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DOMINATION DEFECT IN GRAPHS: GUARDING WITH FEWER GUARDS

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In this paper, we introduce a new graph parameter called the domination defect of a graph. The domination number γ of a graph G is the minimum number of vertices required to dominate the vertices of G. Due to the minimality of γ, if a set of vertices of G has cardinality less than γ then there are vertices of G that are not dominated by that set. The k-domination defect of G is the minimum number of vertices which are left un-dominated by a subset of $\gamma - k$ vertices of G. We study different bounds on the k-domination defect of a graph G with respect to the domination number, order, degree sequence, graph homomorphisms and the existence of efficient dominating sets. We also characterize the graphs whose domination defect is 1 and find exact values of the domination defect for some particular classes of graphs.

Key words : Minimum dominating set; efficient dominating set; vertex removal; changing domination; domination defect.

1. INTRODUCTION

Let G be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. We will denote the *order* of G as $|V|$ or $|V(G)|$ depending on the context. The *open neighborhood* of a vertex $v \in V$ is $N(v) = \{u \in V \mid uv \in E\}$, and the *closed neighbourhood* of v is $N[v] = \{v\} \cup N(v)$. The degree of a vertex v is $|N(v)|$. Let $\delta(G)$ and $\Delta(G)$ denote the minimum and maximum degrees of the vertices of G, respectively. For any $S \subseteq V$, we denote the subgraph of G induced by S as $\langle S \rangle$. The *open*

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neighbourhood of a set $S \subseteq V$ is the set $N(S) = \cup_{v \in S} N(v)$, and the *closed neighbourhood of a set* S is the set $N[S] = N(S) \cup S = \cup_{v \in S} N[v]$.

A set S of vertices of G is a *dominating set* of G if every vertex in $V \setminus S$ is adjacent to at least one vertex in S, that is, $N[S] = V$. The *domination number* $\gamma(G)$ is the minimum cardinality of any dominating set of G. A minimum cardinality dominating set of G will be referred to as a $\gamma(G)$-set. If the graph G is clear from the context, we will denote $\gamma(G)$ as γ and $V(G)$ as V.

The study of dominating sets, domination number and other domination parameters of a graph e.g., [1, 2, 5] forms an integral part of both theoretical as well as practical aspects of graph theory. The origins of the study of domination theory begin with the Five Queens problem from chess. Given a standard 8×8 chess board, the question was asked "How many queens would be needed so that every square can either be attacked by or occupied by a queen?". The minimum number was determined to be five. The study of domination theory was further motivated by the practical application of determining the optimum assignment of guards to guard an installation.

Consider the following hypothetical example. Imagine that you are the curator of an art museum and you wish to determine the minimum number of guards you need to guard the exhibits. Clearly a guard can guard an exhibit that he/she is standing near, and any exhibit in the museum that they can clearly see. In order to model the security situation in your facility, you would construct a graph G in the following manner. Each vertex represents an exhibit location and two vertices u and v are adjacent if and only if the locations they represent are visible from each other, that is, a person standing at the exhibit modeled by vertex u can clearly see the location of the exhibit modeled by vertex v, and vice versa. Suppose that security requirements mandate that a staff of guards are positioned at locations such that every art exhibit is protected by a guard that can see it, and budget restrictions make it desirable to hire as few guards as possible. In this case, the most economical solutions, that is, the minimum guards for possible guard location configurations, correspond to the γ-sets. Suppose that due to budgetary concerns, as curator, you are strictly limited to hiring exactly γ guards.

While this is the optimum solution economically, in a practical sense it leaves much to be desired. There will be days when guards are ill, guards need a day off or some of them institute a labor action and go on strike. As curator you need to know how vulnerable your facility will be if you have to run with fewer than the minimum number of guards necessary.

It is with this problem in mind, that we introduce in this paper the concept of the *domination defect* of a graph. Given a graph G and an integer $1 \leq k < \gamma$, the k-*domination defect* of G is the minimum number of vertices of G left un-dominated by any subset of vertices with cardinality

$\gamma - k$. As a museum curator, knowing the k-domination defect of the graph representing your security situation would immediately tell you how many exhibits will be left unguarded if k workers do not show up for work on a particular day. In other words, the k-domination defect of a graph G is the measure of the maximum surveillance possible by $\gamma - k$ guards.

As a side note, it is interesting to point out that the introduction of many of the different variants of domination parameters were motivated by the problems associated with guarding facilities or placing monitoring devices in networks. The reader interested in studying the related domination parameters and their applications is urged to consult [3, 4].

1.1 *Notation and Definitions*

Let $G = (V, E)$ be a graph with n vertices and domination number $\gamma > 1$. For two vertices a and b in V, we write $a \sim b$ or $a \not\sim b$ accordingly as a and b are *adjacent* or *non-adjacent* in G. A graph G with domination number γ is said to be γ-*vertex critical* or γ-*critical* if for any $v \in V$, $\gamma(G - v) < \gamma(G)$. A subset $S \subset V$ is said to be a *clique dominating set* if the subgraph induced by S, $\langle S \rangle$ is a clique and S dominates G. The minimum size γ_{cl} of a clique dominating set, if it exists, is called the *clique domination number* of the graph. The boundary of a set $S \subset V$ is defined by $B(S) = (V - S) \cap N(S)$, i.e., the vertices in $V - S$ which are dominated by S. Authors in [6] defined the differential of a set S, $\partial(S) = |B(S)| - |S|$ and the differential of a graph G as $\partial(G) = \max\{\partial(S) : S \subset V\}$. For definitions and other undefined graph theoretic concepts used, readers are referred to [7].

