essentially bounded, namely $g(d) \ll_\varepsilon d^\varepsilon$, $\forall \varepsilon > 0$, and vanishes outside $[1, Q]$ for some $Q \in \mathbb{N}$, that is

$$f(n) = \sum_{\substack{d|n \\ d \le Q}} g(d).$$

Note that $f = g * \mathbf{1}$ is essentially bounded if and only if g is.

As usual, $\ll$ is Vinogradov's notation, synonymous to Landau's O-notation. In particular, $\ll_\varepsilon$ means that the implicit constant might depend on an arbitrarily small and positive real number ε, which might change at each occurrence.

In this paper we continue our study of the distribution of a real sieve function f over short *arithmetic bands*, i.e. over the set

$$\bigcup_{1 \le a \le H} \{n \in (N, 2N] : n \equiv a\,(q)\}, \text{ with } H = o(N).$$

Henceforth we write $n \equiv a\,(q)$ to abbreviate $n \equiv a \pmod q$ and omit $a \ge 1$ in sums like $\sum_{a \le H}$. Let us recall that in [2] we have proved that the inequality

$$T_f(q, N, H) \stackrel{def}{=} \sum_{a \le H} \Big(\sum_{\substack{n \sim N \\ n \equiv a\,(q)}} f(n) - \frac{\widehat{f_N}(0)}{q} \Big) \ll_\varepsilon N^\varepsilon \Big(\frac{N}{q} + q + Q \Big) \tag{1.1}$$

holds for every real sieve function f of range $Q \ll N$, after assuming that $H = o(q)$, as $q \to \infty$, and $q = o(N)$, as $N \to \infty$. Such conditions are required in order to avoid overlapping and *sporadicity* of the arithmetic bands, respectively.

Here we generalize the previous inequality for

$$T_f(q, r, b, N, H) \quad \stackrel{def}{=} \quad \sum_{a \le H} \Big(\sum_{\substack{n \sim N \\ n \equiv ar+b\,(q)}} f(n) - \frac{\widehat{f_N}(0)}{q} \Big), \tag{1.2}$$

where $r, b \in \mathbb{Z}$ are such that $(r, q) = 1$ (hereafter $(r, q) = \text{g.c.d.}(r, q)$, as usual). In particular, note that $T_f(q, 1, 0, N, H) = T_f(q, N, H)$. Beyond the fact that we give a more accurate proof, in the present paper we also take the chance to provide with two further corollaries and discuss the optimality of the following inequality.

***Theorem* 1.1** — *Let $q, N, H, Q \in \mathbb{N}$ be such that $Q \ll N$, $H = o(q)$ and $q = o(N)$. For every sieve function $f : \mathbb{N} \to \mathbb{R}$ of range Q and all $r, b \in \mathbb{Z}$ such that $(r, q) = 1$, one has*

$$T_f(q, r, b, N, H) \ll_\varepsilon N^\varepsilon (N/q + q + Q). \tag{1.3}$$

Remark 1.1 : Being $T_f(q,r,b,N,H) \ll N^{1+\varepsilon}H/q$ a trivial bound, both (1.1) and (1.3) are non-trivial once the *width* $\theta \stackrel{def}{=} \frac{\log H}{\log N}$ is positive, $q \ll \sqrt{N^{1-\delta}H}$ and $qQ \ll N^{1-\delta}H$, for a suitable $\delta > 0$. Since it is assumed that $Q \ll q \ll Q$ hereafter, we get a bound $Q \ll \sqrt{N^{1-\delta}H}$ and thus we can go beyond the square-root of N (for $\theta > 0$). Consistently with the terminology introduced in [2], we stop at the *barrier* $\frac{\log Q}{\log N} < \frac{1+\theta}{2}$. In the final section of [2], our study of the distribution of sieve functions in arithmetic bands is compared to the classical results on the distribution in arithmetic progressions, which deal mostly with the overcoming of the so-called *level* $1/2$. Our present results, insofar they generalize our previous ones, already go beyond level $1/2$. In this paper we discuss the possibility of going beyond $1/2+\theta/2$ in the present contest of the sieve functions in arithmetic bands, namely by taking $Q > \sqrt{N^{1+\delta}H}$ for a certain small and fixed $\delta > 0$, so that $N^{1+\varepsilon}H/Q = o(Q)$, provided $\delta > \varepsilon$. In §3 we exhibit a particular sieve function whose behavior in arithmetic bands confirms the optimality of such level.

