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MINIMAXNESS AND FINITENESS PROPERTIES OF LOCAL HOMOLOGY AND LOCAL COHOMOLOGY MODULES

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We prove some results concerning minimaxness and finiteness of local homology modules and by Matlis duality we extend some results for the minimaxness and finiteness of local cohomology modules. We introduce the concept of $\mathcal{C}$-minimax R-modules, and we discuss the maximum and minimum integers such that local homology and local cohomology modules are $\mathcal{C}$-minimax. As a consequence, we find minimum integers such that local homology and local cohomology modules are of finite length.

Key words : Local homology; local cohomology, minimax.

1. Introduction

Throughout this paper, $(R, \mathfrak{m})$ is a commutative Noetherian local ring, $\mathfrak{a}$ is an ideal of R. Let M be an R-module. In [3] Cuong and Nam defined the local homology modules $\mathrm{H}_i^{\mathfrak{a}}(M)$ with respect to $\mathfrak{a}$ by

$$\mathrm{H}_i^{\mathfrak{a}}(M) = \varprojlim_n \mathrm{Tor}_i^R(R/\mathfrak{a}^n, M).$$

This definition is dual to Grothendieck's definition of local cohomology modules and coincides with the definition of Greenless and May [6] when M is an artinian R-module. For each $i \geq 0$, the i-th local cohomology module of M with respect to an ideal $\mathfrak{a}$ is defined as $\mathrm{H}_{\mathfrak{a}}^i(M) = \varinjlim_n \mathrm{Ext}_R^i(R/\mathfrak{a}^n, M)$. For basic results about local homology, we refer the reader to [3, 4] and [15]; for local cohomology refer to [2]. Throughout the paper, $\mathrm{D}(.)$ denotes the Matlis duality functor $\mathrm{Hom}_R(., \mathrm{E}(R/\mathfrak{m}))$.

The R-module M is said to be a minimax module if there is a finite (i.e. finitely generated) submodule N of M, such that M/N is artinian. The class of minimax modules was introduced by Zöschinger [17], and he has given in [17, 18] many equivalent conditions for a module to be minimax. The class of minimax modules includes all finite and all artinian modules. Moreover it is closed under taking submodules, quotients and extensions, i.e., it is a Serre subcategory of the category of R-modules.

In [19], Zöschinger defined and investigated coatomic modules over commutative Noetherian rings. A module M is called coatomic, if every proper submodule of M is contained in a maximal submodule of M. Over Noetherian rings, the coatomic modules are closed under taking quotients, submodules and extensions. It is clear that every finitely generated R-module is coatomic and that every coatomic, artinian module has finite length.

By using the concept of coatomic modules we define $\mathcal{C}$-minimax modules. We say that an R-module M is $\mathcal{C}$-minimax, if there is a coatomic submodule N of M, such that M/N is artinian. Clearly, the class of $\mathcal{C}$-minimax modules includes minimax modules. Moreover, we will show that it is closed under taking submodules, quotients and extensions, i.e., it is a Serre subcategory of the category of R-modules. (See proposition 2.5).

Nam [10] studied minimaxness of local homology and local cohomology modules. In this paper we obtain some results for the $\mathcal{C}$-minimaxness, minimaxness and finiteness of local homology modules and by Matlis duality we get some results for local cohomology modules. Here we extend some main results of [10] to non-complete noetherian local rings. In fact, we see that there exists a close relation between $\mathcal{C}$-minimaxness and minimaxness of local homology and local cohomology modules.

A non-zero R-module M is called secondary if its multiplication map by any element a of R is either surjective or nilpotent. A secondary representation for an R-module M is an expression for M as a finite sum of secondary modules. If such a representation exists, we will say that M is representable. A prime ideal $\mathfrak{p}$ of R is said to be an attached prime of M if $\mathfrak{p} = (N :_R M)$ for some submodule N of M. If M admits a reduced secondary representation, $M = S_1 + S_2 + \ldots + S_k$, then the set of attached primes $\operatorname{Att}_R(M)$ of M is equal to $\{\sqrt{0 :_R S_i} \, : \, i = 1, \ldots, k\}$ (see [9]). Recall that, an R-module M is called good if its zero submodule has a primary decomposition.

