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THE UPPER CONNECTED VERTEX DETOUR MONOPHONIC NUMBER OF A GRAPH

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For any vertex x in a connected graph G of order $n \geq 2$, a set $S_x \subseteq V(G)$ is an x-detour monophonic set of G if each vertex $v \in V(G)$ lies on an x-y detour monophonic path for some element y in S_x. The minimum cardinality of an x-detour monophonic set of G is the x-detour monophonic number of G, denoted by $dm_x(G)$. A connected x-detour monophonic set of G is an x-detour monophonic set S_x such that the subgraph induced by S_x is connected. The minimum cardinality of a connected x-detour monophonic set of G is the connected x-detour monophonic number of G, denoted by $cdm_x(G)$. A connected x-detour monophonic set S_x of G is called a minimal connected x-detour monophonic set if no proper subset of S_x is a connected x-detour monophonic set. The upper connected x-detour monophonic number of G, denoted by $cdm_x^+(G)$, is defined to be the maximum cardinality of a minimal connected x-detour monophonic set of G. We determine bounds and exact values of these parameters for some special classes of graphs. We also prove that for positive integers r, d and k with $2 \leq r \leq d$ and $k \geq 2$, there exists a connected graph G with monophonic radius r, monophonic diameter d and upper connected x-detour monophonic number k for some vertex x in G. Also, it is shown that for positive integers j, k, l and n with $2 \leq j \leq k \leq l \leq n-3$, there exists a connected graph G of order n with $dm_x(G) = j, dm_x^+(G) = k$ and $cdm_x^+(G) = l$ for some vertex x in G.

Key words : Detour monophonic path; vertex detour monophonic number; connected vertex detour monophonic number; upper connected vertex detour monophonic number.

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1. INTRODUCTION

By a *graph* $G = (V, E)$ we mean a finite undirected connected graph without loops or multiple edges. The *order* and *size* of G are denoted by n and m respectively. For basic graph theoretic terminology we refer to Harary [7]. For vertices x and y in a connected graph G, the *distance* $d(x, y)$ is the length of a shortest x-y path in G. An x-y path of length $d(x, y)$ is called an x-y *geodesic*. The *neighbourhood* of a vertex v is the set $N(v)$ consisting of all vertices u which are adjacent with v. The *closed neighbourhood* of a vertex v is the set $N[v] = N(v) \cup \{v\}$. A vertex v is an *extreme vertex* of G if the induced subgraph $\langle N[v] \rangle$ is complete.

The *closed interval* $I[x, y]$ consists of all vertices lying on some x-y geodesic of G, while for $S \subseteq V$, $I[S] = \bigcup_{x,y \in S} I[x, y]$. A set S of vertices is a *geodetic set* if $I[S] = V$, and the minimum cardinality of a geodetic set is the *geodetic number* $g(G)$. A geodetic set of cardinality $g(G)$ is called a g-*set* of G. The geodetic number of a graph was introduced in [1, 8] and further studied in [2, 4].

Chartrand *et al*. [5] introduced the concept of geodomination number of a graph. A pair of vertices x, y is said to geodominate a vertex v if either $v \in \{x, y\}$ or v lies on some x-y geodesic of G. A subset S of V is called a geodominating set of G if every vertex of G is geodominated by some pair of vertices in S. The cardinality of a minimum geodominating set in G is the geodomination number of G and its denoted by $g(G)$.

The concept of vertex geodomination number was introduced in [9] and further studied in [10]. Let x be a vertex of a connected graph G. A set S of vertices of G is an x-*geodominating set* of G if each vertex v of G lies on an x-y geodesic in G for some element y in S. The minimum cardinality of an x-geodominating set of G is defined as the x-*geodomination number* of G and is denoted by $g_x(G)$.

A *chord* of a path P is an edge joining any two non-adjacent vertices of P. A path P is called a *monophonic path* if it is a chordless path. A longest x-y monophonic path P is called an x-y *detour monophonic path*. For any two vertices u and v in a connected graph G, the *monophonic distance* $d_m(u, v)$ from u to v is defined as the length of a longest u-v monophonic path in G. The *monophonic eccentricity* $e_m(v)$ of a vertex v in G is $e_m(v) = max\{d_m(v, u) : u \in V(G)\}$. The *monophonic radius*, $rad_m(G)$ of G is $rad_m(G) = min\{e_m(v) : v \in V(G)\}$ and the *monophonic diameter*, $diam_m(G)$ of G is $diam_m(G) = max\{e_m(v) : v \in V(G)\}$. The monophonic distance was introduced in [11] and further studied in [12].

The detour distance $D(u, v)$ between two vertices u and v is defined to be the length of a longest u-v path. Several basic results on detour distance and related concepts such as detour eccentricity, detour radius, detour diameter, detour centre and detour periphery are given in [6] and [3].

