume that F_i satisfies $(\mathbf{A_1}) - (\mathbf{A_3})$. For any $i = 1, 2, \ldots, M$, given $r \in \mathbf{R}^M$, then for any $\alpha > 0$, there exists a unique continuous, space-time periodic solution w_i^α of

$$\frac{\partial w_i^\alpha}{\partial t} + F_i(x, y, t, r, D_x w_i^\alpha, D_y w_i^\alpha) + \alpha w_i^\alpha = 0 \text{ in } \mathbf{R}^{N+1} \times \mathbf{R}. \tag{3.2}$$

PROOF : Fix $i \in \{1, 2, \ldots, M\}$. According to conditions $(\mathbf{A_1}) - (\mathbf{A_3})$, (3.2) satisfies a comparison principle for any $\alpha > 0$ and therefore (3.2) admits a unique continuous, periodic solution w_i^α. The proof is similar with Theorem 1 and therefore we skip it.

Proposition 3.2 — Under the assumptions of Proposition 1, there exists a constant $K = K(F, |r|)$, such that

$$\max_{\mathbf{R}^{N+1}\times\mathbf{R}} w_i^\alpha - \min_{\mathbf{R}^{N+1}\times\mathbf{R}} w_i^\alpha \leq K$$

where $K = K(C_r, C_1, C_2, l)$ and $C_r = \{\sup |F_i(x, y, t, r, 0, 0)|; i = 1, 2, \ldots, M\}$

PROOF : It is easy to see that $\frac{C_r}{\alpha}$ is a super-solution and $-\frac{C_r}{\alpha}$ is a sub-solution of (3.2). By comparison principle, we have

$$-\frac{C_r}{\alpha} \leq w_i^\alpha \leq \frac{C_r}{\alpha}, \ i.e. - C_r \leq \alpha w_i^\alpha \leq C_r. \tag{3.3}$$

Using the method of [5] and in view of the proof of Theorem 2.1, we can easily obtain the conclusion.

***Theorem* 3.1** — *Under the assumptions of Proposition 3.2, there exists a unique constant λ_i such that equation (3.1) has periodic viscosity solution.*

PROOF : Fix $i \in \{1, 2, \ldots, M\}$ temporarily . By Proposition 3.2, we notice that $w_i^\alpha - \min_{\mathbf{R}^{N+1}\times\mathbf{R}} w_i^\alpha$ is bounded and so is $\alpha w_i^\alpha - \alpha \min_{\mathbf{R}^{N+1}\times\mathbf{R}} w_i^\alpha$. Again, $|\alpha w_i^\alpha| \leq C_r$, it follows that $\alpha \min_{\mathbf{R}^{N+1}\times\mathbf{R}} w_i^\alpha$ is bounded. Without loss of generality, we may assume that $-\alpha \min w_i^\alpha$ converges to a constant $\lambda_i (= \lambda_{i,r})$. Along the same subsequence, we use the half-relaxed limits method and set

$$\text{limsup}^*(w_i^\alpha - \min w_i^\alpha) = \overline{w}_i, \quad \text{liminf}_*(w_i^\alpha - \min w_i^\alpha) = \underline{w}_i.$$

Then $\overline{w}_i$ and $\underline{w}_i$ are respectively a bounded sub-solution and super-solution of (3.1).

Indeed, if $(\overline{w}_i - \phi)(x, y, t)$ achieves its maximum at $(\overline{x}, \overline{y}, \overline{t})$, where ϕ is C^1, and $w_i^\alpha - \min w_i^\alpha - \phi$ achieves maximum at $(x_\alpha, y_\alpha, t_\alpha)$ with the same ϕ, then by the definition of $\overline{w}_i$, we obtain $(x_\alpha, y_\alpha, t_\alpha) \to (\overline{x}, \overline{y}, \overline{t})$, as $\alpha \to 0$.