In [12], we have proved that for an artinian R-module M,

$$\inf\{i \in \mathbb{N} : \mathfrak{a}^n \operatorname{H}_i^{\mathfrak{a}}(M) \neq 0 \text{ for all } n \in \mathbb{N}\} = \inf\{i \in \mathbb{N} : \operatorname{H}_i^{\mathfrak{a}}(M) \text{ is not representable}\}.$$

and for any finite R-module M we have:

$$\inf\{i \in \mathbb{N} : \mathfrak{a}^n \operatorname{H}^i_{\mathfrak{a}}(M) \neq 0 \text{ for all } n \in \mathbb{N}\} = \inf\{i \in \mathbb{N} : \operatorname{H}^i_{\mathfrak{a}}(M) \text{ is not good}\}.$$

In this paper, among other things, we extend the above results. In fact we show that for any artinian R-module M we have:

$$\inf\{i \in \mathbb{N} : \mathfrak{a}^n \operatorname{H}_i^{\mathfrak{a}}(M) \text{ is not finite for all } n \in \mathbb{N}\} = \inf\{i \in \mathbb{N} : \operatorname{H}_i^{\mathfrak{a}}(M) \text{ is not } \mathcal{C} - \text{minimax}\}.$$

$$\sup\{i \in \mathbb{N} : \mathfrak{a}^n \operatorname{H}_i^{\mathfrak{a}}(M) \text{ is not finite for all } n \in \mathbb{N}\} = \sup\{i \in \mathbb{N} : \operatorname{H}_i^{\mathfrak{a}}(M) \text{ is not } \mathcal{C} - \text{minimax}\}.$$

and for any finite R-module M we have:

$$\inf\left\{i \in \mathbb{N} : \mathfrak{a}^n \operatorname{H}^i_{\mathfrak{a}}(M) \text{ is not artinian for all } n \in \mathbb{N}\right\} = \inf\{i \in \mathbb{N} : \operatorname{H}^i_{\mathfrak{a}}(M) \text{ is not } \mathcal{C} - \text{minimax}\}.$$

$$\sup\left\{i \in \mathbb{N} : \mathfrak{a}^n \operatorname{H}^i_{\mathfrak{a}}(M) \text{ is not artinian for all } n \in \mathbb{N}\right\} = \sup\{i \in \mathbb{N} : \operatorname{H}^i_{\mathfrak{a}}(M) \text{ is not } \mathcal{C} - \text{minimax}\}.$$

Also, we prove the following results about finite length local homology and local cohomology modules:

$$\inf\{i \in \mathbb{N} : \operatorname{H}_i^{\mathfrak{a}}(M) \text{ is not of finite length}\} = \inf\{i \in \mathbb{N} : \mathfrak{m} \nsubseteq \sqrt{(0 : \operatorname{H}_i^{\mathfrak{a}}(M))}\}.$$

$$\inf\left\{i \in \mathbb{N} : \operatorname{H}^i_{\mathfrak{a}}(M) \text{ is not of finite length}\right\} = \inf\{i \in \mathbb{N} : \mathfrak{m} \nsubseteq \sqrt{(0 : \operatorname{H}^i_{\mathfrak{a}}(M))}\}.$$

2. MINIMAXNESS AND FINITENESS OF LOCAL HOMOLOGY AND LOCAL COHOMOLOGY MODULES

We will use the concept of Noetherian dimension of an R-module in the proof of some results. Let M be an artinian R-module. The Noetherian dimension of M, $\operatorname{Ndim}_R(M)$, is defined by induction. If $M = 0$, we put $\operatorname{Ndim}_R(M) = 1$. For any integer $t \geq 0$, if $\operatorname{Ndim}_R(M) < t$ is false and whenever $M_1 \subseteq M_2 \subseteq \cdots$ is an ascending chain of submodules of M then there exists an integer m_0 such that $\operatorname{Ndim}_R(M_{m+1}/M_m) < t$ for all $m \geq m_0$, then we put $\operatorname{Ndim}_R(M) = t$. In case M is an artinian module, $\operatorname{Ndim}_R(M) < \infty$. (See [11] and [7]).

The following Proposition is used in the sequel.

Proposition 2.1 — Let $(Q_n)_{n\geq 1}$ be an inverse system of R-modules, with maps $\varphi_{mn} : Q_m \to Q_n$ for $m \geq n$. Let $\mathfrak{a}$ be an ideal of ring R such that $u^k Q_k = 0$ for all $u \in \mathfrak{a}$ and all $k \in \mathbb{N}$. If N is an arbitrary submodule of $\varprojlim_n Q_n$ such that $\varprojlim_n Q_n/N$ is non-zero and representable, then $\mathfrak{a} \subseteq \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Att}_R(\varprojlim_n Q_n/N)$.