1.1.1 *k-Domination Defect* : Let k be a positive integer less than γ. Let $S \subset V$ with $|S| = \gamma - k$ where $1 \leq k < \gamma$. We define the k-defect of S as

$$\zeta_k(S) = |V \setminus N[S]| = n - |N[S]|.$$

Among all sets $S \subseteq V$ such that $|S| = \gamma - k$, we define the *k-domination defect* of G, denoted $\zeta_k(G)$, as the minimum cardinality of $V \setminus N[S]$. Specifically among all subsets of V with cardinality $\gamma - k$, $\zeta_k = \min_{S \subseteq V} \{\zeta_k(S)\}$. A set of cardinality $\gamma - k$ for which $|V(G) \setminus N[S]| = \zeta_k(G)$, i.e., $n - |N[S]| = \zeta_k$ is denoted as a ζ_k-*set* of G. Therefore, $\langle N[S] \rangle$ is an induced subgraph of G with $n - \zeta_k$ vertices and domination number $\gamma - k$. By the minimality of γ, it follows that, $\zeta_k \geq k$ and $1 \leq \zeta_1 < \zeta_2 < \cdots < \zeta_{\gamma-1}$. Also note that if $|S| = 1$, then choosing S to be a single vertex of maximum degree shows that $\zeta_{\gamma-1} = n - \Delta - 1 = \delta(\overline{G})$. Thus, we have

$$1 \leq \zeta_1 < \zeta_2 < \cdots < \zeta_{\gamma-1} = \delta(\overline{G}).$$

Moreover, two consecutive ζ_i's differ at least by 1 and hence, $\zeta_k(G) \leq \delta(\overline{G}) - \{\gamma(G) - k - 1\} = \delta(\overline{G}) + k + 1 - \gamma(G)$, for every $k \in \{1, \ldots, \gamma(G) - 1\}$.

Remark 1.1 : Note that if $\gamma(G) = 1$ for any graph G of order n, then for $k = 1$, any set of cardinality $\gamma - k$ is empty and as such $\zeta_k(G) = n$. Thus to avoid triviality, for any graph G in this work we shall assume that $\gamma(G) \geq 2$. It is also obvious that if G has k isolated vertices, then $\zeta_j(G) = j$, for every $j \in \{1, 2, \ldots, k\}$ and $\zeta_i(G) = k + \zeta_{i-k}(G')$, for $k < i \leq \gamma - 1$ where G' is the graph G with the isolated vertices deleted. So without loss of generality, we study the graphs with no isolated vertices.

1.2 *The k-Domination Defect of Some Known Families of Graphs*

In this section, we find ζ_k for some well-known families of graphs.

***Theorem* 1.1** — *If P_n is a path with n vertices, then* $\zeta_k(P_n) = \begin{cases} 3k-2, & \text{if } n = 3t+1 \\ 3k-1, & \text{if } n = 3t+2 \\ 3k, & \text{if } n = 3t \end{cases}$

PROOF : We denote the vertices of P_n as $\{1, 2, \ldots, n\}$ and we use the known fact that $\gamma(P_n) = \lceil \frac{n}{3} \rceil$.

Case 1 : $[n = 3t + 1]$. In this case, $\gamma = t + 1$ and we choose a $(t - k + 1)$-element set $S = \{2, 5, \ldots, 3t - 3k + 2\}$. Observe that $\zeta_k(S) = n - 3(t - k + 1) = 3k - 2$ and hence it is the minimum value for any set S with $t - k + 1$ vertices. Thus, $\zeta_k = 3k - 2$.

Case 2 : $[n = 3t + 2]$. In this case also, $\gamma = t + 1$ and we choose the same set S as in Case-1. However, in this case $\zeta_k(S) = n - 3(t - k + 1) = 3k - 1$ and this is the minimum value of ζ_k that can be achieved by any set with $t - k + 1$ vertices and hence $\zeta_k = 3k - 1$.

Case 3 : $[n = 3t]$. In this case, $\gamma = t$ and we choose a $(t - k)$-element set $S = \{2, 5, \ldots, 3t - 3k - 1\}$. Now $\zeta_k(S) = n - 3(t - k) = 3k$ and hence it is the minimum value of ζ_k that can be achieved by any set with $t - k$ vertices and hence $\zeta_k = 3k$. □

***Theorem* 1.2** — *If C_n is a cycle with n vertices, then* $\zeta_k(C_n) = \begin{cases} 3k-2, & \text{if } n = 3t+1 \\ 3k-1, & \text{if } n = 3t+2 \\ 3k, & \text{if } n = 3t \end{cases}$

PROOF : The proof follows along the same lines as that of Theorem 1.1 using the known fact that $\gamma(C_n) = \lceil \frac{n}{3} \rceil$. □

***Theorem* 1.3** — *For the Petersen graph P, $\zeta_1(P) = 3$ and $\zeta_2(P) = 6$.*

PROOF : We begin by noting that $\gamma(P) = 3$ and for any two non-adjacent vertices $u, v \in V(P)$, $|N[u] \cap N[v]| = 1$ and $|N[u] \cup N[v]| = 7$. In like manner, for any two adjacent vertices $u, v \in V(P)$, $|N[u] \cap N[v]| = 2$ and $|N[u] \cup N[v]| = 6$. Since $|V| = 10$, it follows that $\zeta_1(P) = 3$. Moreover,

$\zeta_2(P) = |V| - (\Delta + 1) = 10 - 4 = 6.$ □

From the above families of graphs, it may seem that dropping carefully chosen vertices from a γ-set of G will yield a ζ_k-set of G. But this is not true in general, as discussed in Example 1.1 and Remark 1.2.

Example 1.1 : In Figure 1 (left), $\gamma = 2$ and $D = \{1, 4\}$ is a minimum dominating set. Now deleting any one of the vertices of D will result in two un-dominated vertices. It is to be noted that $\zeta_1 = 1$ and $S = \{5\}$ is a ζ_1-set. Moreover, in this case, there exists a minimum dominating set $D' = \{2, 5\}$ containing S. However, this may not be the case always. (See Remark 1.2.)

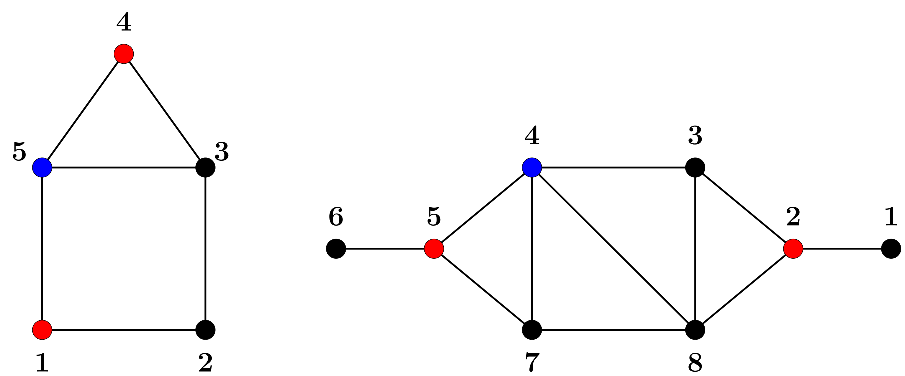


Figure 1 : Examples of 1-Domination Defect

Remark 1.2 : It may happen that none of ζ_k-sets of a graph G are contained in any γ-set of G. In Figure 1 (right), $\gamma = 2$, $\zeta_1 = 3$, $\{2, 5\}$ is the unique γ-set of G and $\{4\}$ is a ζ_1-set of G that is not contained in $\{2, 5\}$. The other ζ_1-set $\{8\}$ is also not contained in $\{2, 5\}$.