The distribution in arithmetic bands of truncated functions of the type

$$f(n) = \sum_{\substack{d|n \\ d\le Q}} g(d)$$

is somehow involved in the study of convolution sums like

$$\sum_{n\le N} f_1(n) f_2(n-a).$$

Indeed, by assuming that $f_1 = g_1 * \mathbf{1}$ and $f_2 = g_2 * \mathbf{1}$, the sum

$$\begin{aligned} \sum_{n\le N} f_1(n) f_2(n-a) &= \sum_{n\le N}\sum_{d|n} g_1(d) \sum_{q|n-a} g_2(q) \\ &= \sum_{d\le N} g_1(d) \sum_{q\le N-a} g_2(q) \sum_{\substack{n\le N \\ n\equiv 0\ (d) \\ n\equiv a\ (q)}} 1 \end{aligned}$$

will not be altered if one sets

$$f_1(n) = \sum_{\substack{d|n \\ d\le N}} g_1(d) \quad \text{and} \quad f_2(n) = \sum_{\substack{d|n-a \\ d\le N-a}} g_2(d).$$

Further, if the *shift* a is confined to $a \ll H$, then the conditions $n \le N$ and $H = o(N)$ clearly yield $\max(n, n-a) \le N + |a| \le 2N$. Moreover, if f_1 and f_2 are essentially bounded, then trivially

$$\sum_{n\le N} f_1(n) f_2(n) \ll N^{1+\varepsilon}$$

and for any $a \ll H$ one has

$$\sum_{n\le N} f_1(n)f_2(n-a) = \sum\sum_{\substack{n_1,n_2\le N\\ n_2-n_1=a}} f_1(n_1)f_2(n_2) + O_\varepsilon\left(N^\varepsilon H\right).$$

However, only under the more severe conditions $g_i(d) \ll d^{-\varepsilon}$ for some $\varepsilon > 0$, ($i = 1,2$), Murty, Saha and the first author [4] have recently established more explicit asymptotic formulae for such convolution sums in terms of finite Ramanujan expansions (see also [5] and [7] for a deeper understanding of such a connection). Regarding the case of sieve functions f_1, f_2 we have to content ourself with the asymptotic formula given in [2], Corollary 1.7, for the *first generation* average sum

$$\sum_{a\le H}\sum_{n\le N} f_1(n)f_2(n-a).$$

By applying Theorem 1.1, here we give a slight generalization of such an asymptotic formula for the correlation of real arithmetic functions f_1 and f_2 in $\mathcal{N}$, i.e.

$$\mathcal{C}_{f_1,f_2}(a) \overset{def}{=} \sum_{n\sim N} f_1(n)f_2(n-a).$$

Corollary 1.1 — Let q, N, H, Q_1, Q_2 be positive integers such that $H = o(q)$, $q = o(N)$ and $Q_1 \le Q_2 \ll N$, as $q, N \to \infty$. For any real and essentially bounded arithmetic functions g_1 and g_2 supported in $[1, Q_1]$ and $[1, Q_2]$, respectively, and for all $r \in \mathbb{N}$, $b \in \mathbb{Z}$ such that $(r, b) = 1$ and $H - b \ge r$, one has

$$\sum_{\substack{a\le H\\ a\equiv b\,(r)}} \mathcal{C}_{f_1,f_2}(a) = R_1(f_1)R_1(f_2)N(H-b)/r + O_\varepsilon\big(N^\varepsilon(N + Q_2^2 + Q_1H)\big),$$

where, for $j = 1,2$, we set $f_j = g_j * \mathbf{1}$ and $R_1(f_j) \overset{def}{=} \sum_{d\le Q_j} \frac{g_j(d)}{d}$ is the so-called first Ramanujan coefficient of f_j.