PROOF : Let $\varprojlim_n Q_n/N = S_1 + S_2 + ... + S_n$ be a minimal secondary representation of $\varprojlim_n Q_n/N$ where S_j is $\mathfrak{p}_j$-secondary for $j = 1, 2, ..., n$. Suppose that there exists an integer $j \in \{1, \ldots, n\}$ such that $\mathfrak{a} \nsubseteq \mathfrak{p}_j$ and look for a contradiction. Then there exists $u \in \mathfrak{a} \setminus \mathfrak{p}_j$. Take $0 \neq g = N + (g_k) \in S_j \subseteq \varprojlim_n Q_n/N$. Let g_k be the first non-zero component of g. Since $u \notin \mathfrak{p}_j$, we have $uS_j = S_j$. But $u^k S_j \subseteq u^k(\varprojlim_n Q_n/N)$, and so $S_j \subseteq u^k(\varprojlim_n Q_n/N)$. As $u^k Q_k = 0$, it follows that the k-th component of each element of $u^k(\varprojlim_n Q_n/N)$ is zero. But $g \in u^k(\varprojlim_n Q_n/N)$ and the k-th component of g is non-zero, which is a contradiction. □

Corollary 2.2 — Let $\mathfrak{a}$ be an ideal of a local ring $(R, \mathfrak{m})$ and M an artinian R-module. Let i be a natural number. If N is a submodule of $\mathrm{H}_i^{\mathfrak{a}}(M)$ such that $\mathrm{H}_i^{\mathfrak{a}}(M)/N$ is artinian, then there exists a positive integer t such that $\mathfrak{a}^t(\mathrm{H}_i^{\mathfrak{a}}(M)) \subseteq N$.

PROOF : Since $\mathrm{H}_i^{\mathfrak{a}}(M) = \varprojlim_n \mathrm{Tor}_i^R(R/\mathfrak{a}^n, M)$ and $u^k \mathrm{Tor}_i^R(R/\mathfrak{a}^k, M) = 0$ for all $u \in \mathfrak{a}$ and all $k \in \mathbb{N}$, Proposition 2.1 implies that $\mathrm{Att}_R(\mathrm{H}_i^{\mathfrak{a}}(M)/N) \subseteq V(\mathfrak{a})$. But, by [2, 7.2.11], $\sqrt{(0 : \mathrm{H}_i^{\mathfrak{a}}(M)/N)} = \bigcap_{\mathfrak{p} \in \mathrm{Att}(\mathrm{H}_i^{\mathfrak{a}}(M)/N)} \mathfrak{p}$. Hence $\mathfrak{a} \subseteq \sqrt{(0 : \mathrm{H}_i^{\mathfrak{a}}(M)/N)}$ and we conclude that there exists a positive integer t such that $\mathfrak{a}^t(\mathrm{H}_i^{\mathfrak{a}}(M)/N) = 0$ and so $\mathfrak{a}^t(\mathrm{H}_i^{\mathfrak{a}}(M)) \subseteq N$. □

The modules which are extension of a coatomic module by an artinian module and we call $\mathcal{C}$-minimax in the following definition, were studied by Rudolf [13]. He has given in [13, 14] many equivalent conditions for a module to be $\mathcal{C}$-minimax. This class of R-modules includes minimax modules and coatomic modules.

Definition 2.3 — An R-module M is said to be $\mathcal{C}$-minimax if there is a coatomic submodule N of M, such that M/N is artinian.

The following result is a characterization of $\mathcal{C}$-minimax modules.

***Theorem* 2.4** — *[13, Theorem 3.3]. Let $(R, \mathfrak{m})$ be a local ring. For an R-module M the following are equivalent:*

(i) There is an $n > 1$ such that $\mathfrak{m}^n M$ is a minimax module;

(ii) M is an extension of a coatomic module by an artinian module i.e M is $\mathcal{C}$-minimax module.

Proposition 2.5 — Let

$$0 \to L \to M \to N \to 0$$

be an exact sequence of R-modules. Then

(i) M is minimax if and only if L and N are both minimax.

(ii) M is $\mathcal{C}$-minimax if and only if L and N are both $\mathcal{C}$-minimax.

PROOF : (i) By [8, Lemma 1.23].