1.3 *Relationship between the Domination Defect and the Differential of a Graph*

Let $1 \leq k \leq \gamma - 1$. Analogous to the definition of the differential of a graph seen in [6], we define the $(\gamma - k)$ differential of a graph G of order n as

$$\partial_{\gamma-k}(G) = \max\{\partial(S) : S \subset V; |S| = \gamma - k\}.$$

It is to be noted that $\partial_{\gamma-k}(G) \leq \partial(G)$. Also note that $N[S] = B(S) \cup S$ and $B(S) \cap S = \emptyset$, i.e., $N[S]$ is the disjoint union of $B(S)$ and S. Thus for any $S \subset V(G)$ with $|S| = \gamma - k$,

$$\zeta_k(S) = n - |N[S]| = n - [|B(S)| + |S|] = n - [\partial(S) + |S| + |S|]$$

$$\text{i.e., } \zeta_k(S) = n - \partial(S) - 2\gamma + 2k.$$

Also $\zeta_k = \zeta_k(G) = \min\{\zeta_k(S) : S \subset V; |S| = \gamma - k\}$. Thus to minimize $\zeta_k(S)$, we need to maximize $\partial(S)$ over S with $|S| = \gamma - k$, i.e.,

$$\zeta_k(G) = n - \partial_{\gamma-k}(G) - 2\gamma + 2k.$$

2. Bounds in Terms of Order and Size

We start with some basic results and bounds on the k-domination defect.

***Theorem* 2.1** — *If G is a graph of order n, then $\zeta_k \leq k(1+\Delta)$.*

PROOF : Let S be a γ-set of G. Since any vertex in S can dominate at most $1+\Delta$ vertices, dropping k vertices from S can leave at most $k(1+\Delta)$ vertices un-dominated. Thus, we have $\zeta_k \leq k(1+\Delta)$. □

***Theorem* 2.2** — *If G is a graph of order n, then $\zeta_k \leq (n+k) - (\gamma+\Delta)$.*

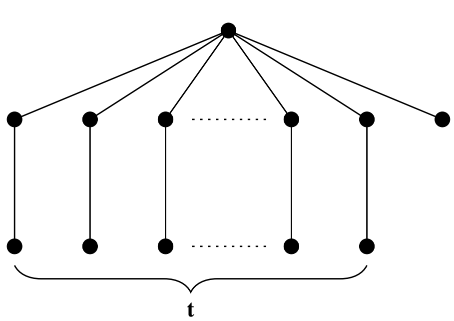
PROOF : Let v be a vertex of maximum degree and $S \subseteq V(G) \setminus N[v]$ be any set of $\gamma - k - 1$ vertices. Since S dominates at least $|S|$ vertices of $V(G) \setminus N[v]$, it follows that $S \cup \{v\}$ dominates at least $(\Delta + 1 + \gamma - k - 1)$ vertices of G. As such, $\zeta_k \leq (n+k) - (\Delta+\gamma)$. □

A *wounded spider* is a graph formed from a star $K_{1,s-1}$ by subdividing at most $s-2$ edges exactly once. This gives us the following corollary.

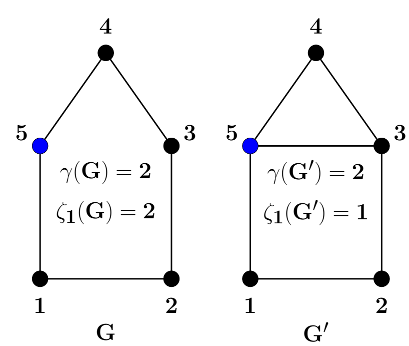
Corollary 2.1 — If G is a wounded spider, then $\zeta_k = k$.

PROOF : By the minimality of γ, $\zeta_k \geq k$. Let G be an n-vertex wounded spider with maximum degree Δ. Since either every leaf or its support vertex must be contained in every minimum dominating set, it follows that $\gamma = n - \Delta$. By Theorem 2.2, we have $\zeta_k \leq (n+k) - (\gamma+\Delta)$. Thus, $\zeta_k \leq k$ and the result follows. □

For an example of a wounded spider, consider the graph in Figure 2a. In this case, $n = 2t+2$, $\gamma = \Delta = t+1$ and $\zeta_k = k$.



(a) Wounded Spider



(b) Adding Edge

Figure 2 : k-Domination Defect

Corollary 2.2 — If G is a graph with size m, then $\zeta_k \leq m + k - \Delta$.

PROOF : Let G be an n-vertex graph with domination number γ. By Theorem 2.2, we have $\zeta_k \leq (n+k) - (\gamma + \Delta)$. Also we know that $\gamma \geq n - m$. (See [3], pg. 55). Combining these we get, $\zeta_k \leq m + k - \Delta$. □

In the next three theorems, we investigate the effect of removal of a vertex and addition of edges on the k-domination defect of a graph G.

***Theorem* 2.3** — *Let G be a graph on n vertices and G' be a graph obtained by deleting a vertex v in G such that $\gamma(G) > \gamma(G') \geq 2$. Then $\zeta_{k+1}(G) \leq \zeta_k(G')+1$, for all k satisfying $1 \leq k \leq \gamma(G)-2$.*

PROOF : Since deleting a vertex can reduce the domination number at most by 1, we have $\gamma(G) = \gamma(G') + 1$. Let $1 \leq k \leq \gamma(G') - 1$ and S be a $\zeta_k(G')$-set. Therefore, $|S| = \gamma(G') - k$ and $\zeta_k(G') = (n-1) - |N_{G'}[S]|$, i.e., $|N_{G'}[S]| = (n-1) - \zeta_k(G')$. On the other hand, since $|S| = \gamma(G') - k = \gamma(G) - (k+1)$, we have $\zeta_{k+1}(G) \leq n - |N_G[S]|$.