The concept of vertex detour monophonic number was introduced in [13]. Let x be a vertex of a connected graph G. A set S of vertices of G is an *x-detour monophonic set* of G if each vertex v of G lies on an x-y detour monophonic path in G for some element y in S. The minimum cardinality of an x-detour monophonic set of G is defined as the *x-detour monophonic number* of G and is denoted by $dm_x(G)$. An x-detour monophonic set of cardinality $dm_x(G)$ is called a dm_x-set of G. Several results regarding the vertex detour monophonic number and interesting applications are given in [13]. The concept of upper vertex detour monophonic number was introduced in [14]. An x-detour monophonic set S_x is called a *minimal x-detour monophonic set* if no proper subset of S_x is an x-detour monophonic set. The *upper x-detour monophonic number*, denoted by $dm_x^+(G)$, is defined as the maximum cardinality of a minimal x-detour monophonic set of G. The concept of vertex detour set has interesting applications in Channel Assignment Problem in radio technologies. Also the detour matrix of a connected graph is used to discuss the applications of the detour index and hyper-detour index of a class of graphs, which in turn, capture different aspects of certain molecular graphs associated with molecules arising in special situations in Chemistry. Also there are applications of vertex detour monophonic sets to security based communication network design. This motivated us to introduce and investigate vertex detour monophonic sets in [13].

The concept of connected vertex detour monophonic number was introduced in [15]. Let x be a vertex of a connected graph G. A *connected x-detour monophonic set* of G is an x-detour monophonic set S_x such that the subgraph induced by S_x is connected. The minimum cardinality of a connected x-detour monophonic set of G is defined as the *connected x-detour monophonic number* of G and is denoted by $cdm_x(G)$. A connected x-detour monophonic set of cardinality $cdm_x(G)$ is called a cdm_x-set of G.

The following theorems will be used in the sequel.

***Theorem* 1.1** [13]. — *Let x be any vertex of a connected graph G.*

(i) Every extreme vertex of G other than the vertex x (whether x is extreme or not) belongs to every x-detour monophonic set.

(ii) No cutvertex of G belongs to any dm_x-set.

***Theorem* 1.2** [13]. — *For any non-trivial tree T with k endvertices, $dm_x(T) = k$ or $k-1$ according as x is a cutvertex or not.*

***Theorem* 1.3** [14]. — *For any non-trivial tree T with k endvertices, $dm_x^+(T) = k$ or $k-1$*

according as x is a cutvertex or not.

Throughout this paper G denotes a connected graph with at least two vertices.

2. MINIMAL CONNECTED VERTEX DETOUR MONOPHONIC SETS

Definition 2.1 — Let x be any vertex of a connected graph G. A connected x-detour monophonic set S_x is called a *minimal connected x-detour monophonic set* if no proper subset of S_x is a connected x-detour monophonic set. The *upper connected x-detour monophonic number* of G is the maximum cardinality of a minimal connected x-detour monophonic set of G and is denoted by $cdm_x^+(G)$.

Example 2.2 : For the graph G given in Figure 2.1, minimum connected x-detour monophonic sets, connected x-detour monophonic number, minimal connected x-detour monophonic sets and the upper connected x-detour monophonic number for all the vertices x of G are given in Table 2.1.

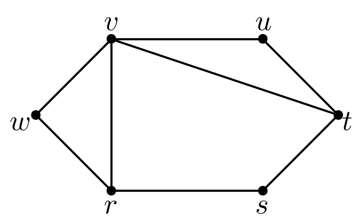


Figure 2.1

vertex x	cdm_x-set	$cdm_x(G)$	minimal connected x-detour monophonic sets	$cdm_x^+(G)$
r	$\{u, w, v\}$	3	$\{u, w, v\}, \{u, w, r, s, t\}$	5
s	$\{u, w, v\}$	3	$\{u, w, v\}, \{u, w, r, s, t\}$	5
t	$\{u, w, v\}$	3	$\{u, w, v\}, \{u, w, r, s, t\}$	5
u	$\{w, v\}$	2	$\{w, v\}, \{w, r, s\}$	3
v	$\{u, w, s, t, r\}$, $\{u, w, s, v, r\}$, $\{u, w, s, t, v\}$	5	$\{u, w, s, t, r\}$, $\{u, w, s, v, r\}$, $\{u, w, s, t, v\}$	5
w	$\{u, v\}$	2	$\{u, v\}, \{u, s, t\}$	3

Table 2.1

For any vertex x in a connected graph G, every minimum connected x-detour monophonic set is a minimal connected x-detour monophonic set, but the converse is not true. For the graph G given in Figure 2.1, $\{w, r, s\}$ is a minimal connected u-detour monophonic set but it is not a minimum connected u-detour monophonic set.