Leting $\varphi = \min w_i^\alpha + \phi$, we have $\varphi_t = \phi_t, \varphi_x = \phi_x, \varphi_y = \phi_y$. Since w_i^α is a viscosity sub-solution of (3.2), we have

$$\begin{aligned}&\varphi_t(x_\alpha, y_\alpha, t_\alpha) + F_i(x_\alpha, y_\alpha, t_\alpha, r, D_x\varphi(x_\alpha, y_\alpha, t_\alpha), D_y\varphi(x_\alpha, y_\alpha, t_\alpha)) + \\ &\alpha w_i^\alpha(x_\alpha, y_\alpha, t_\alpha) \leq 0\end{aligned}$$

i.e.

$$\begin{aligned}&\phi_t(x_\alpha, y_\alpha, t_\alpha) + F_i(x_\alpha, y_\alpha, t_\alpha, r, D_x\phi(x_\alpha, y_\alpha, t_\alpha), D_y\phi(x_\alpha, y_\alpha, t_\alpha)) + \\ &\alpha(w_i^\alpha(x_\alpha, y_\alpha, t_\alpha) - \min w_i^\alpha) \leq -\alpha \min w_i^\alpha.\end{aligned}$$

In view of $(\mathbf{A_1})$, letting $\alpha \to 0$, we have

$$\phi_t(\overline{x}, \overline{y}, \overline{t}) + F_i(\overline{x}, \overline{y}, \overline{t}, r, D_x\phi(\overline{x}, \overline{y}, \overline{t}), D_y\phi(\overline{x}, \overline{y}, \overline{t})) \leq \lambda_i,$$

which proves that $\overline{w}_i$ is a viscosity sub-solution of (3.1). Similarly we can also prove that $\underline{w}_i$ is a viscosity super-solution of (3.1). By comparison, $\overline{w}_i \leq \underline{w}_i$ and again by the definition of $\overline{w}_i, \underline{w}_i$,

$\overline{w}_i \geq \underline{w}_i$, we can set $w_i := \underline{w}_i = \overline{w}_i$. Then w_i is a viscosity solution of (3.1). Finally, it's possible to prove that λ_i for which (3.1) admits a periodic solution is univocally defined.

4. Homogenization of Non-coercive Equations

In order to prove the homogenization result, we need to prove that the effective Hamiltonians satisfy some properties.

Proposition 4.1 — Assume that H_i satisfies the assumptions $(\mathbf{A_1}) - (\mathbf{A_3})$ with $l = 0$ in $(\mathbf{A_3})$. For any $i = 1, 2, \ldots, M$ and for any $(r, p_x, p_y) \in \mathbf{R}^M \times \mathbf{R}^N \times \mathbf{R}$, there exists a unique $\overline{H}_i(r, p_x, p_y)$ such that (1.3) has a bounded space-time periodic solution and $\overline{H}_i(r, p_x, p_y)$ satisfies.

$(\mathbf{i})$ $\overline{H}_i(r, p_x, p_y)$ *is a continuous function of* (r, p_x, p_y).

$(\mathbf{ii})$ *If* $(H_i^k)_k$ *is a sequence of functions satisfying the same assumptions as* H_i *and uniformly in* k*, which converges locally uniformly to* H_i*, then* $\overline{H}_i^k \to \overline{H}_i$ *locally uniformly in* $\mathbf{R}^M \times \mathbf{R}^N \times \mathbf{R}$.

$(\mathbf{iii})$ *If* H_i *satisfies* $(\mathbf{A_4})$*, then* $\overline{H}_i$ *satisfies* $(\mathbf{A_4})$.

