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QUASI-PERIODIC SOLUTIONS FOR NON-AUTONOMOUS MKDV EQUATION

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In this paper, we consider the non-autonomous modified Korteweg-de Vries (mKdV) equation

$$u_t = u_{xxx} - 6f(\omega t)u^2 u_x, x \in \mathbb{R}/2\pi\mathbb{Z},$$

where $f(\omega t)$ is real analytic and quasi-periodic in t with frequency vector $\omega = (\omega_1, \omega_2, \cdots, \omega_m)$. Basing on an abstract infinite dimensional KAM theorem dealing with unbounded perturbation vector-field, we obtain the existence of Cantor families of smooth quasi-periodic solutions.

Key words : Quasi-periodic solution; non-autonomous mKdV equation; KAM theory; normal form.

1. INTRODUCTION AND MAIN RESULTS

Since the first Korteweg-de Vries (KdV) equation [15] was proposed to describe shallow water waves of long wavelength and small amplitude in 1895, the family of KdV equations has been widely used to model a variety of nonlinear phenomena, such as ion acoustic waves, ion-acoustic shock waves, waves in other homogeneous, weakly nonlinear and weakly dispersive, etc. The KdV equations appear in three, five or more order forms. The family of third order KdV is the form of

$$u_t + g(u)u_x + u_{xxx} = 0, \tag{1}$$

where $u(x,t)$ is a function of space x and the time variable t. When the nonlinear term $g(u) = 6u^2$, (1) is called the mKdV equation.

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The mKdV equation usually appears in electric circuits and multi-component plasmas. As an example, Ablowitz [1, 2] investigated the stability and instability conditions of algebraic solutions of the mKdV equation. More conclusions about the family of the KdV equation can be found in [6-10, 21, 31, 34] and the references therein.

In this paper, we consider the non-autonomous mKdV equation

$$u_t = u_{xxx} - 6f(\omega t)u^2 u_x \tag{2}$$

subject to periodic boundary conditions

$$u(t, x + 2\pi) = u(t, x), -\infty < t < \infty, \tag{3}$$

where $f(\omega t) = f(\theta)$ is real analytic in $\mathbb{T}^m$ and quasi-periodic in t with frequency vector $\omega = (\omega_1, \omega_2, \cdots, \omega_m)$.

Non-autonomous differential equations arise in many areas of mathematics and applied mathematics. For example, many problems in control theory are modelled by linear differential equations with time-varying coefficients.

In ordinary differential equation (ODE), the study of non-autonomous differential equation attracted much attention, see [12, 13, 29, 36]. A natural question is that what about the non-autonomous partial differential equations (PDEs). In fact, the existence of quasi-periodic solutions for the non-autonomous PDEs has also attracted great attention. The readers can refer to [4, 11, 22, 28, 30, 32, 37].

In this paper, we will prove the existence of quasi-periodic solutions for equation (2) by using an infinite dimensional KAM theorem, which deals with unbounded perturbation. To this end, we introduce some background of infinite dimensional KAM theorem.

Consider the Hamiltonian partial differential equation (HPDE)

$$\dot{w} = Aw + F(w). \tag{4}$$

For some Sobolev space $\mathcal{H}^p \ni w$, linear operator A maps $\mathcal{H}^p \to \mathcal{H}^{p-d}$ and nonlinear term F sends some neighbourhood of $\mathcal{H}^p \to \mathcal{H}^{p-\delta}$. One calls $d \geq 1$ and $\delta \in \mathbb{R}$ the orders of A and F, respectively. If $\delta \leq 0$, the vector-field F is called bounded perturbation. If $\delta > 0$, the vector-field F is called unbounded perturbation. It is reasonable to suppose that $0 < \delta \leq d - 1$ (see [14, 18, 20]) to guarantee the existence of KAM tori.

In the 1990s, the KAM theory has been generalized to infinite dimensional Hamiltonian perturbation, so as to prove that there are quasi-periodic solutions for some class of PDEs. The first KAM theory was constructed by Kuksin and Wayne [16, 33], after then on the existence of KAM tori of HPDEs with bounded perturbation was focused by many mathematicians. The study in this field is too much to list here, we give only two survey papers [5, 19]. Contrarily, fewer results of KAM theory for HPDEs with unbounded perturbation have been obtained up to now. The earliest KAM theorem for unbounded perturbation is still due to Kuksin [17, 18] where it is supposed that $d-1>\delta$. In [18], Kuksin proved the persistence of the finite-gap solutions of KdV equation with periodic boundary conditions. Another KAM theorem was built by Bambusi-Graffi in [3], as so to deal with the spectrum properties of the linear Schrödinger operator with unbounded linear perturbation. Recently, a new estimate for the small-denominators equation with critical unbounded variable coefficients was established by Liu and Yuan [24]. In [23], a KAM theorem for infinite dimensional Hamiltonian systems with $d-1\geq\delta$ was given by using the new estimate. Thereafter, the study of PDEs with unbounded perturbation was developed quickly, see [22, 25, 26, 27, 35] and the references therein.

In this paper, basing on the KAM theory for unbounded perturbation vector-field in [14], we can obtain the existence of many time-periodic solutions for equation (2). Our main work is to transform the Hamiltonian into its Birkhoff normal form up to order four to extract parameters. The first plague is that the four order term G depends on the time t which bring the difficulty of estimating the small divisor. In order to guarantee the regularity of the vector field of symplectic coordinate transformation, we require the small divisor satisfies

$$|\langle -k,\omega\rangle + j^3+l^3+m^3+n^3| \geq \frac{\epsilon_0^2\max\{|j|,|l|,|m|,|n|\}}{|k|^{\tau+4}}.$$

The second plague is that the normal frequencies Ω_j depend on the angular variables. Notice that the linearized homological equation is a variable coefficients equation in the process of KAM iteration, we find that the dependence does not affect the application of the KAM theorem in [14]. The last plague is that the forcing term exercises a great influence on the frequency which result in the difficulty of checking the condition (17). Fortunately, by choosing proper index set and function $f(\theta)$, we can guarantee that the condition is satisfied, see section 3 for more details.

After overcoming above difficulties, we have the following results.

***Theorem* 1.1** — *Consider the non-autonomous mKdV equation (2). The function $f(\theta)$ satisfies the following conditions:*

(H1) The mean value of $f(\theta)$ is not zero, that is

$$f(\theta) = \sum_{k\in\mathbb{Z}^m} f_k e^{i\langle k,\theta\rangle},\ f_0 \neq 0,$$

(H2) For some $s > 0$, $f(\theta)$ extends analytically in θ to the domain

$$D(2s) := \{\theta \in \mathbb{C}^m || \Im\theta| < 2s\}.$$

Let $[j_1, j_2, \cdots j_n]$ denote the least common multiple of $j_1, j_2, \cdots, j_n$. Then there exists a parameter subset $\widetilde{\Pi}_1 \subset [\varrho, 2\varrho]^m$ with $\mathrm{meas}\ \widetilde{\Pi}_1 > 0$, for any fixed positive integer n and index set $\mathcal{J} = \{j_1 < j_2 < \cdots < j_n\} \subset \mathbb{N} = \{1, 2, 3, \cdots\}$ with $j_n \leq \Gamma$ and $[j_1, j_2, \cdots j_n] \neq j_n$ (where Γ is mentioned in Lemma 2.6), and a given set $\Pi_2 \subset \mathbb{R}^n$ with positive Lebesgue measure, there exist

(1) a set $\Gamma_1^* \times \Gamma_2^* \subset \widetilde{\Pi}_1 \times \Pi_2$ with $\mathrm{meas}\ \Gamma_1^* \times \Gamma_2^* > 0$;

(2) a Lipschitz family of real analytic torus embeddings

$$\Phi : \mathbb{T}^n \times \Gamma_2^* \to \mathscr{S}^n_{N+\frac{1}{2}},$$

where $\mathscr{S}^n_{N+\frac{1}{2}} = \mathbb{T}^n \times \mathbb{R}^n \times \ell^2_{N+\frac{1}{2}} \times \ell^2_{N+\frac{1}{2}}$ and $\ell^2_{N+\frac{1}{2}}$ is a Hilbert space of all complex-valued sequences with norm (10);

(3) a Lipschitz map

$$\chi : \Gamma_2^* \to \mathbb{R}^n,$$

such that for each $(\varphi, \xi) \in \mathbb{T}^n \times \Gamma_2^*$, the curve $u(t) = \Phi(\varphi + \chi(\xi)t, \xi)$ is a quasi-periodic solution of (2) winding around the invariant torus $\Phi(\mathbb{T}^n \times \{\xi\})$. Moreover, each such torus is linearly stable.