(ii) We may suppose for the proof that L is a submodule of M and that $N = M/L$. Assume that M is $\mathcal{C}$-minimax. There exists a coatomic submodule U of M such that M/U is artinian. Then $L \cap U$ is coatomic and $L/(L \cap U) \simeq (L+U)/U$ is artinian, since it is a submodule of M/U. Thus L is $\mathcal{C}$-minimax. On the other hand, since $(U+L)/L$ is coatomic and $(M/L)/((U+L)/L) \simeq M/(U+L)$ is artinian, we conclude that M/L is $\mathcal{C}$-minimax.

Now suppose that L and M/L are $\mathcal{C}$-minimax. Thus there exists a coatomic submodule T of L such that L/T is artinian and there exists a coatomic submodule U/L of M/L such that M/U is artinian. Since U/L is coatomic $(U/T)/(L/T)$ is coatomic. Let K/T be a complement of L/T in U/T. Then K/T is coatomic, being an essential cover of the coatomic module $(U/T)/(L/T)$, and $U/T = L/T + K/T$. Now from $U/K \simeq (U/T)/(K/T) \simeq (L/T)/(L/T \cap K/T)$ we deduce that U/K is artinian and so M/K is artinian since M/U is artinian. But K is coatomic since K/T and T are coatomic. It follows that M is $\mathcal{C}$-minimax. □

***Theorem* 2.6** — *Let $\mathfrak{a}$ be an ideal of a local ring $(R, \mathfrak{m})$ and M an artinian R-module. Let i be a natural number.*

(i) If $\mathrm{H}_i^{\mathfrak{a}}(M)$ is minimax, then there exists a positive integer t such that $\mathfrak{a}^t \mathrm{H}_i^{\mathfrak{a}}(M)$ is finite.

(ii) If $\mathrm{H}_i^{\mathfrak{a}}(M)$ is $\mathcal{C}$-minimax, then there exists a positive integer t such that $\mathfrak{a}^t \mathrm{H}_i^{\mathfrak{a}}(M)$ is coatomic.

PROOF : (i) Since $\mathrm{H}_i^{\mathfrak{a}}(M)$ is minimax, there exists a finite submodule N of $\mathrm{H}_i^{\mathfrak{a}}(M)$ such that $\mathrm{H}_i^{\mathfrak{a}}(M)/N$ is artinian. By corollary 2.2 there exists a positive integer t such that $\mathfrak{a}^t(\mathrm{H}_i^{\mathfrak{a}}(M)) \subseteq N$. Therefore $\mathfrak{a}^t \mathrm{H}_i^{\mathfrak{a}}(M)$ is finite.

(ii) Since $\mathrm{H}_i^{\mathfrak{a}}(M)$ is $\mathcal{C}$-minimax, there exists a coatomic submodule N of $\mathrm{H}_i^{\mathfrak{a}}(M)$ such that $\mathrm{H}_i^{\mathfrak{a}}(M)/N$ is artinian. Corollary 2.2 implies that there exists a positive integer t such that $\mathfrak{a}^t(\mathrm{H}_i^{\mathfrak{a}}(M)) \subseteq N$. Therefore $\mathfrak{a}^t \mathrm{H}_i^{\mathfrak{a}}(M)$ is coatomic. □

Corollary 2.7 — Let $\mathfrak{a}$ be an ideal of a local ring $(R, \mathfrak{m})$ and M an artinian R-module. Let i be a natural number. If $\mathrm{H}_i^{\mathfrak{a}}(M)$ is $\mathcal{C}$-minimax, then there exists a positive integer k such that $\mathfrak{a}^k \mathrm{H}_i^{\mathfrak{a}}(M)$ is finite.

PROOF : By Theorem 2.6 (ii), there exists a positive integer t such that $\mathfrak{a}^t \mathrm{H}_i^{\mathfrak{a}}(M)$ is coatomic. Thus by [19, Satz 2.4], there exists a positive integer s such that $\mathfrak{m}^s \mathfrak{a}^t \mathrm{H}_i^{\mathfrak{a}}(M)$ is finite. Therefore $\mathfrak{a}^{s+t} \mathrm{H}_i^{\mathfrak{a}}(M)$ is finite, as required.

***Theorem* 2.8** — *Let $\mathfrak{a}$ be an ideal of a local ring $(R, \mathfrak{m})$ and M an artinian R-module and n an* *integer. Let t be an arbitrary integer.*

(i) If $\mathrm{H}_i^{\mathfrak{a}}(M)$ is minimax for all $i < n$, then $\mathrm{H}_n^{\mathfrak{a}}(M)/\mathfrak{a}^t\,\mathrm{H}_n^{\mathfrak{a}}(M)$ is minimax.