Now two cases may arise: $|N_G[S]| = |N_{G'}[S]|$ or $|N_G[S]| = 1 + |N_{G'}[S]|$. If $|N_G[S]| = |N_{G'}[S]|$, we have $(n-1) - \zeta_k(G') = |N_{G'}[S]| = |N_G[S]| \leq n - \zeta_{k+1}(G)$, i.e., $(n-1) - \zeta_k(G') \leq n - \zeta_{k+1}(G)$. Thus,

$$\zeta_{k+1}(G) \leq \zeta_k(G') + 1. \tag{1}$$

If $|N_G[S]| = 1+|N_{G'}[S]|$, we have $(n-1)-\zeta_k(G') = |N_{G'}[S]| = |N_G[S]|-1 \leq n-\zeta_{k+1}(G)-1$, i.e., $(n-1) - \zeta_k(G') \leq n - \zeta_{k+1}(G) - 1$. Thus,

$$\zeta_{k+1}(G) \leq \zeta_k(G'). \tag{2}$$

Hence, in either case, equations 1 and 2 give the desired result. □

***Theorem* 2.4** — *Let G be a graph and G' be a graph obtained by adding any number of edges to $E(G)$, such that $\gamma(G) = \gamma(G')$. Then $\zeta_k(G') \leq \zeta_k(G)$, for all k satisfying $1 \leq k \leq \gamma(G) - 1$.*

PROOF : Let $1 \leq k \leq \gamma(G) - 1$ and S be a ζ_k-set of G. Therefore, $|S| = \gamma(G) - k$ and $\zeta_k(G) = n - |N_G[S]|$. Since $|S|$ is also equal to $\gamma(G') - k$ and $N_G[S] \subset N_{G'}[S]$, we have $\zeta_k(G') \leq n - |N_{G'}[S]| \leq n - |N_G[S]| = \zeta_k(G)$. □

In Figure 2b, $n = 5$, $\gamma(G) = \gamma(G') = 2$, $S = \{5\}$ and $\zeta_1(G) = 2, \zeta_1(G') = 1$.

***Theorem* 2.5** — *Let G be a graph and G' be a graph obtained by adding an edge e to $E(G)$ such that $\gamma(G) > \gamma(G')$. Then $\zeta_{k-1}(G') \leq \zeta_k(G)$.*

PROOF : Since G' is obtained by adding an edge e to $E(G)$ such that $\gamma(G) > \gamma(G')$, we have $\gamma(G') = \gamma(G) - 1$. Let S be a ζ_k-set of G. Therefore, $|S| = \gamma(G) - k = \gamma(G') - (k-1)$ and $\zeta_k(G) = n - |N_G[S]|$. Since $N_G[S] \subset N_{G'}[S]$, we have $\zeta_{k-1}(G') \leq n - |N_{G'}[S]| \leq n - |N_G[S]| = \zeta_k(G)$. □

For our next result, we obtain a bound of the k-domination defect number in terms of the order and the domination number.

***Theorem* 2.6** — *For any graph G of order n such that $\gamma(G) \geq 2$ and for all k such that $1 \leq k \leq \gamma(G) - 1$, $\zeta_k(G) \leq k\frac{n}{\gamma(G)}$.*

PROOF : Let S be any $\gamma(G)$-set, $r = \lceil \frac{\gamma(G)}{k} \rceil$ and $\pi = \{X_1, \ldots, X_r\}$ be a partition of S such that for $1 \leq i \leq r$, $|X_i| \leq k$. By the definition of the k-domination defect, for all $1 \leq i \leq r$, there exists a set $Y_i \subseteq \{V(G) \setminus S\} \cup X_i$ such that $|Y_i| \geq \zeta_k(G)$ and no vertex of X_j for $i \neq j$ is adjacent to a vertex of Y_i. Given the way each set Y_i is defined, it follows that $Y_i \cap Y_j = \emptyset$ if $i \neq j$. This implies that the order of G must be at least as large as $\Sigma_{i=1}^r |Y_i|$. Hence $n \geq \Sigma_{i=1}^r |Y_i| \geq \Sigma_{i=1}^r \zeta_k(G) = \lceil \frac{\gamma(G)}{k} \rceil \zeta_k(G)$. Thus $\zeta_k(G) \leq \frac{n}{\lceil \frac{\gamma(G)}{k} \rceil} \leq k\frac{n}{\gamma(G)}$. □

Remark 2.1 : The bound in the above theorem is sharp, as can be seen by the following examples.

- Consider G as the disjoint union of t copies of K_2, where $t \geq 2$. In this case, $n = 2t$, $\gamma(G) = t$ and $\zeta_k(G) = 2k$ for $1 \leq k \leq \gamma(G) - 1$.

- Consider the n-vertex path P_n where $n = 3t$ for $t \geq 2$. In this case, $\gamma(G) = t$ and $\zeta_k(G) = 3k$ for $1 \leq k \leq \gamma(G) - 1$.

***Theorem* 2.7** — *If G is an r-regular graph, then $\zeta_k = k(r+1)$ if and only if G has an efficient dominating set.*

Let G be an r-regular graph of order n and D be an efficient dominating set of G. Since $\{N[u] : u \in D\}$ partitions V, we have $n = \gamma(r+1)$. Let S be obtained by deleting any k vertices from D. Then $\zeta_k(S) = n - |N[S]| = k(r+1)$ and hence $\zeta_k(G) \leq k(r+1)$. On the other hand, since any vertex can dominate exactly $r+1$ vertices, $\zeta_k(G) = k(r+1)$.