2. THE BIRKHOFF NORMAL FORM

We study equation (2) as an infinite dimensional Hamiltonian system. Our aim is to get time quasi-periodic solutions of small amplitude. To attain this, we introduce the phase space

$$\mathcal{H}_0^N = \{u \in L^2(\mathbb{T}, \mathbb{R}) : \hat{u}(0) = 0, \|u\|_N^2 = \sum_{j\in\mathbb{Z}\setminus\{0\}} |j|^{2N} |\hat{u}(j)|^2 < \infty\}$$

for any $N > \frac{3}{2}$, where $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, and

$$\hat{u}(j) = \int_0^{2\pi} u(x) e_{-j}(x) dx, e_j(x) = \frac{1}{\sqrt{2\pi}} e^{ijx}.$$

Under the standard inner product on $L^2(\mathbb{T})$, equation (2) can be given in the form

$$u_t = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial H}{\partial u}\right), \tag{5}$$

with Hamiltonian

$$H(u,t) = \int_{\mathbb{T}}\left(\frac{1}{2}u_x^2 - \frac{1}{2}f(\omega t)u^4\right)\mathrm{d}x. \tag{6}$$

Firstly, we introduce a pair of action-angle variables $(I,\theta) \in \mathbb{R}^m \times \mathbb{T}^m$ to obtain an equivalent Hamiltonian that does not depend on the time variable t, and to reach an autonomous formulation of problem. The equivalent system is

$$\dot{\theta} = \omega, \dot{I} = -\frac{\partial H}{\partial \theta}, \dot{u} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial H}{\partial u}\right) \tag{7}$$

with the Hamiltonian

$$H(\theta, I, u) = \langle \omega, I\rangle + \int_{\mathbb{T}}\left(\frac{1}{2}u_x^2 - \frac{1}{2}f(\theta)u^4\right)\mathrm{d}x. \tag{8}$$

In order to write the Hamiltonian system (7) in infinitely many coordinates, we make the ansatz

$$u(t,x) = \sum_{j\neq 0}\gamma_j q_j(t) e_j(x), \tag{9}$$

where $\gamma_j = \sqrt{|j|}, e_j(x) = \frac{1}{\sqrt{2\pi}}e^{ijx}$. The coordinates are taken from the Hilbert space $\ell^2_{N+\frac{1}{2}}$ of all complex-valued sequences $(q_j)_{j\neq 0}$ with

$$\|q\|^2_{N+\frac{1}{2}} = \sum_{j\neq 0}|j|^{2N+1}|q_j|^2 < \infty, \ \ q_{-j} = \bar{q}_j. \tag{10}$$

Then (7) can be written as

$$\dot{\theta} = \omega, \dot{I} = -\frac{\partial H}{\partial \theta}, \dot{q}_j = i\sigma_j\left(\frac{\partial H}{\partial q_{-j}}\right), \sigma_j = \begin{cases} 1, & j \geq 1, \\ -1, & j \leq -1 \end{cases} \tag{11}$$

with the Hamiltonian

$$H(q,t) = \Lambda + G, \tag{12}$$

where

$$\Lambda = \langle \omega, I\rangle + \sum_{j\geq 1} j^3|q_j|^2,$$

$$G = -\frac{f(\theta)}{4\pi} \sum_{\substack{j+l+m+n=0 \\ j,l,m,n\neq 0}} \gamma_j\gamma_l\gamma_m\gamma_n q_j q_l q_m q_n.$$

Lemma 2.1 — For all $N > 1$, the gradient $\partial_q G$ is real analytic as a map from some neighbourhood of origin in $\ell^2_{N+\frac{1}{2}}$ into $\ell^2_{N-\frac{1}{2}}$, with $\|\partial_q G\|_{N-\frac{1}{2}} = O(\|q\|^3_{N+\frac{1}{2}})$.

PROOF : Direct calculation results in

$$\frac{\partial G}{\partial q_{-j}} = -\frac{f(\theta)}{4\pi}\gamma_j \sum_{l+m+n=j} \gamma_l\gamma_m\gamma_n q_l q_m q_n = -\frac{f(\theta)}{4\pi}\gamma_j g_j$$

where

$$g_j = \sum_{l+m+n=j} \gamma_l\gamma_m\gamma_n q_l q_m q_n.$$

As $f(\theta)$ is real analytic function in $\mathbb{T}^m$, we know that $f(\theta)$ is bounded. This implies that there exists a positive constant C_0 such that $|f(\theta)| \leq C_0$.

Putting $w = (w_j)_j = (\gamma_j q_j)_j$, $g = (g_j)$, we see that $g = w * w * w$. For $q \in \ell^2_{N+\frac{1}{2}}$, we have $w \in \ell^2_N$. Hence we get

$$\|g\|_N = \|w * w * w\|_N \leq C\|q\|^3_{N+\frac{1}{2}}$$

and consequently

$$\|\partial_q G\|_{N-\frac{1}{2}} \leq C\|g\|_N \leq C\|q\|^3_{N+\frac{1}{2}}.$$

The proof is completed. □

Lemma 2.2 — Suppose j, k, l are nonzero integers with $j + k + l = 0$. Then

$$j^3 + k^3 + l^3 = 3jkl \neq 0.$$

PROOF : The proof can be found in [14]. □

Lemma 2.3 — Suppose $j, l, m, n \in \mathbb{Z} \setminus \{0\}$, and define

$$\Delta = \{(j, l, m, n) \in \mathbb{Z}^4 \setminus \{0\} | j + l + m + n = 0\},$$

$$\Delta_1 = \{(j, l, m, n) \in \Delta | j + l, j + m, j + n \neq 0\}.$$

If $(j, l, m, n) \in \Delta_1$, we have

$$|j^3 + l^3 + m^3 + n^3| \geq \frac{3}{2} \max\{|j|, |l|, |m|, |n|\}.$$

PROOF : If $(j,l,m,n) \in \Delta_1$, we have $n = -j-l-m$, and

$$j^3 + l^3 + m^3 + n^3 = 3(j+l)(l+m)(l+n) \neq 0.$$

Without loss of generality, we will assume that $|j| = \max\{|j|, |l|, |m|, |n|\}$, the proof of conclusion in Lemma 2.3 can be divided into two cases.

Case 1 : There are three components have the same sign, then

$$|j| = |l| + |m| + |n|,$$

consequently,

$$\begin{aligned} |j^3 + l^3 + m^3 + n^3| &= 3|(j+l)(l+m)(l+n)| \geq 3|(l+m)(l+n)| \\ &\geq 3 \cdot \frac{2|l| + |n| + |m|}{2} > \frac{3|j|}{2}. \end{aligned}$$

Case 2 : There are two components have the same sign, let l and j have the same sign, then we have

$$|j^3 + l^3 + m^3 + n^3| = |3(j+l)(l+m)(l+n)| \geq 3|j+l| \geq 3|j| > \frac{3|j|}{2}. \square$$

Lemma 2.4 — There is a subset $\Theta_1 \subset [\varrho, 2\varrho]^m (\varrho > 0)$, such that every $\omega \in \Theta_1$ satisfies that

$$|\langle k, \omega \rangle| \geq \frac{\epsilon_0}{|k|^\tau}, \text{ for } \forall 0 \neq k \in \mathbb{Z}^m$$

and

$$\mathrm{meas}\ \Theta_1 \geq (1 - \mathrm{C}_1 \epsilon_0) \varrho^{\mathrm{m}},$$

where C_1 is a positive constant depending on m, ϱ and $\tau > m$, $0 < \epsilon_0 \ll 1$ is a small constant.

PROOF : We see that ω is just the classical Diophantine vectors in $[\varrho, 2\varrho]^m$ which forms a set of full measure in $[\varrho, 2\varrho]^m$. So we omit the details. $\square$

Lemma 2.5 — Assume that $j, l, m, n \in \Delta$. For the parameter set $[\varrho, 2\varrho]^m$, there is a subset $\Theta_2 \subset [\varrho, 2\varrho]^m$ with $\mathrm{meas}\ \Theta_2 \geq \varrho^{\mathrm{m}}(1 - \mathrm{C}_2\epsilon_0)$, satisfying that for $\forall \omega \in \Theta_2, (j,l,m,n) \in \Delta_1$, we have

$$|\langle -k, \omega \rangle + j^3 + l^3 + m^3 + n^3| \geq \frac{\epsilon_0^2 \max\{|j|, |l|, |m|, |n|\}}{|k|^{\tau+4}}.$$

PROOF : We prove this claim in two cases.