Proof : The existence and uniqueness of $\overline{H}_i(r, p_x, p_y)$ are immediate consequences of Theorem 3.1 applied to

$$F_i(x, y, t, r, q_x, q_y) = H_i(x, y, t, r, p_x + q_x, p_y + q_y).$$

About the proof of the continuity of $\overline{H}_i$: we need only to argue that for any (r, p_x, p_y), if the sequence $((r^k, p_x^k, p_y^k))_k$ converges to (r, p_x, p_y), then, up to a subsequence, $\overline{H}_i(r^k, p_x^k, p_y^k)$ converges to $\overline{H}_i(r, p_x, p_y)$.

Since $r^k \to r,\ p_x^k \to p_x$ and $p_y^k \to p_y$, we have $|r^k|_\infty \leq M_r,\ |p_x^k| \leq M_{p_x}$ and $|p_y^k| \leq M_{p_y}$ uniformly in k, where M_r, M_{p_x} and M_{p_y} are constants respectively with r, p_x and p_y. Therefore the continuity of $\overline{H}_i$ follows from the estimates: first we have a bound on $\overline{H}_i(r^k, p_x^k, p_y^k)$ which is coming from (3.3) namely

$$|\overline{H}_i(r^k, p_x^k, p_y^k)| \leq \sup\{|H_i(x, y, t, s, q_x, q_y)| : x \in \mathbf{R}^N, y \in \mathbf{R}, t \in \mathbf{R}, |s| \leq M_r, \\ |q_x| \leq M_{p_x}, |q_y| \leq M_{p_y}\}$$

and then we also have the bounds on the oscillations of the associated sub-solution and super-solution $\overline{v}_i, \underline{v}_i$ which depend only on (r, p_x, p_y) and H_i, $l = p_y$. We refer the reader to [5] for the complete proof. The proof for the $\overline{H}_i^k \to \overline{H}_i$ relies on the same type of arguments and we refer to [5] for details.

We now prove that $\overline{H}_i$ satisfies the monotone condition $(\mathbf{A_4})$. We argue by contradiction, assuming that there exist $r, s \in \mathbf{R}^M$ such that

$$r_i - s_i = \max_{j=1,2,\ldots,M}\{r_j - s_j\} \geq 0,$$

and

$$\overline{H}_i(r, p_x, p_y) < \overline{H}_i(s, p_x, p_y),$$

for some $(p_x, p_y) \in \mathbf{R}^N \times \mathbf{R}$. Let v, w be two periodic functions such that

$$v_t + H_i(x, y, t, r, p_x + D_x v, p_y + D_y v) = \overline{H}_i(r, p_x, p_y),$$

$$w_t + H_i(x, y, t, s, p_x + D_x w, p_y + D_y w) = \overline{H}_i(s, p_x, p_y).$$

Since v, w are bounded, by adding a constant, we can assume that $v > w$ without loss of generality.

By $(\mathbf{A_4})$, we have

$$\begin{aligned} & v_t + H_i(x, y, t, r, p_x + D_x v, p_y + D_y v) \\ = & \overline{H}_i(r, p_x, p_y) \\ < & \overline{H}_i(s, p_x, p_y) \\ = & w_t + H_i(x, y, t, s, p_x + D_x w, p_y + D_y w) \\ \leq & w_t + H_i(x, y, t, r, p_x + D_x w, p_y + D_y w) \end{aligned}$$

and for α small enough

$$\begin{aligned} & \alpha v + v_t + H_i(x, y, t, r, p_x + D_x v, p_y + D_y v) \\ < & \alpha w + w_t + H_i(x, y, t, r, p_x + D_x w, p_y + D_y w). \end{aligned}$$

From comparison principle, $v \leq w$. We obtain a contradiction.