Conversely, let G be an r-regular graph and assume $\zeta_k = k(r+1)$. Let D be a γ-set of G and let $S \subseteq D$ such that $|S| = \gamma - k$. It follows that the k-domination defect of S, i.e., $\zeta_k(S)$ is at most $k(r+1)$. Hence $\zeta_k(S) \leq k(r+1)$. However, since $\zeta_k(G) = k(r+1)$, we have $\zeta_k(S) = k(r+1)$. Since D is a dominating set of G, the k vertices in $D \setminus S$ dominate $k(r+1)$ vertices of G and hence each vertex in $D \setminus S$ privately dominates exactly $r+1$ vertices of G. Finally, since S is an arbitrary set of $\gamma - k$ vertices of D, every vertex in D privately dominates exactly $r+1$ vertices of G and hence D is an efficient dominating set of G. □

3. Bounds in Terms of the Complement

In this section, we find bounds on the k-domination defect of a graph G in terms of various parameters and properties like the domination number, planarity and the diameter of the complement of G.

***Theorem* 3.1** — *For any graph G with $\gamma(G) \geq 3$, if $k \geq 2$ then $\zeta_k(G) \geq \gamma(\overline{G})$.*

Proof : Let S be any $\zeta_k(G)$-set. Let $A = V(G) \setminus N[S]$. If $A \subseteq N(v)$ for some vertex $v \in V(G) \setminus A$, then $S \cup \{v\}$ would be a dominating set of G with cardinality less than $\gamma(G)$, a contradiction. Thus A is not contained in the open neighborhood of any vertex $v \in V(G) \setminus A$. This implies that A dominates $\overline{G}$. Hence $\zeta_k(G) = |A| \geq \gamma(\overline{G})$. □

***Theorem* 3.2** — *If $\overline{G}$ is a planar graph of order n, then $\zeta_k(G) \leq 5$ for any $1 \leq k \leq \gamma(G) - 1$.*

Proof : It is known that since $\overline{G}$ is planar, $m(\overline{G}) \leq 3n - 6$. (See Theorem 6.1.23, pg. 241, [7]) This implies that $\overline{G}$ has some vertex v with degree at most five. Accordingly, $\zeta_k(G) \leq 5$. □

A minor result can be made about the domination defect number of a graph in terms of the diameter of its complement. For ease of notation, if G is a disconnected graph, we say that the diameter of G is ∞.

***Theorem* 3.3** — *For any graph G of order n such that $\gamma(G) \geq 2$, if $1 \leq k \leq \gamma(G) - 1$ then the following holds.*

$$\zeta_k(G) = \begin{cases} \delta(\overline{G}) & \text{if } diam(\overline{G}) \geq 3 \\ k & \text{if } diam(\overline{G}) = 1. \end{cases}$$

Proof : In the first case, suppose that $diam(\overline{G}) \geq 3$. Let u and v be any vertices of $\overline{G}$ that do not share a common neighbour. The set $\{u, v\}$ dominates G. Since $\gamma(G) = 2$, it follows that $k = 1$ is the only integer for which $\zeta_k(G)$ is defined. As such, any set of vertices of cardinality $\gamma(G) - 1$ has exactly one vertex and can dominate at most $\Delta(G) + 1$ vertices of G. Hence $\zeta_1(G) = n - \Delta(G) - 1 = \delta(\overline{G})$.

In the second case, suppose that $diam(\overline{G}) = 1$. Hence $\overline{G} = K_n$ and $G = \overline{K_n}$. Any dominating set of G must contain all of the vertices of G. Thus any set of vertices of cardinality $\gamma(G) - k$ will dominate all but k vertices of G and $\zeta_k(G) = k$. □

In the next two theorems, we prove some Nordhaus-Gaddum type relations for the k-domination defect of a graph and its complement.

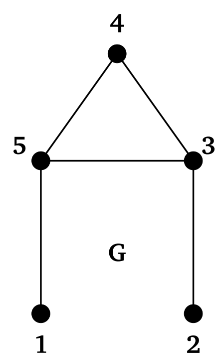
***Theorem* 3.4** — *Let G and $\overline{G}$ be graphs with $\gamma(G), \gamma(\overline{G}) > 1$. Then*

$$2k \leq \zeta_k(G) + \zeta_k(\overline{G}) \leq n - 1, \text{ where } 1 \leq k \leq \min\{\gamma(G), \gamma(\overline{G})\} - 1.$$

PROOF : Since $\zeta_k(G) \leq \delta(\overline{G})$ and $\zeta_k(\overline{G}) \leq \delta(G)$, and $\delta(G) + \delta(\overline{G}) \leq \delta(G) + \Delta(\overline{G}) = n - 1$, we have the upper bound. The lower bound follows from the fact that for any graph G, $\zeta_k(G) \geq k$.

For sharpness of the upper bound, consider a regular self-complementary graph G. Let $\gamma(G) = \gamma(\overline{G}) = \gamma$ and $k = \gamma - 1$, then $\zeta_k(G) = \delta(\overline{G}) = \Delta(\overline{G})$ and $\zeta_k(\overline{G}) = \delta(G)$. Thus the upper bound is sharp.

For sharpness of the lower bound, consider the graph G with two pendant vertices attached to two vertices of K_3 (See Figure 3). Here G is self-complementary and $\gamma(G) = \gamma(\overline{G}) = 2$ and $\zeta_1(G) = \zeta_1(\overline{G}) = 1$. Thus the lower bound is sharp. □



Sharp Lower Bound for
Nordhaus-Gaddum Inequality

Figure 3 : Sharpness of Lower Bound

The lower bound in the above inequality can be slightly improved if $\gamma(G), \gamma(\overline{G}) \geq 3$.

***Theorem* 3.5** — *Let G and $\overline{G}$ be graphs with $\gamma(G), \gamma(\overline{G}) \geq 3$. Then*

$$\zeta_k(G) + \zeta_k(\overline{G}) \geq 2k + 2, \text{ where } 2 \leq k \leq \min\{\gamma(G), \gamma(\overline{G})\} - 1.$$

PROOF : Since $\gamma(G) \geq 3$, for $k \geq 2$ we have, from Theorem 3.1, $\zeta_k(G) \geq \gamma(\overline{G})$. Similarly, $\zeta_k(\overline{G}) \geq \gamma(G)$. Adding them and using the fact that $\gamma(G), \gamma(\overline{G}) \geq k + 1$, we get $\zeta_k(G) + \zeta_k(\overline{G}) \geq \gamma(G) + \gamma(\overline{G}) \geq 2k + 2$. □

4. GRAPH HOMOMORPHISMS AND THE k-DOMINATION DEFECT

In this section, we explore the relationship between graph homomorphisms and the domination defect of graphs.