(ii) If $\mathrm{H}_i^{\mathfrak{a}}(M)$ is $\mathcal{C}$-minimax for all $i < n$, then $\mathrm{H}_n^{\mathfrak{a}}(M)/\mathfrak{a}^t\,\mathrm{H}_n^{\mathfrak{a}}(M)$ is $\mathcal{C}$-minimax.

PROOF : (i) We use induction on n. When $n = 0$, for all positive integers k the canonical epi-morphisms $M \to M/\mathfrak{a}^k M$ induces an epi-morphism $M \to \Lambda_{\mathfrak{a}}(M)$ where $\Lambda_{\mathfrak{a}}(M) = \varprojlim_k M/\mathfrak{a}^k M$ denotes the $\mathfrak{a}$-adic completion of M. Thus $\Lambda_{\mathfrak{a}}(M)$ is homomorphic image of an artinian module and so is minimax. Since $\mathrm{H}_0^{\mathfrak{a}}(M) \cong \Lambda_{\mathfrak{a}}(M)$, the result follows in this case. Now suppose, inductively that $n > 0$ and the result is true for $n-1$. By [3, Corollary 4.5], we can replace M by $\cap_{n>0}\mathfrak{a}^n M$. But $\cap_{n>0}\mathfrak{a}^n M = \mathfrak{a}^k M$ for some $k \in \mathbb{N}$ and so we may assume that $\mathfrak{a}M = M$. Now $\mathrm{Att}_R(M/\mathfrak{a}M) = \mathrm{V}(\mathfrak{a}) \cap \mathrm{Att}_R M = \phi$ implies that $\mathfrak{a} \nsubseteq \cup_{\mathfrak{p}\in \mathrm{Att}_R(M)}\mathfrak{p}$. Thus there exists $x \in \mathfrak{a}$ such that $xM = M$ by [2, Proposition 7.2.11(i)] and so $x^t M = M$. From the exact sequence

$$0 \to (0 :_M x^t) \to M \xrightarrow{x^t} M \to 0$$

and using [3, Corollary 4.2], we obtain the following long exact sequence

$$\cdots \to \mathrm{H}_n^{\mathfrak{a}}(M) \xrightarrow{x^t} \mathrm{H}_n^{\mathfrak{a}}(M) \xrightarrow{f} \mathrm{H}_{n-1}^{\mathfrak{a}}(0 :_M x^t) \xrightarrow{g} \mathrm{H}_{n-1}^{\mathfrak{a}}(M) \to \mathrm{H}_{n-1}^{\mathfrak{a}}(M) \to \cdots .$$

From the above exact sequence we get

$$0 \to \mathrm{Im}\, f \to \mathrm{H}_{n-1}^{\mathfrak{a}}(0 :_M x^t) \to \mathrm{Im}\, g \to 0,$$

$$\cdots \to \mathrm{H}_n^{\mathfrak{a}}(M) \xrightarrow{x^t} \mathrm{H}_n^{\mathfrak{a}}(M) \to \mathrm{Im}\, f \to 0.$$

Thus we have the following exact sequences:

$$\cdots \to \mathrm{Tor}_1^R(R/\mathfrak{a}^t, \mathrm{Im}\, g) \to \mathrm{Im}\, f/\mathfrak{a}^t\,\mathrm{Im}\, f \to \mathrm{H}_{n-1}^{\mathfrak{a}}(0 :_M x^t)/\mathfrak{a}^t\,\mathrm{H}_{n-1}^{\mathfrak{a}}(0 :_M x^t) \to \cdots ,$$

$$\cdots \to \mathrm{H}_n^{\mathfrak{a}}(M)/\mathfrak{a}^t\,\mathrm{H}_n^{\mathfrak{a}}(M) \xrightarrow{x^t} \mathrm{H}_n^{\mathfrak{a}}(M)/\mathfrak{a}^t\,\mathrm{H}_n^{\mathfrak{a}}(M) \to \mathrm{Im}\, f/\mathfrak{a}^t\,\mathrm{Im}\, f \to 0.$$