***Theorem* 4.1** — *Assume that H_i satisfies the assumptions $(\mathbf{A_1}) - (\mathbf{A_5})$ with $l = 0$ in $(\mathbf{A_3})$, then, the continuous viscosity solution u^ϵ of (1.1) converges to the viscosity solution u of (1.2) locally uniformly in $\mathbf{R}^{N+1} \times [0, +\infty)$ as $\epsilon \to 0$.*

PROOF : By Theorem 2.2, we know that the bound for u_ϵ is independent of ϵ, so we can define the half-relaxed limits:

$$\overline{u}(x, y, t) = \limsup_{\epsilon \to 0, (x_\epsilon, y_\epsilon, t_\epsilon) \to (x, y, t)} u^\epsilon(x_\epsilon, y_\epsilon, t_\epsilon),$$

$$\underline{u}(x,y,t) = \liminf_{\epsilon\to 0,(x_\epsilon,y_\epsilon,t_\epsilon)\to(x,y,t)} u^\epsilon(x_\epsilon,y_\epsilon,t_\epsilon).$$

We first show that $\overline{u}(x,y,t)$ is a viscosity sub-solution of the system (1.2). We assume that there exist $i \in \{1,2,\ldots,M\}$ and $\varphi \in C^1$ such that $\overline{u}_i - \varphi$ has a strict maximum at some point $(\overline{x},\overline{y},\overline{t})$ with $\overline{t} > 0$. Without loss of generality, we may as well assume that $i = 1$. Let v be a bounded space-time periodic super-solution of (1.3) associated to $(\overline{u}(\overline{x},\overline{y},\overline{t}), p_x, p_y)$, where $p_x = D_x\varphi(\overline{x},\overline{y},\overline{t}),\ p_y = D_y\varphi(\overline{x},\overline{y},\overline{t})$ and to the function

$$H_1^k(x,y,t,r,q_x,q_y) = \min\{H_1(x',y,t',r,q_x',q_y) : |x'-x|, |t'-t|, |q_x'-q_x| \le k^{-1}\}$$

We use "the perturbed test-function" method (see[11]), defining the function

$$\Phi(x,x',y,y',t,t') = u_1^\epsilon(x,y,t) - \varphi(x,y,t) - \epsilon v(\epsilon^{-1}x', \epsilon^{-1}y', \epsilon^{-1}t') \\ - \frac{|x-x'|^2}{2\delta^2} - \frac{|y-y'|^2}{2\delta^2} - \frac{|t-t'|^2}{2\delta^2}.$$

By classical results on viscosity solutions and since $u_1 - \varphi$ has a strict maximum at $(\overline{x},\overline{y},\overline{t})$, up to extract subsequence, there exist $(x_{\epsilon,\delta}, x'_{\epsilon,\delta}, y_{\epsilon,\delta}, y'_{\epsilon,\delta}, t_{\epsilon,\delta}, t'_{\epsilon,\delta})$, denoted by (x,x',y,y',t,t') for simplicity (depending on ϵ,δ) and $(x_\epsilon,y_\epsilon,t_\epsilon)$ such that (x,x',y,y',t,t') is a local maximum point near $(\overline{x},\overline{x},\overline{y},\overline{y},\overline{t},\overline{t})$ of Φ and

$$(x,x',y,y',t,t') \to (x_\epsilon,x_\epsilon,y_\epsilon,y_\epsilon,t_\epsilon,t_\epsilon), \text{ as } \delta\to 0;$$
$$(x_\epsilon,y_\epsilon,t_\epsilon) \to (\overline{x},\overline{y},\overline{t}), \text{ as } \epsilon\to 0;$$
$$\lim_{\epsilon\to 0}\lim_{\delta\to 0} u_1^\epsilon(x,y,t) = u_1(\overline{x},\overline{y},\overline{t}).$$

We set

$$q_x = \frac{x-x'}{\delta^2}, \quad q_y = \frac{y-y'}{\delta^2}, \quad q_t = \frac{t-t'}{\delta^2}.$$