Lemma 4.1 — If $\varphi : G \to H$ is an onto graph homomorphism from a graph G to a graph H, then for any subset S of vertices in G, $|V(H)| - |N_H[\varphi(S)]| \leq |V(G)| - |N_G[S]|$.

PROOF :

$$h \in H - N_H[\varphi(S)] \Rightarrow h \in H \text{ and } h \notin \varphi(S) \text{ and } h \nsim \varphi(s), \forall s \in S$$

Choose and fix one pre-image g^* of h.

$$\Rightarrow \varphi(g^*) \in H \text{ and } \varphi(g^*) \notin \varphi(S) \text{ and } \varphi(g^*) \nsim \varphi(s), \forall s \in S$$

$$\Rightarrow g^* \in G \text{ and } g^* \notin S \text{ and } g^* \nsim s, \forall s \in S \Rightarrow g^* \in G - N_G[S]$$

Define a function $F : H - N_H[\varphi(S)] \longrightarrow G - N_G[S]$ by $F(h) = g^*$. As shown above, F is well-defined. Now, $F(h_1) = F(h_2) \Rightarrow g_1^* = g_2^* \Rightarrow \varphi(g_1^*) = \varphi(g_2^*) \Rightarrow h_1 = h_2$. Thus F is injective. Hence, $|V(H) - N_H[\varphi(S)]| \leq |V(G) - N_G[S]|$. Since $N_H[\varphi(S)] \subset H$ and $N_G[S] \subset G$, we have $|V(H)| - |N_H[\varphi(S)]| \leq |V(G)| - |N_G[S]|$. □

Using the fact that for any given onto homomorphism φ, $\gamma(G) \geq \gamma(H)$ and Lemma 4.1, we prove the following result.

***Theorem* 4.1** — *Let $\varphi : G \rightarrow H$ be an onto graph homomorphism from a graph G to a graph H. If $\gamma(H) = \gamma(G) - k$, then $\zeta_{k+1}(H) \leq \zeta_{k+1}(G)$.*

PROOF : Let S be a ζ_{k+1}-set in G, i.e., $|S| = \gamma - k - 1$ and $\zeta_{k+1}(G) = \zeta_{k+1}(S)$. Now $|\varphi(S)| \leq \gamma - k - 1$ and let $|\varphi(S)| = \gamma - k - t$ with $t \geq 1$. Using Lemma 4.1, we get

$$\zeta_{k+1}(G) = \zeta_{k+1}(S) = |V(G)| - |N_G[S]| \geq |V(H)| - |N_H[\varphi(S)]| = \zeta_{k+t}(\varphi(S)) \geq \zeta_{k+1}(H). \quad \square$$

5. Domination Defect Sequence

Recall we observed earlier that the domination defect sequence $\zeta_1, \zeta_2, \ldots, \zeta_{\gamma-1}$ is a strictly increasing one. We define a strictly increasing sequence $a_1 < a_2 < a_3 < \cdots < a_t$ to be *hyper-increasing* if $2a_1 \leq a_2$ and $a_{k+2} - a_{k+1} \geq a_{k+1} - a_k$, i.e., $2a_{k+1} \leq a_k + a_{k+2}$ for all $k \in \{1, 2, \ldots, t-2\}$.

***Theorem* 5.1** — *If $a_1 < a_2 < a_3 < \cdots < a_t$ is any finite hyper-increasing sequence of positive integers, then there exists a tree T with domination number $t+1$ such that $\zeta_i(G) = a_i$, for $i = 1, 2, \ldots, t$.*

PROOF : Let $a_1 < a_2 < a_3 < \cdots < a_t$ be a hyper-increasing sequence of positive integers. Set $b_1 = a_1$ and $b_k = a_k - a_{k-1}$ for all $2 \leq k \leq t$. Since the a_i's are hyper-increasing, $b_1 \leq b_2 \leq \cdots \leq b_t$, i.e., the b_i's are a non-decreasing sequence of positive integers. Now, choose a positive integer b strictly greater than b_t and consider the disjoint union of stars $K_{1,b_1}, K_{1,b_2}, \ldots, K_{1,b_t}$ and $K_{1,b}$.

Construct a tree T by joining the center vertex of each $K_{1,b_1}, K_{1,b_2}, \ldots, K_{1,b_t}$ to the center vertex of $K_{1,b}$ by an edge. (See Figure 4). It is clear that $\gamma(T) = t+1$ and $\zeta_i(G) = a_i$, for $i = 1, 2, \ldots, t$. (Rigorous proof is omitted.)

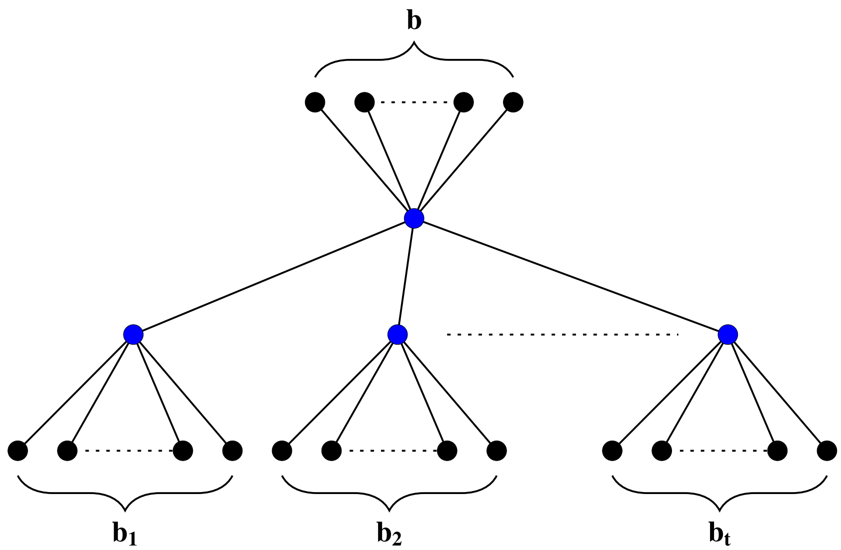


Figure 4 : Attaining Hyper-Increasing Sequence

6. 1-DOMINATION DEFECT

In this section, we focus on the case $k = 1$, i.e., the 1-domination defect or the simply domination defect and denote it as ζ_1 or simply ζ.