Induction hypothesis implies that $\mathrm{H}_{n-1}^{\mathfrak{a}}(0 :_M x^t)/\mathfrak{a}^t\,\mathrm{H}_{n-1}^{\mathfrak{a}}(0 :_M x^t)$ is minimax. On the other hand, it is easy to see that since $\mathrm{Im}\, g$ is minimax $\mathrm{Tor}_1^R(R/\mathfrak{a}^t, \mathrm{Im}\, g)$ is minimax by Proposition 2.5(i). Hence $\mathrm{Im}\, f/\mathfrak{a}^t\,\mathrm{Im}\, f$ is minimax. Since $x^t \in \mathfrak{a}^t$, the first map in the above exact sequence is zero, and therefore $\mathrm{H}_n^{\mathfrak{a}}(M)/\mathfrak{a}^t\,\mathrm{H}_n^{\mathfrak{a}}(M) \simeq \mathrm{Im}\, f/\mathfrak{a}^t\,\mathrm{Im}\, f$. Hence, $\mathrm{H}_n^{\mathfrak{a}}(M)/\mathfrak{a}^t\,\mathrm{H}_n^{\mathfrak{a}}(M)$ is minimax, as desired.

(ii) The proof is similar to the above proof. □

In the next Theorem we obtain some equivalent conditions for minimaxness of local homology modules.

***Theorem* 2.9** — *Let $\mathfrak{a}$ be an ideal of a local ring $(R, \mathfrak{m})$ and M an artinian R-module, and let $n \in \mathbb{N}$. Then the following statements are equivalent:*

(i) $\mathrm{H}_i^{\mathfrak{a}}(M)$ *is minimax for all $i < n$;*

(ii) There exists a positive integer t such that $\mathfrak{a}^t \, \mathrm{H}_i^{\mathfrak{a}}(M)$ is finite for all $i < n$;

(iii) There exists a positive integer t such that $\mathfrak{a}^t \, \mathrm{H}_i^{\mathfrak{a}}(M)$ is minimax for all $i < n$;

(iv) There exists a positive integer t such that $\mathfrak{a}^t \, \mathrm{H}_i^{\mathfrak{a}}(M)$ is $\mathcal{C}$-minimax for all $i < n$;

(v) $\mathrm{H}_i^{\mathfrak{a}}(M)$ *is $\mathcal{C}$-minimax for all $i < n$;*

(vi) There exists a positive integer t such that $\mathfrak{a}^t \, \mathrm{H}_i^{\mathfrak{a}}(M)$ is coatomic for all $i < n$.

PROOF : $(i) \Rightarrow$ (ii): By Theorem 2.6(i).

$(ii) \Rightarrow (iii) \Rightarrow (iv)$: Any finite R-module is minimax and any minimax R-module is $\mathcal{C}$-minimax.

$(iv) \Rightarrow (v)$: We proceed by induction on n. When $n = 0$, for all positive integers t the canonical epi-morphisms $M \to M/\mathfrak{a}^t M$ induces an epi-morphism $M \to \mathrm{H}_0^{\mathfrak{a}}(M)$. Thus $\mathrm{H}_0^{\mathfrak{a}}(M)$ is homomorphic image a minimax module and so is minimax. The result follows in this case. Now suppose, inductively that $n > 0$ and the result is true for $n - 1$. It is sufficient to show that $\mathrm{H}_{n-1}^{\mathfrak{a}}(M)$ is $\mathcal{C}$-minimax. By the inductive hypothesis $\mathrm{H}_i^{\mathfrak{a}}(M)$ is $\mathcal{C}$-minimax for all $i < n - 1$. Thus by Theorem 2.8(ii) $\mathrm{H}_{n-1}^{\mathfrak{a}}(M)/\mathfrak{a}^t \, \mathrm{H}_{n-1}^{\mathfrak{a}}(M)$ is $\mathcal{C}$-minimax. But $\mathfrak{a}^t \, \mathrm{H}_{n-1}^{\mathfrak{a}}(M)$ is $\mathcal{C}$-minimax. Therefore $\mathrm{H}_{n-1}^{\mathfrak{a}}(M)$ is $\mathcal{C}$-minimax, by Proposition 2.5(ii).

$(v) \Rightarrow (i)$: We proceed by induction on n. Suppose, inductively that $n > 0$ and the result is true for $n - 1$. We must show that $\mathrm{H}_{n-1}^{\mathfrak{a}}(M)$ is minimax. Since $\mathrm{H}_{n-1}^{\mathfrak{a}}(M)$ is $\mathcal{C}$-minimax, there exists an integer t such that $\mathfrak{a}^t \, \mathrm{H}_{n-1}^{\mathfrak{a}}(M)$ is minimax, by Theorem 2.4. By the inductive hypothesis $\mathrm{H}_i^{\mathfrak{a}}(M)$ is minimax for all $i < n - 1$. Thus, by Theorem 2.8(i), $\mathrm{H}_{n-1}^{\mathfrak{a}}(M)/\mathfrak{a}^t \, \mathrm{H}_{n-1}^{\mathfrak{a}}(M)$ is minimax. Now, Proposition 2.5(i) implies that $\mathrm{H}_{n-1}^{\mathfrak{a}}(M)$ is minimax and the proof is complete.