Finally, in [1] we studied the so-called *length inertia* property for *weighted Selberg integrals* (see [2] for the link between such integrals and the distribution of a sieve function in arithmetic bands). Such a property permits transfer of non-trivial bounds in short intervals of length h, say, to similar bounds in longer intervals of length H. Now we can show that a length inertia property holds also for the distribution of a sieve function in arithmetic bands. To this aim, for $H = \infty(h)$ (that is $h = o(H)$, as $H \to \infty$) let us consider (recall (1.2))

$$T_f(q,1,0,N,[H/h]h) = \sum_{a\le[H/h]h}\Big(\sum_{\substack{n\sim N\\ n\equiv a\,(q)}} f(n) - \frac{\widehat{f_{\mathcal{N}}}(0)}{q}\Big),$$

where $[x]$ is the integer part of $x \in \mathbb{R}$.

Corollary 1.2 — Let $q, N, H, Q \in \mathbb{N}$ be such that $Q \ll N$, $h = o(q)$, for some $h = o(H)$, and $q = o(N)$. For every sieve function $f : \mathbb{N} \to \mathbb{R}$ of range Q, one has

$$T_f(q,1,0,N,[H/h]h) = \sum_{j\le[H/h]} T_f(q,1,(j-1)h,N,h) \ll_\varepsilon N^\varepsilon(N/q+q+Q)H/h.$$

2. Proofs of Theorem 1.1 and Its Corollaries

In [2] we have established some formulæ that relate the ℓth Ramanujan coefficient of a real sieve function $f = g * \mathbf{1}$ of range Q, i.e.

$$R_\ell(f) \overset{def}{=} \sum_{\substack{d\le Q\\ d\equiv 0\ (\ell)}} \frac{g(d)}{d},$$

to the values of $\widehat{f_{\mathcal{N}}}$ attained at rational numbers, whereas it is easily seen that

$$R_\ell(f) = \frac{1}{\ell}\sum_{m\le Q/\ell} \frac{g(\ell m)}{m} \ll_\varepsilon \frac{Q^\varepsilon}{\ell}. \tag{2.1}$$

In order to prove Theorem 1.1 and Corollary 1.1, we need to quote such properties that here are included in the following lemma.

Lemma 2.1 — Let $f = g * \mathbf{1}$ be a sieve function of range $Q \ll N$, with $Q, N \in \mathbb{N}$. Then

$$\widehat{f_{\mathcal{N}}}(0) = R_1(f)N + O_\varepsilon\left(Q^{1+\varepsilon}\right); \tag{2.2}$$

$$\widehat{f_{\mathcal{N}}}\left(\frac{j}{\ell}\right) = R_\ell(f)N + O_\varepsilon((\ell Q)^\varepsilon(Q+\ell)), \tag{2.3}$$

$$\forall \ell > 1, \forall j \in \mathbb{Z}_\ell^* = \{1 \le j \le \ell :\ (j,\ell) = 1\}.$$

Proof : From

$$\begin{aligned}
\sum_{\substack{n\sim N\\ n\equiv a\ (q)}} f(n) &= \sum_{\substack{n\sim N\\ n\equiv a\ (q)}} \sum_{\substack{d\le Q\\ d|n}} g(d) = \sum_{d\le Q} g(d) \sum_{\substack{n\sim N\\ n\equiv a\ (q)\\ n\equiv 0\ (d)}} 1\\
&= \sum_{\substack{d\le Q\\ (d,q)|a}} g(d) \sum_{\substack{n\sim N/d\\ nd\equiv a\ (q)}} 1 = \sum_{\substack{d\le Q\\ (d,q)|a}} g(d)\left(\frac{N}{dq}(d,q)+O(1)\right)\\
&= \frac{N}{q}\sum_{\substack{d\le Q\\ (d,q)|a}} \frac{g(d)}{d}(d,q) + O_\varepsilon\left(Q^{1+\varepsilon}\right),
\end{aligned}$$