***Theorem* 2.3** — *Let x be any vertex of a connected graph G. If $y \neq x$ is an extreme vertex of G, then y belongs to every minimal connected x-detour monophonic set of G.*

PROOF : Clearly y is not an internal vertex of any detour monophonic path starting from x so that y belongs to every minimal connected x-detour monophonic set of G. □

Corollary 2.4 — For any vertex x in the complete graph K_n of order $n \geq 2$, $cdm_x^+(K_n) = n - 1$.

***Theorem* 2.5** — *For any vertex x in the cycle C_n of order $n \geq 4$, we have*

$$cdm_x^+(C_n) = \begin{cases} 1 & \text{if } n = 4 \\ 2 & \text{if } n > 4. \end{cases}$$

PROOF : Let $C_n = (u_1, u_2, \ldots, u_n, u_1)$ be a cycle of order $n \geq 4$ and let $x = u_1$. If $n = 4$, then $S_x = \{u_3\}$ is the unique minimal connected x-detour monophonic set of C_n and so $cdm_x^+(C_n) = 1$. If $n > 4$ and n is even, then $S_1 = \{u_{\frac{n}{2}+1}\}, S_2 = \{u_2, u_3\}$ and $S_3 = \{u_{n-1}, u_n\}$ are the minimal connected x-detour monophonic sets of C_n. If n is odd, then $S_1 = \{u_2, u_3\}, S_2 = \{u_{n-1}, u_n\}$ and $S_3 = \{u_{\frac{n+1}{2}}, u_{\frac{n+3}{2}}\}$ are the minimal connected x-detour monophonic sets of C_n. Hence $cdm_x^+(C_n) = 2$ if $n > 4$. □

Proposition 2.6 — If G is any connected graph of order n at least 4, then $cdm_x^+(G + K_1) = n$, where $\{x\} = V(K_1)$.

PROOF : No vertex of $G + K_1$ is an internal vertex of any detour monophonic path starting from x. Hence $V(G)$ is the minimal connected x-detour monophonic set of $G+x$ and so $cdm_x^+(G+K_1) = n$. □

***Theorem* 2.7** — *Let $W_n = K_1 + C_{n-1} (n \geq 5)$ be the wheel.*

(i) If $n = 5$, then $cdm_x^+(W_n) = 1$ for all x in C_{n-1}.

(ii) If $n > 5$, then $cdm_x^+(W_n) = 3$ for all x in C_{n-1}.

PROOF : Let $C_{n-1} = (u_1, u_2, \ldots, u_{n-1}, u_1)$ be a cycle of order $n - 1$ and let u be the vertex of K_1. Let x be any vertex in C_{n-1}, say $x = u_1$. If $n = 5$, then $S_x = \{u_3\}$ is the unique minimal connected x-detour monophonic set of G and so $cdm_x^+(W_n) = 1$. If $n > 5$ and n is odd, then $S_1 = \{u, u_{\frac{n-1}{2}+1}\}, S_2 = \{u, u_2, u_3\}$ and $S_3 = \{u, u_{n-2}, u_{n-1}\}$ are the minimal connected x-detour monophonic sets of W_n. If n is even, then $S_1 = \{u, u_2, u_3\}, S_2 = \{u, u_{n-2}, u_{n-1}\}$ and $S_3 =$

$\{u, u_{\frac{n}{2}}, u_{\frac{n+2}{2}}\}$ are the minimal connected x-detour monophonic sets of W_n. Hence $cdm_x^+(W_n) = 3$ if $n > 5$. □

***Theorem* 2.8** — *Let* $K_{r,s}$ $(2 \leq r \leq s)$ *be the complete bipartite graph with bipartition* (V_1, V_2). *Then*

(i) $cdm_x^+(K_{2,2}) = 1$ *for any vertex* x.

(ii) $$cdm_x^+(K_{2,s}) = \begin{cases} 1 & \text{if } x \in V_1 \\ s & \text{if } x \in V_2 \text{ and } s \geq 3. \end{cases}$$

(iii) $$cdm_x^+(K_{r,s}) = \begin{cases} r & \text{if } x \in V_1 \\ s & \text{if } x \in V_2 \text{ and } r, s \geq 3. \end{cases}$$

PROOF : (i) Since $K_{2,2}$ is isomorphic to C_4, (i) follows from Theorem 2.5.

(ii) Let $r = 2$ and $s \geq 3$. Let $V_1 = \{v_1, v_2\}$ and $V_2 = \{w_1, w_2, \ldots, w_s\}$ be the bipartition of $K_{2,s}$. Then any vertex w_i of V_2 lies on the v_1-v_2 detour monophonic path (v_1, w_i, v_2) and so $\{v_1\}$ is a minimal connected x-detour monophonic set of $K_{2,s}$ for $x = v_2$. Similarly $\{v_2\}$ is a minimal connected x-detour monophonic set of $K_{2,s}$ for $x = v_1$. Thus $cdm_x^+(K_{2,s}) = 1$.