The viscosity inequalities for the sub-solution u^ε at (x,y,t) of (1.1) and the super-solution v at $(\epsilon^{-1}x', \epsilon^{-1}y', \epsilon^{-1}t')$ of (1.3) read

$$\varphi_t + q_t + H_1(\epsilon^{-1}x, \epsilon^{-1}y, \epsilon^{-1}t, u^\varepsilon(x,y,t), D_x\varphi(x,y,t) + q_x, D_y\varphi(x,y,t) + q_y) \le 0. \tag{4.1}$$

$$q_t + H_1^k(\epsilon^{-1}x', \epsilon^{-1}y', \epsilon^{-1}t', \overline{u}(\overline{x},\overline{y},\overline{t}), p_x + q_x, p_y + q_y) \ge \overline{H}_1^k(\overline{u}, p_x, p_y). \tag{4.2}$$

From the classical results, we have

$$\frac{|x-x'|^2}{\delta^2} + \frac{|y-y'|^2}{\delta^2} + \frac{|t-t'|^2}{\delta^2} \to 0, \text{ as } \delta\to 0, \text{ for any } \epsilon > 0,$$

and therefore $|\epsilon^{-1}x - \epsilon^{-1}x'| + |\epsilon^{-1}t - \epsilon^{-1}t'| = \epsilon^{-1}\delta o(1)$, where $o(1) \to 0$, as $\varepsilon, \delta \to 0$ and on the other hand, since $(x, y, t) \to (\overline{x}, \overline{y}, \overline{t})$, we have

$$|(D_x\varphi(x, y, t) + q_x) - (p_x + q_x)| \to 0 \text{ as } \epsilon, \delta \to 0.$$

Therefore, if we choose $\delta \ll \epsilon$ and ϵ, δ small enough, we have

$$|\epsilon^{-1}x - \epsilon^{-1}x'|,\ |\epsilon^{-1}t - \epsilon^{-1}t'|,\ |(D_x\varphi(x, y, t) + q_x) - (p_x + q_x)| \leq k^{-1}.$$

and furthermore

$$\begin{aligned} &H_1(\epsilon^{-1}x, \epsilon^{-1}y', \epsilon^{-1}t, \overline{u}, D_x\varphi(x, y, t) + q_x, p_y + q_y) \\ \geq\ &H_1^k(\epsilon^{-1}x', \epsilon^{-1}y', \epsilon^{-1}t', \overline{u}, p_x + q_x, p_y + q_y). \end{aligned}$$

Then, we have

$$\begin{aligned} \varphi_t + \overline{H}_1^k(\overline{u}, p_x, p_y) &\leq H_1(\epsilon^{-1}x, \epsilon^{-1}y', \epsilon^{-1}t, \overline{u}, D_x\varphi + q_x, p_y + q_y) \\ &\quad - H_1(\epsilon^{-1}x, \epsilon^{-1}y, \epsilon^{-1}t, u^{\varepsilon}, D_x\varphi + q_x, D_y\varphi + q_y) \\ &= \Gamma_1 + \Gamma_2 + \Gamma_3, \end{aligned}$$

where

$$\begin{aligned} \Gamma_1 &= H_1(\epsilon^{-1}x, \epsilon^{-1}y', \epsilon^{-1}t, \overline{u}, D_x\varphi + q_x, p_y + q_y) - H_1(\epsilon^{-1}x, \epsilon^{-1}y, \epsilon^{-1}t, \overline{u}, D_x\varphi + q_x, p_y + q_y), \\ \Gamma_2 &= H_1(\epsilon^{-1}x, \epsilon^{-1}y, \epsilon^{-1}t, \overline{u}, D_x\varphi + q_x, p_y + q_y) - H_1(\epsilon^{-1}x, \epsilon^{-1}y, \epsilon^{-1}t, \overline{u}, D_x\varphi + q_x, D_y\varphi + q_y), \\ \Gamma_3 &= H_1(\epsilon^{-1}x, \epsilon^{-1}y, \epsilon^{-1}t, \overline{u}, D_x\varphi + q_x, D_y\varphi + q_y) - H_1(\epsilon^{-1}x, \epsilon^{-1}y, \epsilon^{-1}t, u^{\epsilon}, D_x\varphi + q_x, D_y\varphi + q_y). \end{aligned}$$