Case 1 : When $|k| \le K = \frac{\max\{|j|,|l|,|m|,|n|\}}{2m\varrho}$, and $(j,l,m,n) \in \Delta_1$, we have

$$\begin{aligned}|\langle -k,\omega\rangle + j^3 + l^3 + m^3 + n^3| &\ge |j^3 + l^3 + m^3 + n^3| - K|\omega| \\ &\ge \frac{3}{2}\max\{|j|,|l|,|m|,|n|\} - 2Km\varrho \\ &\ge \frac{1}{2}\max\{|j|,|l|,|m|,|n|\}.\end{aligned}$$

Case 2 : When $|k| > K$, we consider the set

$$\Pi_{jlmn,k} = \left\{\omega \in [\varrho, 2\varrho]^m || \langle -k,\omega\rangle + j^3 + l^3 + m^3 + n^3| \le \delta = \frac{\epsilon_0^2 \max\{|j|,|l|,|m|,|n|\}}{|k|^{\tau+4}}\right\}.$$

Choosing a vector $\vartheta = \{-1,1\}^m$ such as $\langle -k,\vartheta\rangle = |k|$, we write $\omega = r\vartheta + \upsilon, r \in \mathbb{R}, \upsilon \in \vartheta^{\perp}$, and

$$g(r) = \langle -k, r\vartheta + \upsilon\rangle + j^3 + l^3 + m^3 + n^3 = r|k| + \langle -k,\upsilon\rangle + j^3 + l^3 + m^3 + n^3.$$

So, we easily get $g'(r) = |k|$, it follows that

$$\{r : r\vartheta + \upsilon \in [\varrho, 2\varrho]^m, |\langle -k, r\vartheta + \upsilon\rangle + j^3 + l^3 + m^3 + n^3| \le \delta\} \subseteq \{r : |r - r_0(\upsilon)| \le \frac{\delta}{|k|}\}.$$

Hence we have

$$\operatorname{meas}(\Pi_{jlmn,k}) \le \frac{\delta}{|k|} \cdot \varrho^{m-1} = \frac{\epsilon_0^2 \max\{|j|,|l|,|m|,|n|\}}{|k||k|^{\tau+4}} \varrho^{m-1} \le \frac{m\epsilon_0 \varrho^m}{|k|^{\tau+4}}.$$

Now summing over all $j,l,m,n,|k| > K$ gives that

$$\begin{aligned}\operatorname{meas}\left(\bigcup_{\substack{jlmn,\\ |k|>K}} \Pi_{jlmn,k}\right) &\le 4^4 m^4 \epsilon_0 \varrho^{m+4} \sum_{|k|>K} \frac{1}{|k|^{\tau}} \\ &\le 4^4 m^4 \epsilon_0 \varrho^m \varrho^4 \sum_{l=K}^{\infty} \frac{(m-1)(2l+1)^{m-1}}{l^{\tau}} \\ &\le C_2 \epsilon_0 \varrho^m,\end{aligned}$$

where C_2 depends on ϱ, m, τ.

Let $\Theta_2 = [\varrho, 2\varrho]^m \setminus (\bigcup_{jlmn,|k|>K} \Pi_{jlmn,k})$, then $\operatorname{meas}\Theta_2 \ge \varrho^{\mathrm{m}}(1 - \mathrm{C}_2\epsilon_0)$. □

Lemma 2.6 — Choose $0 < \epsilon_0 \ll 1$, and fix ϵ_0. Consider the Hamiltonian (12). For some index set Δ_1, there exists a subset $\widetilde{\Pi}_1 \subset [\varrho, 2\varrho]^m$ with $\operatorname{meas}\widetilde{\Pi}_1 > 0$ such that for $\forall\omega \in \widetilde{\Pi}_1$, there is a real analytic symplectic coordinate transformation Φ defined in a complex neighbourhood $D(s) := \{\theta || \Im\theta| < s\}$ of the torus $\mathbb{T}^m$ and a neighbourhood of the origin of $\ell^2_{N+\frac{1}{2}}$ which transforms the Hamiltonian (12) into its Birkhoff normal form up to order four. More precisely,

$$H \circ \Phi = \Lambda + \bar{B} + \bar{P}$$

with

$$\begin{aligned}\bar{B} = &-\frac{3f_0}{\pi}\sum_{j,l\geq 1, j\neq l} jl|q_j|^2|q_l|^2 - \frac{f_0}{\pi}\sum_{j\geq 1} j^2|q_j|^4\\ &-\frac{3(f(\theta)-f_0)}{\pi}\sum_{\substack{j\neq l,\\ \max\{j,l\}>\Gamma}} jl|q_j|^2|q_l|^2 - \frac{(f(\theta)-f_0)}{\pi}\sum_{j>\Gamma} j^2|q_j|^4,\end{aligned}$$

where $\Gamma = \frac{1}{\epsilon_0}$ is an enough large positive constant, and

$$\|\bar{P}_q\|_{N-\frac{1}{2}} = O(\|q\|^5_{N+\frac{1}{2}}). \tag{13}$$

PROOF : Using the transformation $\Phi = X^1_F$, we get

$$\begin{aligned}H_1 &= H\circ\Phi = H\circ X^1_F\\ &= \Lambda + \{\Lambda, F\} + G + \int_0^1 (1-t)\{\{\Lambda, F\}, F\}\circ X^t_F dt + \int_0^1 \{G, F\}\circ X^t_F dt\\ &= \Lambda + \{\Lambda, F\} + G + \bar{P},\end{aligned}$$

Define

$$F = \sum_{\substack{j,l,m,n\neq 0,\\ k\in\mathbb{Z}^m}} F_{k,jlmn} q_j q_l q_m q_n e^{i\langle k,\theta\rangle},$$

by

$$iF_{k,jlmn} = \begin{cases} -\frac{1}{4\pi}\frac{f_k\gamma_j\gamma_l\gamma_m\gamma_n}{\langle -k,\omega\rangle + j^3+l^3+m^3+n^3}, & (j,l,m,n)\in\Delta_1,\\ \frac{1}{4\pi}\frac{f_k\gamma_j\gamma_l\gamma_m\gamma_n}{\langle k,\omega\rangle}, & \substack{\max\{|j|,|l|,|m|,|n|\}\leq\Gamma,\\ (j,l,m,n)\in\Delta\setminus\Delta_1, k\neq 0,}\\ 0, & \text{otherwise,}\end{cases}$$

where f_k is the $k-$th Fourier coefficient of the function $f(\theta)$, Γ is an enough large positive constant. Due to

$$\begin{aligned}G = &-\frac{3f_0}{4\pi}\sum_{j\neq l}|jl||q_j|^2|q_l|^2 - \frac{f_0}{2\pi}\sum_{j\neq 0} j^2|q_j|^4\\ &-\frac{f_0}{4\pi}\sum_{(j,l,m,n)\in\Delta_1}\gamma_j\gamma_l\gamma_m\gamma_n q_j q_l q_m q_n - \frac{1}{4\pi}\sum_{\substack{j,l,m,n\neq 0\\ k\neq 0}} f_k\gamma_j\gamma_l\gamma_m\gamma_n q_j q_l q_m q_n e^{i\langle k,\theta\rangle}\\ = &-B - Q,\end{aligned}$$

where

$$B=\frac{3f_0}{4\pi}\sum_{j\neq l}|jl||q_j|^2|q_l|^2+\frac{f_0}{2\pi}\sum_{j\neq 0}j^2|q_j|^4=\frac{3f_0}{\pi}\sum_{j,l\geq 1,j\neq l}jl|q_j|^2|q_l|^2+\frac{f_0}{\pi}\sum_{j\geq 1}j^2|q_j|^4,$$

and

$$Q=\frac{f_0}{4\pi}\sum_{(j,l,m,n)\in\Delta_1}\gamma_j\gamma_l\gamma_m\gamma_n q_jq_lq_mq_n+\frac{1}{4\pi}\sum_{\substack{j,l,m,n\neq 0\\ k\neq 0}}f_k\gamma_j\gamma_l\gamma_m\gamma_n q_jq_lq_mq_n e^{i\langle k,\theta\rangle}.$$

Then we have

$$\{\Lambda,F\}-Q=\tilde{Q},$$

where

$$\tilde{Q}=\frac{3(f(\theta)-f_0)}{\pi}\sum_{\substack{j\neq l\\ \max\{j,l\}>\Gamma}}jl|q_j|^2|q_l|^2+\frac{f(\theta)-f_0}{\pi}\sum_{j>\Gamma}j^2|q_j|^4.$$