Since every vertex of V_1 is adjacent to w_i, no vertex of V_2 is an internal vertex of any detour monophonic path starting from w_i. Thus every w_i-detour monophonic set of G contains $S = V_2 - \{w_i\}$. Also, any vertex v_j of V_1 lies on an w_i-u detour monophonic path (w_i, v_j, u), where $u \in S$. Hence S is an w_i-detour monophonic set of $K_{2,s}$. Since $s \geq 3$, the subgraph induced by S is not connected and hence $cdm_x^+(K_{2,s}) > s - 1$. Now, $\langle S \cup \{v_1\}\rangle$, is a connected w_i-detour monophonic set of maximum cardinality. Therefore, $cdm_x^+(K_{2,s}) = s$.

(iii) The proof is similar to the second part of the proof of (ii). □

***Theorem* 2.9** — *Let* T *be any tree of order* n.

(i) If x *is a cutvertex of* T, *then* $cdm_x^+(T) = n$.

(ii) *If* x *is a pendant vertex of* T, *then*

$$cdm_x^+(T) = \begin{cases} 1 & \text{if } T \text{ is a path} \\ n - d_m(x, y) & \text{where } y \in V(T) \text{ with } deg\ y \geq 3 \\ & \text{such that } d_m(x, y) \text{ is minimum.} \end{cases}$$

PROOF : (i) Let x be a cutvertex of T and let S_x be any minimal connected x-detour monophonic set of T. By Theorem 2.3, S_x contains all extreme vertices. If $S_x \neq V(T)$, then there exists a cutvertex v of T such that $v \notin S_x$. Let u and w be two endvertices belonging to different components of $T - \{v\}$. Since v lies on the unique u-w path, it follows that the subgraph induced by S_x is not connected, which is a contradiction. Hence $cdm_x^+(T) = n$.

(ii) Let T be a tree which is not a path and let x be an endvertex of T. Let y be the vertex of T with $deg\, y > 3$ such that $d_m(x, y)$ is minimum. Let P be the x-y path in T. Then $S_x = (V(T) - V(P)) \cup \{y\}$ is a connected x-detour monophonic set of T. We claim that S_x is a minimal connected x-detour monophonic set of T. Otherwise, there is a proper subset M_x of S_x such that M_x is a connected x-detour monophonic set of T. By Theorem 2.3, every connected x-detour monophonic set of T contains all extreme vertices except possibly x and hence there exists a cutvertex v of T such that $v \in S_x$ and $v \notin M_x$. Let $B_1, B_2, \ldots, B_m (m \geq 3)$ be the components of $T - \{y\}$. Assume that x belongs to B_1.

Case 1 : $v = y$.

Let $z \in B_2$ and $w \in B_3$ be two endvertices of T. Then v lies on the unique z-w detour monophonic path. Since z and w belong to M_x and $v \notin M_x$, the subgraph induced by M_x is not connected, which is a contradiction.

Case 2 : $v \neq y$.

Let $v \in B_i (i \neq 1)$. Now, choose an endvertex $u \in B_i$ such that v lies on the y-u detour monophonic path. Let $a \in B_j (j \neq i, 1)$ be an endvertex of T. Then y lies on the u-a detour monophonic path. Hence it follows that v lies on the u-a detour monophonic path. Since u and a belong to M_x and $v \notin M_x$, the subgraph induced by M_x is not connected, which is a contradiction.

Hence S_x is a minimal connected x-detour monophonic set of T. Since T is a tree, S_x is the unique minimal connected x-detour monophonic set of T and so $cdm_x^+(T) = n - d_m(x, y)$.

Now, let T be a path. Let x and y be the end vertices of T. Clearly $\{y\}$ is the unique minimal connected x-detour monophonic set of T and so $cdm_x^+(T) = 1$. □

Corollary 2.10 — For any tree T of order $n \geq 3$, $cdm_x^+(T) = n$ if and only if x is a cutvertex of T.

3. BOUNDS AND REALIZATION RESULTS FOR $cdm_x^+(G)$

For any vertex x in a connected graph G of order $n \geq 2$ we have $1 \leq cdm_x(G) \leq cdm_x^+(G) \leq n$.

These bounds are sharp. For the path P_n $(n \geq 2)$, $cdm_x(P_n) = 1$ for an endvertex x in P_n. For any non-trivial tree T with order $n \geq 3$, $cdm_x^+(T) = n$ for any cutvertex x in T. Also, for the complete graph $K_n(n \geq 2)$, $cdm_x(K_n) = cdm_x^+(K_n) = n - 1$. The inequalities can be strict. For the graph G given in Figure 2.1, $cdm_r(G) = 3, cdm_r^+(G) = 5$ and $n = 6$. Thus $1 < cdm_r(G) < cdm_r^+(G) < n$.