we get (2.2) as long as $q=1$. Now, let us assume that $1 \le j < \ell$ with $(j,\ell)=1$, and write

$$\begin{aligned}\widehat{f_{\mathcal{N}}}(j/\ell) &= \sum_{d\le Q} g(d) \sum_{v\sim \frac{N}{d}} e_\ell(jdv) \\ &= \sum_{\substack{d\le Q\\ d\equiv 0\,(\ell)}} g(d)\Big(\frac{N}{d}+O(1)\Big) + O\Big(\sum_{\substack{d\le Q\\ d\not\equiv 0\,(\ell)}} \frac{|g(d)|}{\|jd/\ell\|}\Big),\end{aligned}$$

where $\|r\| \stackrel{def}{=} \min_{n\in\mathbb{Z}} |r-n|$, $\forall r\in\mathbb{R}$. Thus, (2.3) follows from

$$\sum_{\substack{d\le Q\\ d\equiv 0\,(\ell)}} g(d)\Big(\frac{N}{d}+O(1)\Big) = R_\ell(f)N + O_\varepsilon\Big(Q^\varepsilon\Big(\frac{Q}{\ell}+1\Big)\Big)\ ,$$

and

$$\sum_{\substack{d\le Q\\ d\not\equiv 0\,(\ell)}} \frac{1}{\|jd/\ell\|} \le \sum_{0<|r|\le \ell/2} \sum_{\substack{d\le Q\\ jd\equiv r\,(\ell)}} \frac{1}{|r/\ell|} \ll \ell \sum_{r\le \ell/2} \frac{1}{r}\Big(\frac{Q}{\ell}+1\Big) \ll_\varepsilon \ell^\varepsilon (Q+\ell).$$

The proof is completed. □

PROOF OF THEOREM 1.1 : By the orthogonality of additive characters $e_q(t) \stackrel{def}{=} e(t/q)$, ($q\in\mathbb{N}$, $t\in\mathbb{Z}$), we get

$$\begin{aligned}T_f(q,r,b,N,H) &= \frac{1}{q}\sum_{a\le H}\Big(\sum_{n\sim N} f(n)\sum_{k\le q} e_q\big(k(ar+b-n)\big) - \widehat{f_{\mathcal{N}}}(0)\Big) \\ &= \frac{1}{q}\sum_{k<q}\sum_{a\le H} e_q(k(ar+b))\widehat{f_{\mathcal{N}}}(-k/q) \\ &= \frac{1}{q}\sum_{\substack{\ell|q\\ \ell>1}}\sum_{j\in\mathbb{Z}_\ell^*} \widehat{f_{\mathcal{N}}}\left(-\frac{j}{\ell}\right)\sum_{a\le H} e_\ell(j(ar+b)),\end{aligned}$$

where we have set $\ell = q/(k,q)$ and $j = k/(k,q)$. Since $(r,q)=1$, for any $\ell|q$ there exists an integer $\overline{r}$ such that $r\overline{r}\equiv 1 \pmod{\ell}$. Therefore we write

$$T_f(q,r,b,N,H) = \frac{1}{q}\sum_{\substack{\ell|q\\ \ell>1}}\sum_{j\in\mathbb{Z}_\ell^*} \widehat{f_{\mathcal{N}}}\left(-\frac{j\overline{r}}{\ell}\right) e_\ell(j\overline{r}b)\sum_{a\le H} e_\ell(ja).$$