Next our goal is to establish the regularity of the vector field X_F. From Lemma 2.4 and 2.5, we can get

$$|F_{k,jlmn}|\leq\frac{1}{4\pi}\cdot\frac{|k|^\tau f_k\gamma_j\gamma_l\gamma_m\gamma_n}{\epsilon_0}\leq\frac{1}{4\pi}\cdot\frac{|k|^\tau f_k\gamma_j\gamma_l\gamma_m\gamma_n}{\epsilon_0^2\max\{|j|,|l|,|m|,|n|\}}\leq\frac{1}{4\pi}\cdot\frac{|k|^{\tau+4}f_k\gamma_j\gamma_l\gamma_m\gamma_n}{\epsilon_0^2\max\{|j|,|l|,|m|,|n|\}}$$

for $\omega\in\Theta_1$ and $\max\{|j|,|l|,|m|,|n|\}\leq\Gamma=\frac{1}{\epsilon_0}$, and

$$|F_{k,jlmn}|\leq\frac{1}{4\pi}\cdot\frac{|k|^{\tau+4}f_k\gamma_j\gamma_l\gamma_m\gamma_n}{\epsilon_0^2\max\{|j|,|l|,|m|,|n|\}}$$

for $\omega\in\Theta_2$ and $(j,l,m,n)\in\Delta_1$. Moreover, the $j-$th element of gradient ∂_qF explicitly reads

$$\frac{\partial F}{\partial q_{-j}}=\sum_{\substack{l+m+n=j,\\ k\in\mathbb{Z}^m}}\left(F_{k,(-j)lmn}+F_{k,l(-j)mn}+F_{k,lm(-j)n}+F_{k,lmn(-j)}\right)q_lq_mq_n.$$

Thus, for the parameter $\omega\in\widetilde{\Pi}_1=\Theta_1\cap\Theta_2$, we get the estimate in a complex neighbourhood $\theta\in D(s)$ of $\mathbb{T}^m$

$$\left|\frac{\partial F}{\partial q_{-j}}\right|\leq\frac{C_3}{\epsilon_0\gamma_j}\sum_{l+m+n=j}\gamma_l\gamma_m\gamma_n|q_lq_mq_n|:=\frac{C_3}{\gamma_j}r_j,$$

where C_3 only depends on the function $f(\theta)$ and r_j stand for the sum

$$\sum_{l+m+n=j}\gamma_l\gamma_m\gamma_n|q_l||q_m||q_n|.$$

Defining $w = (w_j)_j = (r_j|q_j|)_j$, $r = (r_j)$, then $r_j = (w * w * w)_j$, consequently $r = w * w * w$. For $q \in \ell^2_{N+\frac{1}{2}}$, we have $w \in \ell^2_N$. Hence we have

$$\|r\|_N = \|w * w * w\|_N \leq C\|w\|_N^3 \leq C\|q\|^3_{N+\frac{1}{2}},$$

and therefore

$$\|\partial_q F\|_{N+\frac{1}{2}} \leq C\|r\|_N \leq C\|q\|^3_{N+\frac{1}{2}}.$$

Namely,

$$\|\partial_q F\|_{N+\frac{1}{2}} = O(\|q\|^3_{N+\frac{1}{2}}).$$

Now the Hamiltonian is changed into

$$H_1 = \Lambda + \bar{B} + \bar{P}$$

with $\bar{P}$ in (13). $\mathrm{meas}\,\widetilde{\Pi}_1 \geq \varrho^{\mathrm{m}}(1 - \mathrm{C}_1\epsilon_0 - \mathrm{C}_2\epsilon_0)$, based on this result, $\mathrm{meas}\,\widetilde{\Pi}_1 > 0$ holds true when ϵ_0 is enough small. So, we finish the proof of the Lemma 2.6. □

3. The Proof of Main Results

From the transformation Φ in Lemma 2.6, we get the new Hamiltonian

$$H_1 = \Lambda + \bar{B} + \bar{P}$$

which is analytic in $D(s) \times U$, where $D(s)$ is a complex strip domain of $\mathbb{T}^m$ and U is some neighbourhood of the origin of $\ell^2_{N+\frac{1}{2}}$.

We introduce symplectic polar and real coordinates $(\varphi, \eta, z, \bar{z})$ by setting

$$\begin{cases} q_{j_b} = \sqrt{\xi_b + \eta_b} e^{-i\varphi_b}, q_{-j_b} = \sqrt{\xi_b + \eta_b} e^{i\varphi_b}, & b = 1, 2 \cdots n, \\ q_j = z_j, q_{-j} = \bar{z}_j, & j \in \mathcal{N}_* = \mathbb{N} \setminus \mathcal{J}, \end{cases}$$

where $\xi = (\xi_1, \xi_2, \cdots \xi_n) \in \mathbb{R}^n_+$. Then

$$\Lambda = \langle \omega, I \rangle + \sum_{1 \leq b \leq n} j_b^3(\xi_b + \eta_b) + \sum_{j \in \mathcal{N}_*} j^3(z_j \bar{z}_j),$$

$$\begin{aligned}\bar{B}=&-\frac{3f_0}{\pi}\Bigg(\sum_{1\le a\neq b\le n} j_b j_a(\xi_b+\eta_b)(\xi_a+\eta_a)+2\sum_{1\le b\le n, j\in\mathcal{N}_*} j_b j(\xi_b+\eta_b)(z_j\bar{z}_j)\\&+\sum_{j,j'\in\mathcal{N}_*} jj'(z_j\bar{z}_j)(z_{j'}+\bar{z}_{j'})\Bigg)-\frac{f_0}{\pi}\Bigg(\sum_{1\le b\le n} j_b^2(\xi_b+\eta_b)^2+\sum_{j\in\mathcal{N}_*} j^2(z_j\bar{z}_j)^2\Bigg)\\&-\frac{3(f(\theta)-f_0)}{\pi}\Bigg(2\sum_{\substack{j,j'>\Gamma,\\ j\neq j'}} jj'(z_j\bar{z}_j)(z_{j'}\bar{z}_{j'})+2\sum_{\substack{1\le b\le n,\\ j>\Gamma}} j_b j(\xi_b+\eta_b)(z_j\bar{z}_j)\\&+2\sum_{\substack{j>\Gamma, j'<\Gamma,\\ j\neq j_1,\cdots,j_n}} jj'(z_j\bar{z}_j)(z_{j'}\bar{z}_{j'})\Bigg)-\frac{f(\theta)-f_0}{\pi}\sum_{j>\Gamma} j^2(z_j\bar{z}_j)^2.\end{aligned}$$

Thus the new Hamiltonian, still denoted by H, up to a constant depending on ξ, is derived in the form

$$H=N+P=\langle\omega,I\rangle+\sum_{1\le b\le n}\omega_b\eta_b+\sum_{j\in\mathcal{N}_*}\Omega_j(z_j\bar{z}_j)+\bar{Q}+\bar{P}$$

with symplectic structure

$$\sum_{1\le l\le m} dI_l\wedge d\theta+\sum_{1\le b\le n} d\eta_b\wedge d\varphi_b+\sum_{j\in\mathcal{N}_*} dz_j\wedge d\bar{z}_j,$$

where

$$\omega_b=j_b^3-\frac{6f_0}{\pi}\sum_{\substack{1\le a\le n,\\ a\neq b}} j_a j_b\xi_a-\frac{2f_0}{\pi}j_b^2\xi_b, \tag{14}$$

$$\Omega_j=\begin{cases} j^3-\frac{6f_0}{\pi}\sum_{1\le b\le n} j_b j\xi_b, & j\le\Gamma,\\ j^3-\frac{6f(\theta)}{\pi}\sum_{1\le b\le n} j_b j\xi_b, & j>\Gamma,\end{cases} \tag{15}$$

$$\begin{aligned}\bar{Q}=&\frac{3f_0}{\pi}\Bigg(\sum_{1\le a\neq b\le n} j_b j_a\eta_b\eta_a+2\sum_{\substack{1\le b\le n,\\ j\in\mathcal{N}_*}} j_b j\eta_b(z_j\bar{z}_j)+\sum_{j,j'\in\mathcal{N}_*} jj'(z_j\bar{z}_j)(z_{j'}+\bar{z}_{j'})\Bigg)\\&-\frac{f_0}{\pi}\Bigg(\sum_{1\le b\le n} j_b^2(\eta_b)^2+\sum_{j\in\mathcal{N}_*} j^2(z_j\bar{z}_j)^2\Bigg)\\&-\frac{3(f(\theta)-f_0)}{\pi}\cdot\Bigg(2\sum_{\substack{j,j'>\Gamma,\\ j\neq j'}} jj'(z_j\bar{z}_j)(z_{j'}\bar{z}_{j'})+2\sum_{\substack{1\le b\le n,\\ j>\Gamma}} j_b j(\eta_b)(z_j\bar{z}_j)\\&+2\sum_{\substack{j>\Gamma, j'<\Gamma,\\ j\neq j_1,\cdots,j_n}} jj'(z_j\bar{z}_j)(z_{j'}\bar{z}_{j'})\Bigg)+\frac{f(\theta)-f_0}{\pi}\sum_{j>\Gamma} j^2(z_j\bar{z}_j)^2.\end{aligned}$$