***Theorem* 3.1** — *Let G be a connected graph with at least one cutvertex v and let $x \in V(G)$. Let S_x be an x-detour monophonic set of G. Then every component of $G - v$ contains an element of $S_x \cup \{x\}$.*

PROOF : Suppose there is a component B of $G - v$ such that B contains no vertex of $S_x \cup \{x\}$. Then $x \in V - V(B)$. Let $u \in V(B)$. Since S_x is an x-detour monophonic set, there exists an element $y \in S_x$ such that u lies in some x-y detour monophonic path $P = (x = u_0, u_1, u_2, \ldots, u, \ldots, u_n = y)$ in G. Since v is a cutvertex of G, the x-u subpath of P and the u-y subpath of P both contain v, it follows that P is not a path, which is a contradiction. □

***Theorem* 3.2** — *Let x be any cutvertex of a connected graph G. Then*

$$cdm_x^+(G) \geq 3.$$

PROOF : Let S_x be any connected x-detour monophonic set of G. Since x is a cutvertex of G, $G - x$ has at least two components. By Theorem 3.1, S_x has vertices from each component of $G - x$. Now, since the induced subgraph $G[S_x]$ is connected, S_x must contain x. Thus S_x has at least three vertices and so $cdm_x(G) \geq 3$. Hence $cdm_x^+(G) \geq 3$. □

Remark 3.3 : If x is not a cutvertex of G, then $V - \{x\}$ is a connected x-detour monophonic set of G. Thus $cdm_x^+(G) \leq n - 1$. Hence if $cdm_x^+(G) = n$, then x is a cutvertex of G.

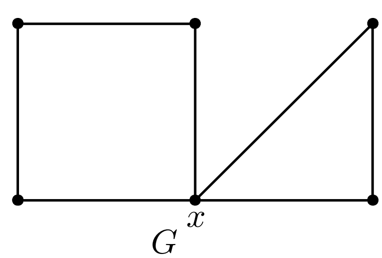


Figure 3.1

However, the converse is not true. For the graph G given in Figure 3.1, $cdm_x^+(G) = 5 < n$ for the cutvertex x in G.

Remark 3.4 : Since any connected graph G contains at least two vertices which are not cutvertices, there is no graph G of order n with $cdm_x^+(G) = n$ for every vertex x.

***Theorem* 3.5** — *For any two integers k and n with $1 \leq k \leq n$ and $n \geq 3$, there exists a connected graph G of order n with $dm_x^+(G) = k$ for some vertex x in G.*

PROOF : We prove this theorem by considering two cases.

Case 1 : $1 \leq k \leq n-1$.

Let G be the graph obtained from the path $P_{n-k} = (u_1, u_2, \ldots, u_{n-k})$ of order $n-k \geq 1$ and the complete graph K_k of order k with vertex set $V(K_k) = \{w_1, w_2, \ldots, w_k\}$ by joining each w_i with u_{n-k} in P_{n-k}. Let $x = u_1$. Now $S = \{x, w_1, w_2, \ldots, w_k\}$ is the set of all extreme vertices of G. Hence by Theorem 2.3, $S_x = S - \{x\}$ is the unique minimal connected x-detour monophonic set of G and so $cdm_x^+(G) = k$.

Case 2 : $k = n$.

Let G be any tree of order n. Then by Theorem 2.9(i), $cdm_x^+(G) = n$ for any cutvertex x in G.□

For every connected graph G, $rad_m(G) \leq diam_m(G)$. It is shown in [11] that any two positive integers a and b with $a \leq b$ are realizable as the monophonic radius and monophonic diameter, respectively, of some connected graph. In the following theorem we extend this result.

***Theorem* 3.6** — *For integers r, d and k with $2 \leq r \leq d$ and $k \geq 2$, there exists a connected graph G with $rad_m(G) = r, diam_m(G) = d$ and $cdm_x^+(G) = k$ for some vertex x in G.*

PROOF : *Case* 1 : $r = d = 2$.

Let $C_4 = (v_1, v_2, v_3, v_4, v_1)$ be a cycle of order 4 and let K_{k-1} be a complete graph of order $k-1$ with vertex set $V(K_{k-1}) = \{w_1, w_2, \ldots, w_{k-1}\}$. Let G be the graph obtained from C_4 and K_{k-1} by joining each vertex w_i with the vertices v_1, v_2 and v_3. The graph G is shown in Figure 3.2.