By applying (2.1), (2.3) and the well-known inequality (see [6], Ch. 25)

$$\sum_{V_1<v\le V_2} e(v\alpha) \ll \min\Big(V_2-V_1, \frac{1}{\|\alpha\|}\Big),$$

we conclude that

$$
\begin{aligned}
T_f(q,r,b,N,H) &\ll_\varepsilon \frac{1}{q}\sum_{\substack{\ell|q\\ \ell>1}}\Big(|R_\ell(f)|N+(\ell Q)^\varepsilon(Q+\ell)\Big)\sum_{j\in\mathbb{Z}_\ell^*}\frac{1}{\|j/\ell\|}\\
&\ll_\varepsilon \frac{N^\varepsilon}{q}\sum_{\substack{\ell|q\\ \ell>1}}\Big(\frac{N}{\ell}+Q+\ell\Big)\ell \ll_\varepsilon N^\varepsilon\Big(\frac{N}{q}+Q+q\Big),
\end{aligned}
$$

that is (1.3). □

Remark 2.1 : Besides (1.3), from the previous proof it transpires that also the upper bound of

$$
\begin{aligned}
T_f(q,r,b,N,H) &= \frac{1}{q}\sum_{\substack{\ell|q\\ \ell>1}}\sum_{j\in\mathbb{Z}_\ell^*}\widehat{f_N}\left(-\frac{j\bar{r}}{\ell}\right)e_\ell(j\bar{r}b)\sum_{a\le H}e_\ell(ja)\\
&\ll \frac{1}{q}\sum_{\substack{\ell|q\\ \ell>1}}\ell\sum_{\substack{j\le\ell/2\\ (j,\ell)=1}}\frac{1}{j}\max_{j\in\mathbb{Z}_\ell^*}|\widehat{f_N}(j/\ell)|
\end{aligned}
$$

is independent of both b and r such that $(r,q)=1$ (that is to say, it is the same upper bound obtained for $r=1$ and $b=0$).

PROOF OF COROLLARY 1.1 : Let us write

$$
\begin{aligned}
\mathcal{C}_{f_1,f_2}(a) &= \sum_{n\sim N}f_1(n)f_2(n-a)=\sum_{n\sim N}f_1(n)\sum_{\substack{q|n-a\\ q\le Q_2}}g_2(q)\\
&= \sum_{q\le Q_2}g_2(q)\sum_{\substack{n\sim N\\ n\equiv a\,(q)}}f_1(n).
\end{aligned}
$$

Now, note that, since without loss of generality we can assume $|b|\le r$, from $H=o(q)$ one has that $(H-b)/r=o(q)$, as $q\to\infty$. Therefore, from (2.2) and Theorem 1.1 we obtain

$$
\begin{aligned}
\sum_{\substack{a\le H\\ a\equiv b\,(r)}}\mathcal{C}_{f_1,f_2}(a) &= \sum_{q\le Q_2}g_2(q)\sum_{s\le(H-b)/r}\sum_{\substack{n\sim N\\ n\equiv sr+b\,(q)}}f_1(n)\\
&= \sum_{q\le Q_2}g_2(q)\left(\frac{H-b}{qr}\sum_{n\sim N}f_1(n)+T_{f_1}\Big(q,r,b,N,(H-b)/(qr)\Big)\right)\\
&= \frac{H-b}{r}\Big(\sum_{q\le Q_2}\frac{g_2(q)}{q}\Big)\left(R_1(f_1)N+O_\varepsilon\left(Q_1^{1+\varepsilon}\right)\right)\\
&\quad +O_\varepsilon\Big(N^\varepsilon\sum_{q\le Q_2}\Big(\frac{N}{q}+q+Q_1\Big)\Big)\\
&= R_1(f_1)R_1(f_2)N(H-b)/r+O_\varepsilon\big(N^\varepsilon(N+Q_2^2+Q_1H)\big),
\end{aligned}
$$

that is the formula of the Corollary. □

PROOF OF COROLLARY 1.2 : Since

$$\begin{aligned} T_f(q,1,0,N,[H/h]h) &= \sum_{j\le[H/h]}\Big(\sum_{(j-1)h<a\le jh}\sum_{\substack{n\sim N\\ n\equiv a\,(q)}} f(n)-\frac{h}{q}\widehat{f_{\mathcal N}}(0)\Big)\\ &= \sum_{j\le[H/h]}\Big(\sum_{a\le h}\sum_{\substack{n\sim N\\ n\equiv a+(j-1)h\,(q)}} f(n)-\frac{h}{q}\widehat{f_{\mathcal N}}(0)\Big)\\ &= \sum_{j\le[H/h]} T_f(q,1,(j-1)h,N,h), \end{aligned}$$

the Corollary follows immediately from Theorem 1.1. □

Remark 2.2 : Note that the above equation holds for any f, not necessarily a sieve function.