To apply the KAM Theorem in Appendix, we introduce a new parameter ς. For $r > 0$ and any $\omega_- \in \widetilde{\Pi}_1$ fixed and $\omega \in \breve{\Pi}_1 := \{\omega \in \widetilde{\Pi}_1 || \omega - \omega_-| < r^{\frac{7}{6}}\}$, we introduce new parameter ς as following

$$\omega = \omega_- + \varsigma, \varsigma \in \Pi_1 := \{\varsigma \in \mathbb{R}_+^m || \varsigma| < r^{\frac{7}{6}}\}.$$

Let $\varpi(\zeta) = \omega(\varsigma) \oplus \tilde{\omega}(\xi)$ with $\zeta = \varsigma \oplus \xi$, $x = \theta \oplus \varphi, y = I \oplus \eta$. We suppose that the parameter domain is

$$\zeta \in \Pi = \Pi_1 \times \Pi_2 = \{\varsigma \in \mathbb{R}_+^m : |\varsigma| < r^{\frac{7}{6}}\} \times \{\xi \in \mathbb{R}_+^n : |\xi| < r^{\frac{7}{6}}\}.$$

Then the Hamiltonian can be written as

$$H = \langle \varpi, y \rangle + \sum_{j \in \mathcal{N}_*} \Omega_j z_j \bar{z}_j + P(x, y, z, \bar{z})$$

with

$$P = \bar{Q}(y, z, \bar{z}) + \bar{P}(x, y, z, \bar{z}).$$

Now consider the phase space domain

$$D(s, r) : |\Im x| < s, |y| < r^2, \|z\|_p + \|\bar{z}\|_p < r, \tag{16}$$

where the definition of norm $\|z\|_p$ can refer to the Appendix. We will adopt lots of notations and definitions from [14], such as the phase space, weighted norm for the Hamilton vector field, etc.. More definitions are presented in Appendix.

Next we will check the assumption A, B and C of the KAM Theorem 4.1 in Appendix.

Regarding Ω as an infinite dimensional column vector with its index $j \in \mathcal{N}_*$, from (15), we know

$$\Omega_j = \breve{\Omega}_j + \tilde{\Omega}_j(\xi),$$

where $\breve{\Omega}_j = j^3$ is independent of ξ. Furthermore, basing on (15), we get

$$|\Omega_j|_{\Pi}^{\text{lip}} \leq \begin{cases} \frac{6f_0}{\pi} \sum_{1 \leq b \leq n} j_b j, & j \leq \Gamma, \\ \frac{6f(\theta)}{\pi} \sum_{1 \leq b \leq n} j_b j, & j > \Gamma, \end{cases}$$

Accordingly, we find

$$|\Omega|_{-1,\Pi}^{\text{lip}} = \sup_{j \in \mathcal{N}_*} j^{-1} |\Omega_j|_{\Pi}^{\text{lip}} \leq \max\left\{\frac{6|f_0|}{\pi} n j_b, \frac{6C_0}{\pi} n j_b\right\} := M_1.$$

That is, assumption A is fulfilled with $d = 3,\ \delta = 1$.

In the following we will check the assumption B.

In view of (14), we know that $\xi \mapsto \tilde{\omega}$ is an affine transformation from Π_2 to $\mathbb{R}^n$. In view of $f_0 \neq 0$, we note that

$$\tilde{\omega} = \breve{\omega} + \frac{2f_0}{\pi} A\xi,$$

where

$$\breve{\omega} = \begin{pmatrix} j_1^3 \\ j_2^3 \\ \vdots \\ j_n^3 \end{pmatrix}, A = \begin{pmatrix} j_1^2 & 3j_1j_2 & \cdots & 3j_1j_n \\ 3j_1j_2 & j_2^2 & \cdots & 3j_2j_n \\ \cdots & & \cdots & \\ 3j_1j_n & 3j_2j_n & \cdots & j_n^2 \end{pmatrix}, \xi = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{pmatrix}.$$

Then we get that $\det A = (-2)^n(3n-2)j_1^2 j_2^2 \cdots j_n^2 \neq 0$. Moreover, by the definition of ϖ, we obtain that

$$\frac{\partial \varpi}{\partial \zeta} = \begin{pmatrix} \frac{\partial \omega}{\partial \varsigma} & 0 \\ \frac{\partial \tilde{\omega}}{\partial \varsigma} & \frac{\partial \tilde{\omega}}{\partial \xi} \end{pmatrix} = \begin{pmatrix} Id_m & 0 \\ \frac{\partial \tilde{\omega}}{\partial \varsigma} & \frac{2f_0}{\pi} A \end{pmatrix}, \zeta \in \Pi,$$

where $\mathrm{I}d_m$ denotes the unit $m \times m-$matrix. Therefore, the real map $\zeta \mapsto \varpi(\zeta)$ is a lipeomorphism between Π and its image. This means that the first part of assumption B is fulfilled with positive M_2 and L only depend on the function $f(\theta)$ and the set $\mathcal{J}$.

Letting

$$\Omega = \check{\Omega} + \frac{2f_0}{\pi} B\xi,$$

where $\check{\Omega}$ is an infinite dimensional column vector and its $j-$th element $\check{\Omega}_j = j^3$, B is a $-\infty \times n$ matrix with its $j-$row

$$B_j = \begin{cases} 3j(j_1, j_2, \cdots, j_n), & j \leq \Gamma, \\ \frac{3f(\theta)}{f_0} j(j_1, j_2, \cdots, j_n), & j > \Gamma, \end{cases}$$

and regarding k and l as $n+m-$dimensional and infinite dimensional row vector respectively, we have to check for every $k \in \mathbb{Z}^{n+m}$ and $l \in \mathbb{Z}^\infty$ with $1 \leq |l| \leq 2$,

$$\text{meas}\{\zeta \in \Pi : \langle k, \varpi(\zeta)\rangle + \langle l, \Omega(\xi)\rangle = 0\} = 0. \tag{17}$$

Letting

$$k = (k_1, k_2), k_1 \in \mathbb{Z}^m, k_2 \in \mathbb{Z}^n,$$

and

$$\mathfrak{g}(\zeta) = \langle k, \varpi(\zeta)\rangle + \langle l, \Omega(\xi)\rangle = \langle k_1, \omega(\varsigma)\rangle + \langle k_2, \tilde{\omega}(\xi)\rangle + \langle l, \Omega(\xi)\rangle,$$

$$\Delta := \{\zeta \in \Pi : \mathfrak{g}(\zeta) = 0\}.$$

To prove that $\mathrm{meas}\,\Delta = 0$, we divide the proof into two cases.

Case 1 : Assume $k_1 \neq 0$, then

$$\frac{\partial \mathfrak{g}(\zeta)}{\partial \varsigma} = \langle k_1, \frac{\partial \omega}{\partial \varsigma}\rangle = \langle k_1, \mathrm{I}d_m\rangle = k_1 \neq 0,$$

which means that $\mathrm{meas}\,\Delta = 0$.