Using $(\mathbf{A_3})$, $\Gamma_1 \leq C_3|\epsilon^{-1}y' - \epsilon^{-1}y||p_y + q_y|$, and therefore

$$\lim_{\epsilon\to 0}\lim_{\delta\to 0}\Gamma_1 = 0.$$

Similarly, $\Gamma_2 \leq C_5|p_y - D_y\varphi| \to 0$, as $\delta, \epsilon \to 0$.

For the estimate of Γ_3, let $\beta > 0$. By definition of $\overline{u}$, for δ, ϵ small enough with $\delta \ll \epsilon$, we have

$$u_j^{\epsilon}(x, y, t) - \overline{u}_j(\overline{x}, \overline{y}, \overline{t}) \leq \frac{\beta}{2},\ \text{for } 2 \leq j \leq \beta.$$

Set $\gamma_\beta = (\overline{u}_1(\overline{x}, \overline{y}, \overline{t}) + \beta, u_2^{\epsilon}(x, y, t), \ldots, u_M^{\epsilon}(x, y, t))$. It follows that

$$\max_j\{\gamma_{\beta,j} - \overline{u}_j\} = \overline{u}_1 + \beta - \overline{u}_1 = \beta > 0$$

which is achieved for the first component. Then by $(\mathbf{A_4})$, we have

$$\begin{aligned} & H_1(\epsilon^{-1}x, \epsilon^{-1}y, \epsilon^{-1}t, \overline{u}, D_x\varphi + q_x, D_y\varphi + q_y) \\ \leq\ & H_1(\epsilon^{-1}x, \epsilon^{-1}y, \epsilon^{-1}t, \gamma_\beta, D_x\varphi + q_x, D_y\varphi + q_y). \end{aligned}$$

Then

$$\begin{aligned} \Gamma_3 \leq\ & H_1(\epsilon^{-1}x, \epsilon^{-1}y, \epsilon^{-1}t, \gamma_\beta, D_x\varphi + q_x, D_y\varphi + q_y) \\ & - H_1(\epsilon^{-1}x, \epsilon^{-1}y, \epsilon^{-1}t, u^\epsilon(x,y,t), D_x\varphi + q_x, D_y\varphi + q_y) \\ & \to 0, \end{aligned}$$

as $\delta, \epsilon, \beta \to 0$, which follows from

$$\lim_{\beta\to 0, \epsilon\to 0, \delta\to 0} \gamma_\beta = \overline{u}(\overline{x}, \overline{y}, \overline{t}) \qquad \text{and} \lim_{\epsilon\to 0, \delta\to 0} u^\epsilon = \overline{u}(\overline{x}, \overline{y}, \overline{t}).$$

Therefore

$$\varphi_t(x,y,t) + \overline{H}_1^k(\overline{u}, p_x, p_y) \leq o(1).$$

Sending $\epsilon, \delta \to 0$, we obtain

$$\varphi_t(\overline{x}, \overline{y}, \overline{t}) + \overline{H}_1(\overline{u}(\overline{x}, \overline{y}, \overline{t}), \varphi_x(\overline{x}, \overline{y}, \overline{t}), \varphi_y(\overline{x}, \overline{y}, \overline{t})) \leq 0$$

which proves that $\overline{u}$ is a sub-solution of (1.2).

Similarly, we can also prove that $\underline{u}$ is a viscosity super-solution of (1.2).

It is easy to see that (1.2) satisfies comparison principle, since $\overline{H}_i$ is continuous, satisfies monotone condition and just depends on (r, p_x, p_y). Then we obtain

$$\overline{u} \leq \underline{u}.$$

It follows that $\overline{u} = \underline{u} := u$, where u is the locally uniform limit of u^ϵ.

We complete the result.

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