3. SUPPORTING THE OPTIMALITY OF THE LEVEL

Let us set $\mathrm{sgn}(x) \stackrel{def}{=} x/|x|$ for all $x\in\mathbb{R}\setminus\{0\}$ and $\mathrm{sgn}(0)\stackrel{def}{=}0$. Then, for any fixed $q\in\mathbb{N}\cap(Q,2Q]$, we define the arithmetic function g as

$$g(d)=g(d,q,N,H)\stackrel{def}{=}\mathrm{sgn}\Big(\sum_{a\le H}\Big(\sum_{\substack{m\sim N/d\\ md\equiv a\,(q)}}1-\frac{1}{q}\sum_{m\sim N/d}1\Big)\Big),$$

if $d\in\mathbb{N}\cap(Q,2Q]$, and $g(d)=0$ otherwise. It is plain that g is the Eratosthenes transform of the sieve function $f(n)=f(n,q,N,H)\stackrel{def}{=}\sum_{d|n}g(d)$ of range $2Q$. After noting that $\mathrm{sgn}(x)x=|x|$ for all $x\in\mathbb{R}$, we write

$$\begin{aligned} |T_f(q,N,H)| &= \Big|\sum_{a\le H}\Big(\sum_{\substack{n\sim N\\ n\equiv a\,(q)}} f(n)-\frac{1}{q}\sum_{n\sim N}f(n)\Big)\Big|\\ &= \Big|\sum_{d\sim Q}g(d)\sum_{a\le H}\Big(\sum_{\substack{m\sim N/d\\ md\equiv a\,(q)}}1-\frac{1}{q}\sum_{m\sim N/d}1\Big)\Big|\\ &= \sum_{d\sim Q}\Big|\sum_{a\le H}\Big(\sum_{\substack{m\sim N/d\\ md\equiv a\,(q)}}1-\frac{1}{q}\sum_{m\sim N/d}1\Big)\Big|. \end{aligned}$$

We want to show that $|T_f(q,N,H)|\gg NH/q$, so we set

$$\mathcal{S}\stackrel{def}{=}\Big\{d\in\mathbb{N}\cap(Q,2Q]:\sum_{a\le H}\sum_{\substack{m\sim N/d\\ md\equiv a\,(q)}}1\ge 1\Big\},\quad \mathcal{E}\stackrel{def}{=}\mathbb{N}\cap(Q,2Q]\setminus\mathcal{S},$$

and prove that $|\mathcal{S}| = o(Q)$, which in turns yields $|\mathcal{E}| \gg Q$. This implies

$$\begin{aligned} |T_f(q,N,H)| &= \sum_{d\in\mathcal{S}}\Big|\sum_{a\le H}\Big(\sum_{\substack{m\sim N/d\\ md\equiv a\,(q)}} 1 - \frac{1}{q}\sum_{m\sim N/d} 1\Big)\Big| + \sum_{d\in\mathcal{E}}\frac{H}{q}\sum_{m\sim N/d} 1 \\ &\gg |\mathcal{E}|\frac{NH}{qQ} \gg \frac{NH}{q}. \end{aligned}$$

Now let us prove that $|\mathcal{S}| = o(Q)$. To this end, after recalling that the divisor function $\mathbf{d}(n) \stackrel{def}{=} \sum_{d|n} 1$ is essentially bounded and $q > Q$, we note that

$$|\mathcal{S}| \le \sum_{d\in\mathcal{S}}\sum_{a\le H}\sum_{\substack{m\sim N/d\\ md\equiv a\,(q)}} 1 \le \sum_{a\le H}\sum_{\substack{n\sim N\\ n\equiv a\,(q)}} \mathbf{d}(n) \ll_\varepsilon \frac{N^{1+\varepsilon}H}{Q}.$$

If $Q > \sqrt{N^{1+\delta}H}$ for a certain $\delta > 0$, then $N^{1+\varepsilon}H/Q = o(Q)$ once $\varepsilon < \delta$, that yields the desired conclusion.