Case 2 : Assume $k_1 = 0$, then

$$\mathfrak{g}(\zeta) = \langle k_2, \tilde{\omega}(\xi)\rangle + \langle l, \Omega(\xi)\rangle = \langle k_2, \breve{\omega}\rangle + \langle l, \check{\Omega}\rangle + \langle k_2, \frac{2f_0}{\pi}A\xi\rangle + \langle l, \frac{2f_0}{\pi}B\xi\rangle.$$

For the nondegeneracy condition (17) we have to check that

$$\langle k_2, \breve{\omega}\rangle + \langle l, \check{\Omega}\rangle \neq 0 \ \text{ or } \ k_2 A + lB \neq 0. \tag{18}$$

Letting

$$\widetilde{B} = BA^{-1} = (\widetilde{B}_j)_{j\in\mathcal{N}_*}, \tag{19}$$

where

$$A^{-1} = \frac{1}{6n-4}\begin{pmatrix} \frac{5-3n}{j_1^2} & \frac{3}{j_1 j_2} & \cdots & \frac{3}{j_1 j_n} \\ \frac{3}{j_1 j_2} & \frac{5-3n}{j_2^2} & \cdots & \frac{3}{j_2 j_n} \\ \cdots & & \cdots & \\ \frac{3}{j_1 j_n} & \frac{3}{j_2 j_n} & \cdots & \frac{5-3n}{j_n^2} \end{pmatrix}$$

is obtained by using a series of elementary transformation. Simple calculation results in

$$\widetilde{B}_j = \begin{cases} \frac{3j}{3n-2}(\frac{1}{j_1}, \frac{1}{j_2}, \cdots, \frac{1}{j_n}), & j \leq \Gamma, \\ \frac{3f(\theta)j}{(3n-2)f_0}(\frac{1}{j_1}, \frac{1}{j_2}, \cdots, \frac{1}{j_n}), & j > \Gamma. \end{cases}$$

Then $k_2 A + lB \neq 0$ holds true for k_2 and l except the following ten cases:

(1) $l_j = \begin{cases} \pm 1, & j \leq \Gamma,\ j = \frac{h(3n-2)}{3}[j_1, \cdots, j_n], \\ 0, & \text{otherwise}, \end{cases}$

and $k_2 = \mp h[j_1, j_2, \cdots, j_n](\frac{1}{j_1}, \frac{1}{j_2}, \cdots, \frac{1}{j_n})$ with $h \in \mathbb{N}$;

(2) $l_j = \begin{cases} \pm 2, & j \leq \Gamma,\ j = \frac{h(3n-2)}{6}[j_1, \cdots, j_n], \\ 0, & \text{otherwise}, \end{cases}$

and $k_2 = \mp h[j_1, j_2, \cdots, j_n](\frac{1}{j_1}, \frac{1}{j_2}, \cdots, \frac{1}{j_n})$ with $h \in \mathbb{N}$;

(3) $l_j = \begin{cases} \pm 1, & j = j', j'' \leq \Gamma, \;\; 3(j' + j'') = h(3n-2)[j_1, \cdots, j_n], \\ 0, & \text{otherwise}, \end{cases}$
and $k_2 = \mp \frac{3(j'+j'')}{3n-2}(\frac{1}{j_1}, \frac{1}{j_2}, \cdots, \frac{1}{j_n})$ with $h \in \mathbb{N}$;

(4) $l_j = \begin{cases} \pm 1, & j = j' \leq \Gamma, \\ \mp 1, & j = j'' < j' \leq \Gamma, \\ & \text{and } 3(j' - j'') = h(3n-2)[j_1, \cdots, j_n], \\ 0, & \text{otherwise}, \end{cases}$
and $k_2 = \mp \frac{3(j'-j'')}{3n-2}(\frac{1}{j_1}, \frac{1}{j_2}, \cdots, \frac{1}{j_n})$ with $h \in \mathbb{N}$;

(5) $l_j = \begin{cases} \pm 1, & j = j' > \Gamma, \;\; \frac{3f(\theta)}{f_0} j' = h(3n-2)[j_1, \cdots, j_n], \\ 0, & \text{otherwise}, \end{cases}$
and $k_2 = \mp \frac{3f(\theta)j'}{f_0(3n-2)}(\frac{1}{j_1}, \frac{1}{j_2}, \cdots, \frac{1}{j_n})$ with $h \in \mathbb{N}$;

(6) $l_j = \begin{cases} \pm 2, & j = j' > \Gamma, \;\; \frac{6f(\theta)}{f_0} j' = h(3n-2)[j_1, \cdots, j_n], \\ 0, & \text{otherwise}, \end{cases}$
and $k_2 = \mp \frac{6f(\theta)j'}{f_0(3n-2)}(\frac{1}{j_1}, \frac{1}{j_2}, \cdots, \frac{1}{j_n})$ with $h \in \mathbb{N}$;

(7) $l_j = \begin{cases} \pm 1, & j = j', j'' > \Gamma, \\ & \text{and } \frac{3f(\theta)}{f_0}(j' + j'') = h(3n-2)[j_1, \cdots, j_n], \\ 0, & \text{otherwise}, \end{cases}$
and $k_2 = \mp \frac{3f(\theta)(j'+j'')}{f_0(3n-2)}(\frac{1}{j_1}, \frac{1}{j_2}, \cdots, \frac{1}{j_n})$ with $h \in \mathbb{N}$;

(8) $l_j = \begin{cases} \pm 1, & j = j' > \Gamma, \\ \mp 1, & j = j'' > \Gamma, j' > j'', \\ & \text{and } \frac{3f(\theta)}{f_0}(j' - j'') = h(3n-2)[j_1, \cdots, j_n], \\ 0, & \text{otherwise}, \end{cases}$
and $k_2 = \mp \frac{3f(\theta)(j'-j'')}{f_0(3n-2)}(\frac{1}{j_1}, \frac{1}{j_2}, \cdots, \frac{1}{j_n})$ with $h \in \mathbb{N}$;

(9) $l_j = \begin{cases} \pm 1, & j = j' > \Gamma, \\ \pm 1, & j = j'' \leq \Gamma, \\ & \text{and } 3(\frac{f(\theta)}{f_0} j' + j'') = h(3n-2)[j_1, \cdots, j_n], \\ 0, & \text{otherwise}, \end{cases}$
and $k_2 = \mp \frac{3(\frac{f(\theta)}{f_0} j' + j'')}{3n-2}(\frac{1}{j_1}, \frac{1}{j_2}, \cdots, \frac{1}{j_n})$ with $h \in \mathbb{N}$;

(10) $$l_j = \begin{cases} \pm 1, & j = j' > \Gamma, \\ \pm 1, & j = j'' \leq \Gamma, \\ & \text{and } 3(\frac{f(\theta)}{f_0}j' - j'') = h(3n-2)[j_1, \cdots, j_n], \\ 0, & \text{otherwise}, \end{cases}$$

and $k_2 = \mp \frac{3(\frac{f(\theta)}{f_0}j' - j'')}{3n-2}(\frac{1}{j_1}, \frac{1}{j_2}, \cdots, \frac{1}{j_n})$ with $h \in \mathbb{N}$.

Now we check that $\langle k_2, \breve{\omega}\rangle + \langle l, \check{\Omega}\rangle \neq 0$ holds true for (1), (3), (4), (5) and (9) under the proper condition, the rest cases can be obtained applying the same method.

Suppose **(1)** holds true. We know that

$$\begin{aligned} \langle k_2, \breve{\omega}\rangle + \langle l, \check{\Omega}\rangle &= \mp h[j_1, j_2, \cdots, j_n](j_1^2 + \cdots + j_n^2) \pm \left(\frac{h(3n-2)}{3}[j_1, \cdots, j_n]\right)^3 \\ &= \mp h[j_1, j_2, \cdots, j_n]\left(j_1^2 + \cdots + j_n^2 - h^2(\frac{3n-2}{3})^3[j_1, \cdots, j_n]^2\right). \end{aligned}$$

Since $[j_1, \cdots, j_n] \neq j_n$, we get that $[j_1, \cdots, j_n] \geq 2j_n$. Then

$$j_1^2 + \cdots + j_n^2 < nj_n^2 \leq \frac{n}{4}[j_1, \cdots, j_n]^2 < h^2(\frac{3n-2}{3})^3[j_1, \cdots, j_n]^2,$$

so $\langle k_2, \breve{\omega}\rangle + \langle l, \check{\Omega}\rangle \neq 0$.

Suppose **(3)** holds true. In this case,

$$\langle k_2, \breve{\omega}\rangle + \langle l, \check{\Omega}\rangle = \mp(j' + j'')\left(\frac{3}{3n-2}(j_1^2 + \cdots + j_n^2) - \frac{j'^3 + j''^3}{j' + j''}\right),$$

when $n \geq 2$,

$$\sqrt{\frac{3}{3n-2}(j_1^2 + \cdots + j_n^2)} < \sqrt{\frac{3nj_n^2}{3n-2}} < \frac{4}{3}j_n \leq \frac{4}{6}[j_1, \cdots, j_n] \leq \frac{3n-2}{6}[j_1, \cdots, j_n].$$

Moreover, $j'^3 + j''^3 > 2(\frac{j'+j''}{2})^3$, then

$$\sqrt{\frac{3}{3n-2}(j_1^2 + \cdots + j_n^2)} < \frac{3n-2}{6}[j_1, \cdots, j_n] \leq \frac{j' + j''}{2} < \sqrt{\frac{j'^3 + j''^3}{j' + j''}}.$$

So $\langle k_2, \breve{\omega}\rangle + \langle l, \check{\Omega}\rangle \neq 0$.