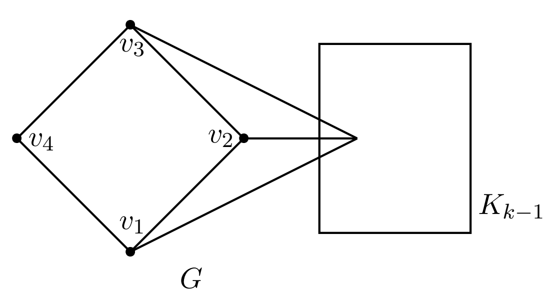


Figure 3.2

It is easily verified that the monophonic eccentricity of each vertex in G is 2 and so $rad_m(G) =$

$diam_m(G) = 2$. Let $x = v_4$. Clearly $w_i (1 \leq i \leq k-1)$ is not an internal vertex of any detour monophonic path starting from x. Therefore, each $w_i (1 \leq i \leq k-1)$ must belong to every minimal connected x-detour monophonic set of G. It is easily verified that $S_x = \{w_1, w_2, \ldots, w_{k-1}, v_2\}$ is the unique minimal connected x-detour monophonic set of G and so $cdm_x^+(G) = |S_x| = k$.

Case 2 : $2 < r = d$ or $2 \leq r < d$.

Let H be the graph obtained from the cycle $C_{r+2} = (v_1, v_2, \ldots, v_{r+2}, v_1)$ and the path $P_{d-r+1} = (u_0, u_1, u_2, \ldots, u_{d-r})$ by identifying the vertex v_{r+1} in C_{r+2} with u_0 in P_{d-r+1} and joining each vertex $u_i (1 \leq i \leq d-r)$ with v_{r+2}. Now, let G be the graph obtained from H by adding $k-2$ new vertices $w_1, w_2, \ldots, w_{k-2}$ and joining each w_i with v_2 and v_{r+2} in H. The graph G is shown in Figure 3.3.

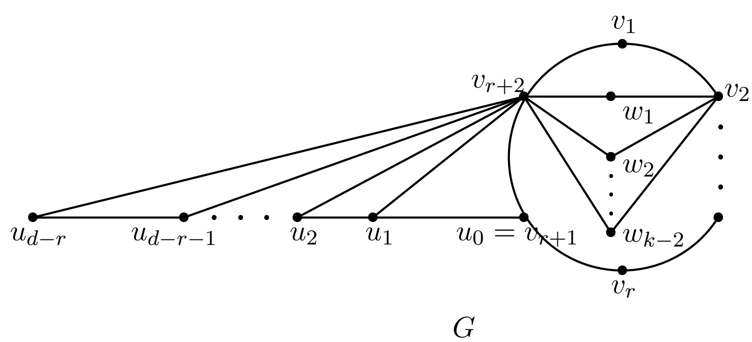


Figure 3.3

It is easily verified that $r \leq e_m(x) \leq d$ for any vertex x in G. Also $e_m(v_{r+2}) = r$ and $e_m(v_1) = d$. Hence $rad_m(G) = r$ and $diam_m(G) = d$. Now, let $x = u_{d-r}$ and let $S = \{v_1, v_{r+2}, w_1, w_2, \ldots, w_{k-2}\}$. Since every vertex of G lies on an x-y detour monophonic path for some $y \in S$ and the induced subgraph $G[S]$ is connected, S is a connected x-detour monophonic set of G. Also any $z \in S$ does not lie on a x-u detour monophonic path for any $u \in S - \{z\}$. Hence S is a minimal x-detour monophonic set of G and so $cdm_x^+(G) \geq k$. Also, any minimal connected x-detour monophonic set of G contains at most k vertices and hence $cdm_x^+(G) \leq k$. Hence $cdm_x^+(G) = k$. □

Since any connected x-detour monophonic set is an x-detour monophonic set, it follows that $dm_x(G) \leq cdm_x(G) \leq cdm_x^+(G)$. Now we have the following realization theorem.

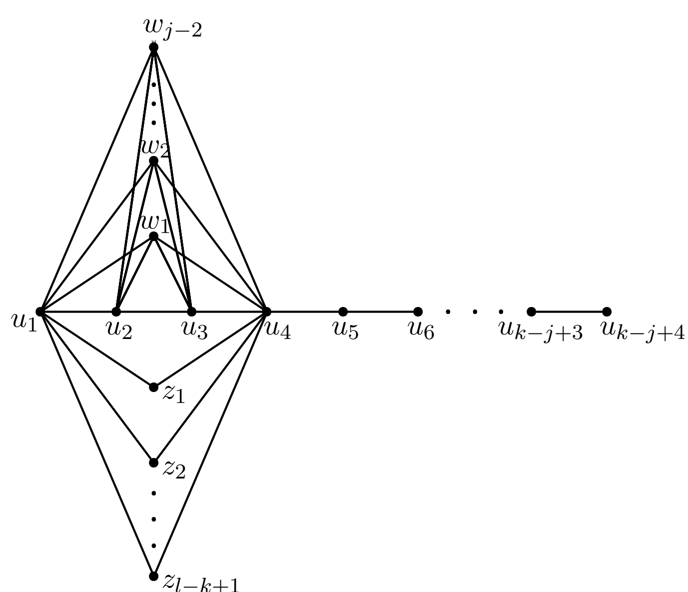
***Theorem* 3.7** — *For any three positive integers j, k and l with $2 \leq j \leq k \leq l$, there exists a connected graph G with $dm_x(G) = j$, $cdm_x(G) = k$ and $cdm_x^+(G) = l$ for some vertex x in G.*

PROOF : We consider two cases.