Summarizing, $Q > \sqrt{N^{1+\delta}H}$ gives a trivial bound for $T_f(q,N,H)$, if $Q < q \le 2Q$. In this sense, the above function f is a counterexample (depending on q, N, H) to the possibility for the variable Q of going beyond the *barrier* $\sqrt{NH}$.

4. Further Remarks

Sieve functions are ubiquitous in analytic number theory. Besides the truncated divisor sum Λ_R (see [8]), that is a linear combination of sieve functions of range R (see [2] for our remarks on Λ_R), we quote the so-called *restricted* divisor function

$$\tau_Q(n) \stackrel{def}{=} (\mathbf{1}_Q * \mathbf{1})(n) = \sum_{d|n, d\le Q} 1,$$

whose Eratosthenes transform is the indicator $\mathbf{1}_Q$ of $[1,Q]\cap\mathbb{N}$. We refer the reader to [10] for results on the distribution of τ_Q in short arithmetic progressions. Here we wish to stress that in [10] one finds also conjectures and average results concerning the distribution in arithmetic bands of the more complicated function

$$\tau_{Q,R}(n) \stackrel{def}{=} (\mathbf{1}_Q * \mathbf{1}_R)(n) = \sum_{\substack{(d,t)\\ dt=n}} \mathbf{1}_Q(d)\mathbf{1}_R(t).$$

Such an essentially bounded function is linked to the pair correlation problem for fractional parts of the quadratic function αk^2, with $k \in \mathbb{N}$ and $\alpha \in \mathbb{R}$ (compare also [9]). While, under suitable conditions on H, Q, R, the search of an upper bound for

$$\sum_{a\le H}\Big(\sum_{n\equiv ar\,(q)}\tau_{Q,R}(n)-\frac{QR}{q}\Big),\ \text{with } (r,q)=1,$$

is pursued in [10], Conjecture 1.2, here Theorem 1.1 leads to an asymptotic formula for the inverse Eratosthenes transform of $\tau_{Q,R}$ in arithmetic bands, namely for

$$\sum_{a\le H}\sum_{n\equiv ar\,(q)}(\tau_{Q,R}*\mu)(n).$$

In [3] we established an asymptotic inequality for the exponential sum associated to the *localized* divisor functions, a family of functions including the aforementioned τ_Q, $\tau_{Q,R}$, and the standard divisor function d_k, $k\ge 2$ (recall that $d_k(n)$ is the number of ways to write n as a product of k positive integers). The particular instance of such an inequality for d_k is

$$\sum_{n\sim N} d_k(n)e(n\alpha)\ll_{k,\varepsilon}(Nq)^{\varepsilon}(N/q+q+N^{1-1/k}),$$

for all $\alpha\in[a/q-1/q^2,a/q+1/q^2]$, with $(a,q)=1, q>1$. Somehow, this can be regarded as being analogous to the inequality which follows by combining (2.3) with (2.1). Such a circumstance is remarkable even in view of the fact that any d_k falls short of being a sieve function, with a sort of Eratosthenes transform which turns out to be the sum of d_{k-1} plus some restricted divisor functions (see the last section of [1]).

Finally, since the function proposed in §3 seems to be rather artificial in that it depends on a fixed modulus $q\in\mathbb{N}\cap(Q,2Q]$, an intriguing open question to ask is how many standard sieve functions might support the optimality of the level accomplished by Theorem 1.1 and the results of [2].

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