In the next theorem, we find the domination defect of a graph with respect to the domination defects of its components.

***Theorem* 6.1** — *Let G be a graph with t components $G_1, G_2, \ldots, G_t$ and let $\zeta(G_i)$ be the domination defect of G_i, for $i = 1, 2, \ldots, t$. Then*

$$\zeta(G) = \min_{1 \le i \le t} \{\zeta(G_i)\}.$$

PROOF : Let $\gamma(G_i)$ and $\gamma(G)$ be the domination number of G_i and G respectively. Then

$$\gamma(G) = \gamma(G_1) + \gamma(G_2) + \cdots + \gamma(G_t). \tag{3}$$

Moreover, if W is a subset of vertices in G with $W = W_1 \cup W_2 \cup \cdots \cup W_t$ where W_i is a subset of vertices in G_i, then $N_G[W]$ is the disjoint union of $N_{G_1}[W_1], N_{G_2}[W_2], \ldots, N_{G_t}[W_t]$. Since $\zeta(G_i)$ is the domination defect of G_i, for $i = 1, 2, \ldots, t$, there exist subsets $T_i \subseteq G_i$ such that $|T_i| = \gamma(G_i) - 1$

and $\zeta(G_i) = \zeta(T_i) = |V(G_i)| - |N_{G_i}[T_i]|$. Let $\zeta(G_j) = \min_{1 \leq i \leq t}\{\zeta(G_i)\}$. Let S_i be any $\gamma(G_i)$-set for $i = 1, 2, \ldots, t$ and

$$S = \left[\bigcup_{i=1, i\neq j}^{t} S_i\right] \cup T_j$$

Clearly $|S| = \gamma(G) - 1$ and

$$|N_G[S]| = |N_{G_j}[T_j]| + \sum_{i=1, i\neq j}^{t} |N_{G_i}[S_i]| = |V(G_j)| - \zeta(G_j) + \sum_{i=1, i\neq j}^{t} |V(G_i)| = \sum_{i=1}^{t} |V(G_i)| - \zeta(G_j).$$

Thus, in G, $\zeta(S) = |V(G)| - |N_G[S]| = \zeta(G_j)$.

Now, we prove that $\zeta(S)$ is the minimum among all subsets of $V(G)$ with cardinality $\gamma(G) - 1$ by showing that $\zeta(G) = \zeta(S) = \zeta(G_j)$. For purposes of contradiction, assume there exists $S' \subseteq V(G)$, such that $|S'| = \gamma(G) - 1$ and $\zeta(S') < \zeta(S)$. Let $S' = S'_1 \cup S'_2 \cup \cdots \cup S'_t$ where S'_i is a subset of vertices in G_i. Since $|S'| = \gamma(G) - 1$, from Equation 3 at least one S'_l is not a dominating set of G_l. Thus $\zeta(S'_l) \geq \zeta(G_l) \geq \zeta(G_j)$. Hence,

$$\zeta(S') = |V(G)| - |N_G[S']| = \sum_{i=1}^{t} \left[|V(G_i)| - |N_{G_i}[S'_i]|\right] \geq |V(G_l)| - |N_{G_l}[S'_l]| \geq \zeta(S'_l) \geq \zeta(G_j) = \zeta(S),$$

i.e., $\zeta(S') \geq \zeta(S)$, a contradiction. □

In the following theorem, we characterize the graphs G with $\zeta = 1$ using the concept of vertex removal and its effect on the domination number.

***Theorem* 6.2** — *If G is a graph of order n, then $\zeta = 1$ if and only if there exists a vertex $v \in V$ such that $\gamma(G - v) = \gamma - 1$.*

PROOF : If $\zeta = 1$, then there exists $S \subseteq V(G)$, such that $|S| = \gamma - 1$ and $|N[S]| = n - 1$. Let v be the sole vertex of G which is not in $N[S]$. Then $G - v = \langle N[S] \rangle$ has domination number $\gamma - 1$.

Conversely, let $v \in V$ such that $\gamma(G - v) = \gamma - 1$. Thus there exists $S \subseteq V(G)$ such that $|S| = \gamma - 1$ and $G - v = \langle N[S] \rangle$. Hence $\zeta(S) = n - (n - 1) = 1$ and $\zeta = 1$. □

Corollary 6.1 — $\zeta > 1$ if and only if $\gamma(G - v) \geq \gamma(G)$, for every $v \in V$.

PROOF : It immediately follows from Theorem 6.2. □

Corollary 6.2 — If G is γ-critical, then $\zeta = 1$.

PROOF : Since G is γ-critical, for any $v \in V$, $\gamma(G - v) = \gamma - 1$. Hence the corollary follows from Theorem 6.2. □

Now, we provide some sufficient conditions for the case of $\zeta = 2$ and in general for $\zeta = k$.

***Theorem* 6.3** — *If G is a graph such that $\gamma(G - v) = \gamma(G)$, for every $v \in V$ and there exist $u, v \in V$ such that $\gamma(G - \{u, v\}) < \gamma(G)$, then $\zeta = 2$.*

PROOF : Let $u, v \in V$ such that $\gamma(G - \{u, v\}) < \gamma(G)$. Then $\gamma(G - \{u, v\}) = \gamma - 1$, since $\gamma(G - v) = \gamma$, for every $v \in V$. Hence there exists $S \subseteq V(G)$ with $|S| = \gamma - 1$ and $G - \{u, v\} = \langle N[S] \rangle$. Thus, $\zeta(S) = n - (n - 2) = 2$. Since $\gamma(G - v) = \gamma$, for every $v \in V$, there does not exist $S \subseteq V(G)$ with $|S'| = \gamma - 1$ such that $\zeta(S') = 1$ and hence $\zeta = 2$. □

Remark 6.1 : The converse of this theorem is not true, i.e., $\zeta = 2$ does not necessarily imply that $\gamma(G - v) = \gamma(G)$, for every $v \in V$. This is illustrated in Figure 5. In Figure 5, G is a 6 vertex graph with $\gamma(G) = 2$ and $\zeta = 2$. Note that, $\gamma(G - \{4\}) = 3 > 2 = \gamma(G)$. However, $\zeta = 2$ implies that there exist $u, v \in V$ such that $\gamma(G - \{u, v\}) < \gamma(G)$. (Proof: Since $\zeta = 2$, there does not exist any subset $S' \subseteq V(G)$ with $|S'| = \gamma - 1$ and $|N[S']| = n - 1$. However, there exists $S \subseteq V(G)$ such that $|S| = \gamma - 1$ and $|N[S]| = n - 2$. Thus, $G - N[S]$ contains exactly two vertices, say u and v, such that $G - \{u, v\} = \langle N[S] \rangle$ and hence $\gamma(G - \{u, v\}) \leq \gamma - 1 < \gamma(G)$.)