$(v) \Rightarrow (vi)$: By Theorem 2.6(ii).

$(vi) \Rightarrow (ii)$: See the proof of Corollary 2.7. □

We need the following Lemma in the proof of the next Theorem.

Lemma 2.10 — Let $(R, \mathfrak{m})$ be a local ring. If M is a coatomic and minimax R-module, then M is a finite R-module.

PROOF : Since M is minimax, there exists a finite submodule N of M, such that M/N is artinian. But M is coatomic and every artinian homomorphic image of a coatomic module has finite length.

Hence M/N has finite length and we conclude that M/N is finite and so M is finite.

***Theorem* 2.11** — *Let $\mathfrak{a}$ be an ideal of a local ring $(R, \mathfrak{m})$ and M an artinian R-module, and let $n \in \mathbb{N}$. Then the following statements are equivalent:*

(i) $\mathrm{H}_i^{\mathfrak{a}}(M)$ is finite for all $i < n$;

(ii) $\mathrm{H}_i^{\mathfrak{a}}(M)$ is coatomic for all $i < n$;

(iii) There exists a positive integer t such that $\mathfrak{m}^t\,\mathrm{H}_i^{\mathfrak{a}}(M)$ is finite for all $i < n$.

PROOF : (i)⇒ (ii): Finite modules are coatomic.

(ii) ⇔ (iii): By [19, Satz 2.4].

(iii) ⇒ (i) : Since $\mathfrak{a}^t\,\mathrm{H}_i^{\mathfrak{a}}(M) \subseteq \mathfrak{m}^t\,\mathrm{H}_i^{\mathfrak{a}}(M)$ for all $i < n$, $\mathfrak{a}^t\,\mathrm{H}_i^{\mathfrak{a}}(M)$ is finite for all $i < n$ and so $\mathrm{H}_i^{\mathfrak{a}}(M)$ is minimax for all $i < n$ by Theorem 2.9. But (ii) ⇔ (iii) implies that $\mathrm{H}_i^{\mathfrak{a}}(M)$ is coatomic for all $i < n$. Now the result follows by Lemma 2.10. □

As a consequence of Theorem 2.11, we derive the following result; which is a main result about finite length local homology modules.

***Theorem* 2.12** — *Let $\mathfrak{a}$ be an ideal of a local ring $(R, \mathfrak{m})$ and M an artinian R-module, and let $n \in \mathbb{N}$. Then the following statements are equivalent:*

(i) $\mathrm{H}_i^{\mathfrak{a}}(M)$ is of finite length for all $i < n$;

(ii) There exists a positive integer t such that $\mathfrak{m}^t\,\mathrm{H}_i^{\mathfrak{a}}(M) = 0$ for all $i < n$.

PROOF : (i)⇒ (ii): It follows by [2, Corollary 7.2.12].

(ii) ⇒ (i): It follows by Theorem 2.11 and [3, Proposition 4.7]. □

Now, we obtain immediately the following corollary.

Corollary 2.13 — Let $\mathfrak{a}$ be an ideal of a local ring $(R, \mathfrak{m})$ and M an artinian R-module. Then

$$\inf\{i \in \mathbb{N} : \mathrm{H}_i^{\mathfrak{a}}(M) \text{ is not of finite length}\} = \inf\{i \in \mathbb{N} : \mathfrak{m} \nsubseteq \sqrt{(0 : \mathrm{H}_i^{\mathfrak{a}}(M))}\}.$$

Lemma 2.14 — Let $\mathfrak{a}$ be an ideal of a local ring $(R, \mathfrak{m})$ and $0 \to N \to M \to L \to 0$ be an exact sequence of R-modules. If there exists an integer n such that $\mathfrak{a}^n N$ and $\mathfrak{a}^n L$ are finite, then there exists an integer t such that $\mathfrak{a}^t M$ is finite.