Suppose **(4)** holds true. In view of

$$\langle k_2, \breve{\omega}\rangle + \langle l, \check{\Omega}\rangle = \mp(j' - j'')\left(\frac{3}{3n-2}(j_1^2 + \cdots + j_n^2) - \frac{j'^3 - j''^3}{j' - j''}\right),$$

according to $j'^3 - j''^3 > (j'-j'')^3$, we have

$$\sqrt{\frac{3}{3n-2}(j_1^2+\cdots+j_n^2)} < \frac{4}{3}j_n < \frac{3n-2}{3}[j_1,\cdots,j_n] < j'-j'' < \sqrt{\frac{j'^3-j''^3}{j'-j''}}.$$

So $\langle k_2,\breve{\omega}\rangle + \langle l,\check{\Omega}\rangle \neq 0$.

Suppose **(5)** holds true. From

$$\langle k_2,\breve{\omega}\rangle + \langle l,\check{\Omega}\rangle = \mp j'\left(\frac{3f(\theta)}{f_0(3n-2)}(j_1^2+\cdots+j_n^2) - j'^2\right),$$

we know

$$\sqrt{\frac{3f(\theta)}{f_0(3n-2)}(j_1^2+\cdots+j_n^2)} < \sqrt{\frac{3nf(\theta)}{f_0(3n-2)}}j_n \le \sqrt{\frac{3}{2}}\sqrt{\frac{f(\theta)}{f_0}}\cdot\frac{1}{2}[j_1,\cdots,j_n]$$
$$\le \frac{4}{\frac{3f(\theta)}{f_0}}[j_1,\cdots,j_n] \le \frac{3n-2}{\frac{3f(\theta)}{f_0}}[j_1,\cdots,j_n] \le j',$$

for $\frac{f(\theta)}{f_0} \le \frac{4\sqrt[3]{2}}{3}$. Hence, $\langle k_2,\breve{\omega}\rangle + \langle l,\check{\Omega}\rangle \neq 0$.

Suppose **(9)** holds true. It's easy to see that $3(\frac{f(\theta)}{f_0}j'+j'')$ is between $3(j'+j'')$ and $\frac{3f(\theta)}{f_0}(j'+j'')$, Let $(\frac{f(\theta)}{f_0}j'+j'') = m(j'+j'')$, then m is between 1 and $\frac{f(\theta)}{f_0}$. In what follows,

$$\langle k_2,\breve{\omega}\rangle + \langle l,\check{\Omega}\rangle = \mp(j'+j'')(\frac{3m}{3n-2}(j_1^2+\cdots+j_n^2) - \frac{j'^3+j''^3}{j'+j''}),$$

when $m^3 \le \frac{32}{27}$, we can obtain

$$\sqrt{\frac{3m}{3n-2}(j_1^2+\cdots+j_n^2)} < \sqrt{\frac{3}{2}}\sqrt{m}j_n \le \sqrt{\frac{3}{2}}\cdot\frac{1}{2}\sqrt{m}[j_1,\cdots,j_n] < \frac{4}{6m}[j_1,\cdots,j_n]$$
$$\le \frac{3n-2}{6m}[j_1,\cdots,j_n] \le \frac{j'+j''}{2} < \sqrt{\frac{j'^3+j''^3}{j'+j''}}.$$

Therefore $\langle k_2,\breve{\omega}\rangle + \langle l,\check{\Omega}\rangle \neq 0$ holds true for $f(\theta)\in\mathbb{R}$ which satisfies

$$0 < \frac{f(\theta)}{f_0} \le \frac{32}{27}.$$

To sum up, we can check that $\langle k_2,\breve{\omega}\rangle + \langle l,\check{\Omega}\rangle \neq 0$ holds true easily for all of the above ten cases when the function $f(\theta)\in\mathbb{R}$ satisfies that

$$0 < \frac{f(\theta)}{f_0} \le \frac{32}{27}. \tag{20}$$

This means that the second part of the assumption B is satisfied.

Remark 1 : There exists function $f(\theta)$ such that the condition (20) is satisfied. For example, $f(\theta) = \frac{1}{6}\sin\theta_1 \sin\theta_2 \cdots \sin\theta_m + 1$.

It remains to check assumption C. We use the notation $i_\zeta X_P$ for X_P evaluated at ζ, and likewise in analogous cases. Consider the perturbation P on $D(s,r)\times\Pi$, where $D(s,r)$ is defined in (16) by taking $p = N + \frac{1}{2}$. For each ζ, the vector field $i_\zeta X_P$, considered as a map from a subset of $\mathscr{S}_p^{m+n}$ to $\mathscr{S}_{p-1}^{m+n}$, is of the order $p-(p-1)=1$, which strictly smaller than $d-1=2$. Moreover, it is easy to see that $i_\zeta X_P$ is real analytic on $D(s,r)$ for each $\zeta\in\Pi$, and $i_w X_P$ is uniformly Lipschitz on Π for each $w\in D(s,r)$. Namely, the assumption C is satisfied.

Next we consider sup norm and Lipschitz semi-norm of the perturbation P on $D(s,r)\times\Pi$. Obviously we have

$$\|X_{\bar{Q}}\|_{r,p-1,D(s,r)\times\Pi} = O(r^2). \tag{21}$$

Moreover, $\bar{P}$ is at least six order of q, we get

$$\|X_{\bar{P}}\|_{r,p-1,D(s,r)\times\Pi} = O((r^{\frac{7}{12}})^6\cdot r^{-2}) = O(r^{\frac{3}{2}}). \tag{22}$$

From (21), (22), we have

$$\|X_P\|_{r,p-1,D(s,r)\times\Pi} = O(r^{\frac{3}{2}}).$$

As X_P is real analytic in ζ, we have

$$\|X_P\|^{\text{lip}}_{r,p-1,D(s,r)\times\Pi} = O(r^{\frac{3}{2}}\cdot r^{-\frac{7}{6}}) = O(r^{\frac{1}{3}}).$$

Choosing

$$\alpha = r^{\frac{4}{3}}\gamma^{-1},$$

where γ is taken from the KAM Theorem 4.1 and setting $M := M_1 + M_2$, which only depends on the function $f(\theta)$ and the set $\mathcal{J}$, we obtain

$$\epsilon := \|X_P\|_{r,p-1,D(s,r)\times\Pi} + \frac{\alpha}{M}\|X_P\|^{\text{lip}}_{r,p-1,D(s,r)\times\Pi} = O(r^{\frac{3}{2}}) \le \alpha\gamma.$$

When r is small enough, which is just the smallness condition (27) in KAM Theorem 4.1. Applying Theorem 4.1 in Appendix, the conclusion of Theorem 1.1 is obtained.

4. Appendix: The KAM Theorem

Consider a small perturbation $H = N + P$ of an infinite dimensional Hamiltonian in the parameter dependent normal form

$$N = \sum_{1\le j\le m} \omega_j(\xi) y_j + \sum_{j\in\mathbb{N}_*} \Omega_j z_j \bar{z}_j \tag{23}$$

on a phase space

$$\mathscr{S}_p^m = \mathbb{T}^m \times \mathbb{R}^m \times \ell_p^2 \times \ell_p^2 \ni (x, y, z, \bar{z}),$$

with symplectic structure

$$\sum_{1\le j\le m} \mathrm{d}x_j \wedge \mathrm{d}y_j + \sum_{j\ge 1} \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_j,$$

where

$$\ell_p^2 = \{x \in \ell^2(\mathbb{N}, \mathbb{R}) : \|x\|^2 = \sum_{j\ge 1} |q_j|^2 j^{2p} < \infty\}.$$

where $p \ge 0$. The tangential frequencies $\omega = (\omega_1, \omega_2, \cdots, \omega_m)$ and normal frequencies $\Omega = (\Omega_1, \Omega_2, \cdots)$ are real analytic in the space coordinates and Lipschitz in the parameters, and for each $\xi \in \Pi$ its Hamiltonian vector field $X_P = (P_y, -P_x, P_{\bar{z}}, -P_z)^T$defines near $T_0 := \mathbb{T}^m \times \{y = 0\} \times \{z = 0\} \times \{\bar{z} = 0\}$ a real analytic map

$$X_P : \mathscr{S}_p^m \to \mathscr{S}_q^m,$$

where

$$p - q = \tilde{d}.$$

We use the notation $i_\xi X_P$ for X_P evaluated at ξ, and likewise in analogous cases.