Case 1 : $2 \leq j < k \leq l$.

Let G be the graph obtained from the path $P_{k-j+4} = (u_1, u_2, \ldots, u_{k-j+4})$ by adding $l-k+j-1$ new vertices $w_1, w_2, \ldots, w_{j-2}, z_1, z_2, \ldots, z_{l-k+1}$ and joining each w_i with u_1, u_2, u_3 and u_4 and joining each z_i with u_1 and u_4 in P_{k-j+4}. The graph G is shown in Figure 3.4. Let $x = u_1$.

We claim that $dm_x(G) = j$. The extreme vertex u_{k-j+4} belongs to every x-detour monophonic set of G. Also $\{u_{k-j+4}\}$ is not an x-detour monophonic set of G and each $w_i, 1 \leq i \leq j-2$, is not an internal vertex of any detour monophonic path starting from x. Thus every x-detour monophonic set of G contains $S_1 = \{u_{k-j+4}, w_1, w_2, \ldots, w_{j-2}\}$. It is clear that no z_i lies on any x-z detour monophonic path for any $z \in S_1$. Thus S_1 is not an x-detour monophonic set of G. Now, since $S_2 = S_1 \cup \{u_3\}$ is an x-detour monophonic set of G, it follows that $dm_x(G) = j$.



G

Figure 3.4

We now claim that $cdm_x(G) = k$. Since every x-detour monophonic set of G contains S_1 and S_1 contains u_{k-j+4} and w_1, every connected x-detour monophonic set of G contains the set $T = \{u_4, u_5, \ldots, u_{k-j+3}\}$. Let $S_3 = S_1 \cup T$. It is clear that no z_i lies on any x-z detour monophonic path for any $z \in S_3$. Thus S_3 is not an x-detour monophonic set of G. Further $S_4 = S_3 \cup \{u_3\}$ is a connected x-detour monophonic set of G and hence $cdm_x(G) = k$.

Next, we show that $cdm_x^+(G) = l$. Clearly $M = S_3 \cup \{z_1, z_2, \ldots, z_{l-k+1}\}$ is a connected x-detour monophonic set of G. We claim that M is a minimal connected x-detour monophonic set of G. Suppose there exists a proper subset N of M such that N is a connected x-detour monophonic set of G. Let $s \in M$ and $s \notin N$. Since every connected x-detour monophonic set of G contains S_3, it follows that $s = z_i$ for some i. Now, z_i does not lie on any x-z detour monophonic path for any vertex

$z \in N$. Hence it follows that N is not an x-detour monophonic set of G, which is a contradiction. Thus M is a minimal connected x-detour monophonic set of G and so $cdm_x^+(G) \geq |M| = l$. Also, it is clear that there is no minimal connected x-detour monophonic set M' of G with $|M'| > l$. Hence $cdm_x^+(G) = l$.

Case 2 : $2 \leq j = k \leq l$.

Let G be the graph obtained from the path $P_4 = (u_1, u_2, u_3, u_4)$ by adding $l - k + j - 1$ new vertices $w_1, w_2, \ldots, w_{j-1}, z_1, z_2, \ldots, z_{l-k}$ and joining each w_i with u_1, u_2, u_3 and u_4 and joining each z_i with u_1 and u_4. The graph G is shown in Figure 3.5. Let $x = u_1$.

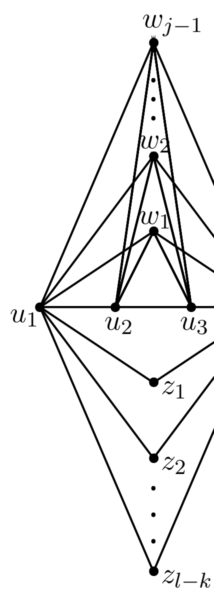


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Figure 3.5

By an argument similar to Case 1, the set $S = \{w_1, w_2, \ldots, w_{j-1}, u_3\}$ is both minimum x-detour monophonic set and minimum connected x-detour monophonic set of G and so $dm_x(G) = cdm_x(G) = j$. Also, $S' = \{w_1, w_2, \ldots, w_{j-1}, z_1, z_2, \ldots, z_{l-k}, u_4\}$ is a minimal connected x-detour monophonic set of G with maximum cardinality and so $cdm_x^+(G) = l$.