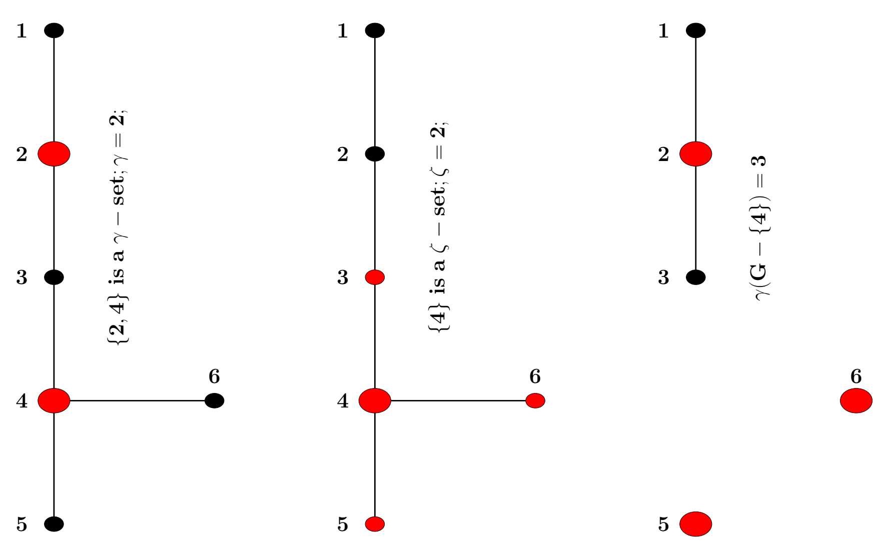


Figure 5 : A graph with $\zeta = 2$ and $\gamma(G - v) > \gamma(G)$

A similar result to that of Theorem 6.3 holds for $\zeta = k$.

***Theorem* 6.4** — *Let G be a graph such that $\gamma(G - T) = \gamma(G)$, for every $T \subset V$ with $|T| \leq k - 1$ and there exist $v_1, v_2, \ldots, v_k \in V$ such that $\gamma(G - \{v_1, v_2, \ldots, v_k\}) < \gamma(G)$. Then $\zeta = k$.*

PROOF : The proof follows exactly along the lines of the proof of Theorem 6.3. □

In the next two theorems, we study the relationship between ζ and the degree sequence of a graph.

***Theorem* 6.5** — *If $d_1 \geq d_2 \geq \cdots \geq d_n$ is the degree sequence of G with domination number γ, then $n-(d_1+d_2+\cdots+d_{\gamma-1})-\gamma+1 \leq \zeta \leq d_\gamma+1$.*

PROOF : Let S be a $\gamma(G)$-set. Consider the set $S\setminus\{v\}$, where v is a vertex with minimum degree in S. Then $deg(v) \leq d_\gamma$ and $|N[v]| \leq d_\gamma+1$. Thus at most $d_\gamma+1$ vertices of G will not be dominated by $S\setminus\{v\}$.

For the lower bound, recall that the defect of a set of vertices S, is defined as $\zeta(S)=n-|N[S]|$. Thus to minimize the defect, we need to maximize $|N[S]|$. Now,

$$\max_{|S|=\gamma-1}\{|N[S]|\} \leq (d_1+1)+(d_2+1)+\cdots+(d_{\gamma-1}+1)=(d_1+d_2+\cdots+d_{\gamma-1})+\gamma-1$$

Therefore, $\zeta \geq n-(d_1+d_2+\cdots+d_{\gamma-1})-\gamma+1$. □

Corollary 6.3 — If G is k-regular graph, then $n-(\gamma-1)(k+1) \leq \zeta \leq k+1$.

PROOF : It follows from above Theorem using $d_i=k$, for every i. □

Corollary 6.4 — If $diam(G)=2$, then $\zeta \geq n-(d_1+d_2+\cdots+d_{\gamma-1}+d_n)+1$.

PROOF : Since, $diam(G)=2$ implies $\gamma \leq \delta = d_n$, the corollary follows from Theorem 6.5. □

***Theorem* 6.6** — *If G is a graph such that the domination number of G is equal to its clique domination number γ_{cl}, then $\zeta \leq d_\gamma+1-\gamma$.*

PROOF : Let S be a γ_{cl}-set in G and v be a minimum degree vertex in S. Then $deg(v) \leq d_\gamma$. Now, out of the $|N[v]|=deg(v)+1$ vertices dominated by v, at least γ vertices (including v) are also dominated by other vertices in S. Therefore, at most $(deg(v)+1-\gamma)$ many vertices may remain un-dominated if we use $S\setminus\{v\}$. Thus, $\zeta \leq deg(v)+1-\gamma \leq d_\gamma+1-\gamma$. □

7. CONCLUSION

In this paper, we introduced a new graph invariant called the domination defect of a graph. From an applications standpoint, it can be interpreted as the measure of the maximum surveillance possible by $\gamma-k$ guards, where γ is the domination number of the graph representing the security situation in a facility. We studied different bounds on the domination defect of a graph G with respect to its order, degree sequence, domination number, graph homomorphisms and the existence of efficient dominating sets. We also characterized the graphs whose domination defect is 1 and found exact values of the domination defect for some particular classes of graphs.

We close with the following open problem. Recall from Theorem 5.1 that we specified that the tree we were constructing had a hyper-increasing sequence. We are strongly suspicious that this is true for all graphs. So in closing, we pose the following:

Problem 7.1 : For any graph G, $\zeta_1, \zeta_2, \ldots, \zeta_{\gamma-1}$ is a hyper increasing sequence.

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