To give the KAM theorem we need to introduce some domains and norms. For $s, r > 0$, we introduce the complex T_0-neighbourhoods

$$D(s, r) = \{|\Im x| < s\} \times \{|y| < r^2\} \times \{\|z\|_p + \|\bar{z}\|_p < r\} \tag{24}$$

$$\subset \mathbb{C}^m \times \mathbb{C}^m \times \ell_{p,\mathbb{C}}^2 \times \ell_{p,\mathbb{C}}^2 = \mathscr{S}_{p,\mathbb{C}}^m. \tag{25}$$

and weighted norm for $W = (X, Y, Z, \bar{Z}) \in \mathscr{S}_{q,\mathbb{C}}^m$

$$\|W\|_{r,q} = |X| + \frac{|Y|}{r^2} + \frac{\|Z\|_q}{r} + \frac{\|\bar{Z}\|_q}{r}$$

where $|\cdot|$ denotes the sup-norm for complex vectors. Furthermore, for a map $W : U \times \Pi \to \mathscr{S}_{q,\mathbb{C}}^m$, such as the Hamiltonian vector field X_P, we define the norms

$$\|W\|_{r,q;U\times\Pi}^{\sup} = \sup_{(w,\xi)\in U\times\Pi} \|W(w, \xi)\|_{r,q},$$

$$\|W\|^{\text{lip}}_{r,q;U\times\Pi} = \sup_{\xi,\zeta\in\Pi,\xi\neq\zeta} \frac{\|\Delta_{\xi\zeta}W\|^{\text{sup}}_{r,q;U}}{|\xi-\zeta|},$$

where $\Delta_{\xi\zeta}W = i_\xi W - i_\zeta W$, and

$$\|i_\xi W\|^{\text{sup}}_{r,q;U} = \sup_{w\in U} \|W(w,\xi)\|_{r,q}.$$

In a completely analogous manner, the Lipschitz semi-norm of the frequencies ω is defined as

$$|\omega|^{\text{lip}}_{\Pi} = \sup_{\xi,\zeta\in\Pi,\xi\neq\zeta} \frac{\|\Delta_{\xi\zeta}\omega\|}{|\xi-\zeta|},$$

and the Lipschitz semi-norm of $\tilde{\Omega} : \Pi \to \ell^\infty_{-\delta}$ is defined as

$$|\tilde{\Omega}|^{\text{lip}}_{-\delta,\Pi} = \sup_{\xi,\zeta\in\Pi,\xi\neq\zeta} \frac{\|\Delta_{\xi\zeta}\tilde{\Omega}\|_{-\delta}}{|\xi-\zeta|}$$

for any real number δ. Note that $|\tilde{\Omega}|^{\text{lip}}_{-\delta,\Pi} = |\Omega|^{\text{lip}}_{-\delta,\Pi}$, since $\bar{\Omega} = \Omega - \tilde{\Omega}$ is independent of ξ.

Suppose the normal form N described above satisfies the following assumptions:

Assumption A: Frequency Asymptotics. There exist two real numbers $d > 1$ and $\delta < d-1$ such that the following holds. First, the frequencies Ω_n are real valued functions of ξ of the form

$$\Omega_n(\xi) = \bar{\Omega}_n + \tilde{\Omega}_n(\xi),$$

where $\bar{\Omega}_n$ is independent of ξ and of the form $\tilde{\Omega}_n = cn^d + \cdots$, where the dots stand for an expansion in lower order terms in n. Second, the functions

$$\xi \mapsto \frac{\tilde{\Omega}_n(\xi)}{n^\delta}, \quad n \geq 1$$

are uniformly Lipschitz on Π, or equivalently, the map

$$\tilde{\Omega} : \Pi \to \ell^\infty_{-\delta}, \;\; \xi \mapsto \tilde{\Omega}(\xi) = (\tilde{\Omega}_n(\xi))_{n\geq 1}$$

is Lipschitz on Π.

Assumption B: Nondegeneracy. The map $\xi \to \omega(\xi)$ between Π and its image is a homeomorphism which is Lipschitz continuous in both directions. Moreover, for every $k \in \mathbb{Z}^m$ and $l \in \mathbb{Z}^\infty$ with $1 \leq |l| \leq 2$ (here $|l| = \sum_{j\geq 1} |l_j|$), the resonance set

$$\Re_{kl} = \{\xi \in \Pi : \langle k, \omega(\xi)\rangle + \langle l, \Omega(\xi)\rangle = 0\} \tag{26}$$

has Lebesgue measure zero.

Assumption C: Regularity. There is a neighbourhood U of T_0 in $\mathscr{S}^m_{p,\mathbb{C}}$ such that P is defined on $U \times \Pi$, and its Hamiltonian vector field defines a map

$$X_P : U \times \Pi \to \mathscr{S}^m_{q,\mathbb{C}},$$

where q satisfies

$$p - q < d - 1.$$

Moreover, $i_\xi X_P$ is real analytic on U for each $\xi \in \Pi$, and $i_w X_P$ is uniformly Lipschitz on Π for each $w \in U$.

We introduce one more constant. By assumption A and B,

$$|\omega|^{\text{lip}}_\Pi + |\Omega|^{\text{lip}}_\Pi \le M < \infty.$$

Finally observe that if X_P satisfies assumption C, then it does so with the T_0-neighbourhoods $D(s, r)$ for all $s > 0, r > 0$ sufficiently small.

Under the above condition, we have the following KAM theorem.

***Theorem* 4.1** — *Suppose N is a family of Hamiltonians of the form (23) defined on a phase space $\mathscr{S}^m_p$ and depending on parameters in Π so that assumption A and B are satisfied. Then there exists a positive constant γ depending only on $m, d, \delta,$ the frequencies ω and Ω and the real number $s > 0$ such that for every perturbed Hamiltonian $H = N + P$ that satisfies assumption C and the smallness condition*

$$\epsilon = \|X_P\|^{\text{sup}}_{r,q,D(s,r)\times\Pi} + \frac{\alpha}{M}\|X_P\|^{\text{lip}}_{r,q,D(s,r)\times\Pi} \le \alpha\gamma \tag{27}$$

for some $r > 0$ and $0 < \alpha < 1$, the following holds. There exist

(i) a Cantor set $\Pi_\alpha \subset \Pi$ with $meas(\Pi \setminus \Pi_\alpha) \to 0$ $(\alpha \to 0)$,

(ii) a Lipschitz family of real analytic torus embeddings $\Phi : \mathbb{T}^m \times \Pi_\alpha \to \mathscr{S}^m_p$,

(iii) a Lipschitz map $\phi : \Pi_\alpha \to \mathbb{R}^m$,

such that for each $\xi \in \Pi_\alpha$, the map Φ restricted to $\mathbb{T}^m \times \{\xi\}$ is a real analytic embedding of a rotational frequencies $\phi(\xi)$ for the perturbed Hamiltonian H at ξ. In other words,

$$t \mapsto \Phi(\theta + t\phi(\xi), \xi), t \in \mathbb{R}$$

is a real analytic, quasi-periodic solution for the Hamiltonian $i_\xi H$ for every $\theta \in \mathbb{T}^m$ and $\xi \in \Pi_\alpha$.

Moreover, each embedding is real analytic on $D(s/2) = \{|\Im x| < s/2\}$, and

$$\|\Phi - \Phi_0\|^{\text{sup}}_{r,p,D(s/2)\times\Pi_\alpha} + \frac{\alpha}{M}\|\Phi - \Phi_0\|^{\text{lip}}_{r,p,D(s/2)\times\Pi_\alpha} \le \frac{c\varepsilon}{\alpha},$$

$$|\phi - \omega|_{\Pi_\alpha} + \frac{\alpha}{M}|\phi - \omega|^{\mathrm{lip}}_{\Pi_\alpha} \leq c\varepsilon,$$

where

$$\Phi_0 : \mathbb{T}^m \times \Pi \to T_0, \quad (x, \xi) \mapsto (x, 0, 0, 0)$$

is the trivial embedding for each ξ*, and* c *is a positive constant which depends on the same parameters as* γ*.*

PROOF : The proof can be found in [18]. □

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