In the following theorem, we construct a graph of prescribed order, monophonic diameter and upper connected vertex detour monophonic number under suitable conditions.

***Theorem* 3.8** — *If* n, d *and* k *are positive integers such that* $2 \leq d \leq n - 1$, $3 \leq k \leq n$ *and* $n - d - k + 1 \geq 0$, *then there exists a connected graph* G *of order* n *with monophonic diameter* d *and* $cdm_x^+(G) = k$ *for some vertex* x *in* G.

PROOF : We prove this theorem by considering two cases.

Case 1 : $d = 2$.

If $k = n$, then for the star $G = K_{1,n-1}$, we have $d = 2$ and $cdm_x^+(G) = n$ for the cutvertex x of G. Now, let $3 \leq k < n$. Let $P_3 = (u_1, u_2, u_3)$ be the path of order 3. Add $n - 3$ new vertices $v_1, v_2, \ldots, v_{n-k-1}, w_1, w_2, \ldots, w_{k-2}$ and join each w_i with u_2 and join each v_i with u_1, u_2 and u_3. Also, join each $v_i, 1 \leq i \leq n-k-2$, with $v_j, i+1 \leq j \leq n-k-1$. The resulting graph G is shown in Figure 3.6. Then G is a graph of order n with monophonic diameter $d = 2$. Let $x = u_1$. Now every minimal connected x-detour monophonic set of G contains the set of all extremal vertices $S = \{w_1, w_2, \ldots, w_{k-2}\}$. Further S is not a minimal connected x-detour monophonic set of G. Also $S \cup \{y\}$, where $y \in V(G) - S$, is not a minimal connected x-detour monophonic set of G. It is clear that $S_1 = S \cup \{u_2, u_3\}$ is a minimal connected x-detour monophonic set of G and so $cdm_x^+(G) \geq k$. Also there is no minimal connected x-detour monophonic set of G with $cdm_x^+(G) > k$. Hence $cdm_x^+(G) = k$.

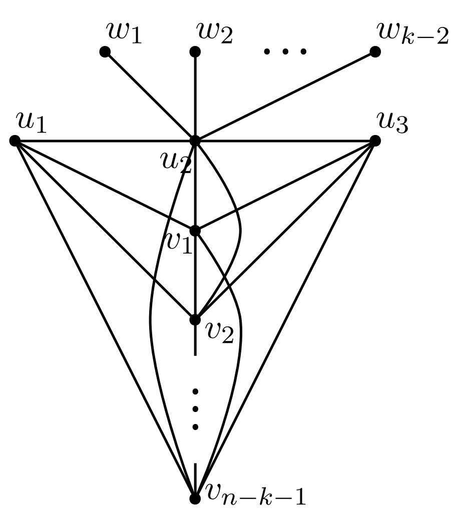

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Figure 3.6

Case 2 : $3 \leq d \leq n-1$.

If $k = n$, then for any tree T of order n and diameter d, we have $cdm_x^+(T) = n$ for any cutvertex x in T. Now, let $3 \leq k < n$. Let G be the graph obtained from the path $P_d = (u_1, u_2, \ldots, u_d)$ by adding $n - d$ new vertices $w_1, w_2, \ldots, w_{k-1}, v_1, v_2, \ldots, v_{n-d-k+1}$ and joining each w_i with u_1 and joining each v_i with u_1 and u_3 (see Figure 3.7). The graph G has order n and monophonic diameter d.

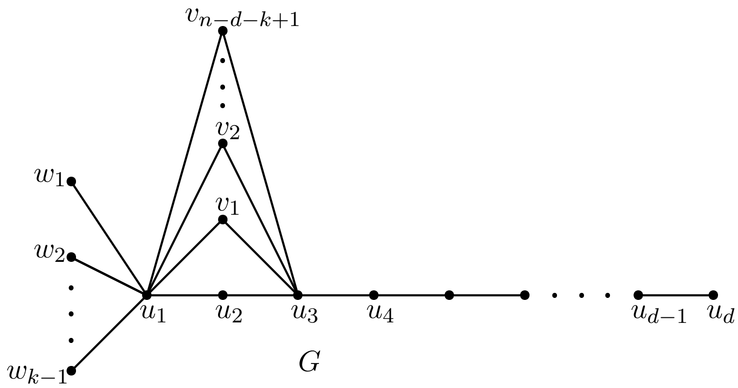

Figure 3.7

Let $x = u_d$. Every minimal connected x-detour monophonic set of G contains the set of all extreme vertices $S = \{w_1, w_2, \ldots, w_{k-1}\}$. Clearly, S is not a minimal connected x-detour monophonic set of G and $S \cup \{u_1\}$ is the unique minimal connected x-detour monophonic set of G. Hence $cdm_x^+(G) = k$. □

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