PROOF : We can assume that N is a submodule of M and $M/N = L$. Thus $\mathfrak{a}^n M/(\mathfrak{a}^n M \cap N) \simeq \mathfrak{a}^n L$ is finite. Let U be a complement of $\mathfrak{a}^n M \cap N$ in $\mathfrak{a}^n M$. Then U is finite, being essential cover of the finite module $\mathfrak{a}^n M/(\mathfrak{a}^n M \cap N)$. But $\mathfrak{a}^n M = (\mathfrak{a}^n M \cap N) + U$ and so $\mathfrak{a}^{2n} M = \mathfrak{a}^n(\mathfrak{a}^n M \cap N) + \mathfrak{a}^n U$. It follows that $\mathfrak{a}^{2n} M$ is finite, as required.

***Theorem* 2.15** — *Let $\mathfrak{a}$ be an ideal of a local ring $(R, \mathfrak{m})$ and M an artinian R-module, and let k be a non-negative integer. Then the following statements are equivalent:*

(i) $\mathrm{H}_i^{\mathfrak{a}}(M)$ is finite for all $i > k$;

(ii) $\mathrm{H}_i^{\mathfrak{a}}(M)$ is minimax for all $i > k$;

(iii) $\mathrm{H}_i^{\mathfrak{a}}(M)$ is $\mathcal{C}$-minimax for all $i > k$;

(iv) $\mathrm{H}_i^{\mathfrak{a}}(M)$ is coatomic for all $i > k$;

(v) There exists a positive integer t such that $\mathfrak{a}^t \mathrm{H}_i^{\mathfrak{a}}(M)$ is finite for all $i > k$.

PROOF : (i) ⇒ (ii) ⇒ (iii): Any finite R-module is minimax and any minimax R-module is $\mathcal{C}$-minimax.

(iii) ⇒ v): By Corollary 2.7 there exists a positive integer t_i such that $\mathfrak{a}^{t_i} \mathrm{H}_i^{\mathfrak{a}}(M)$ is finite for any $i > k$. Since $\mathrm{H}_i^{\mathfrak{a}}(M) = 0$ for all $i > \mathrm{Ndim}_R M$ by [3, Proposition 4.8], we can find an integer t such that $\mathfrak{a}^t \mathrm{H}_i^{\mathfrak{a}}(M)$ is finite for all $i > k$.

(i) ⇒ (iv): Any finite R-module is coatomic.

(iv) ⇒ (iii): Any coatomic R-module is $\mathcal{C}$-minimax.

(v) ⇒ (i): We proceed by induction on $n := \mathrm{Ndim}_R M$. Let $n = 0$. Since $\mathrm{H}_i^{\mathfrak{a}}(M) = 0$ for all $i > 0$, by [3, Proposition 4.8], the result follows in this case. Now suppose, inductively that $n > 0$ and the result is true for $n-1$. By [3, Corollary 4.5], we can replace M by $\cap_{n>0}\mathfrak{a}^n M$. But $\cap_{n>0}\mathfrak{a}^n M = \mathfrak{a}^k M$ for some $k \in \mathbb{N}$ and so we may assume that $\mathfrak{a}M = M$. Since M is artinian, $xM = M$ for some $x \in \mathfrak{a}$ and so $x^t M = M$. Thus for all $i > k$, the exact sequence

$$0 \to (0 :_M x^t) \to M \xrightarrow{x^t} M \to 0$$

implies that

$$\cdots \to \mathrm{H}_{i+1}^{\mathfrak{a}}(M) \to \mathrm{H}_i^{\mathfrak{a}}(0 :_M x^t) \xrightarrow{\varphi_i} \mathrm{H}_i^{\mathfrak{a}}(M) \xrightarrow{x^t} \mathrm{H}_i^{\mathfrak{a}}(M) \to \cdots.$$

Lemma 2.14 implies that there exists an integer s such that $\mathfrak{a}^s \mathrm{H}_i^{\mathfrak{a}}(0 :_M x^t)$ is finite for all $i > k$. Now, since $\mathrm{Ndim}_R(0 :_M x^t) \leq n-1$, (see [4, Lemma 4.7]), the induction hypothesis implies that $\mathrm{H}_i^{\mathfrak{a}}(0 :_M x^t)$ is finite for all $i > k$ and so we have the exact sequence

$$0 \to \mathrm{Im}\,\varphi_i \to \mathrm{H}_i^{\mathfrak{a}}(M) \to x^t \mathrm{H}_i^{\mathfrak{a}}(M) \to 0$$

for all $i > k$. Since $\operatorname{Im} \varphi_i$ and $x^t \operatorname{H}_i^{\mathfrak{a}}(M)$ are finite we conclude that $\operatorname{H}_i^{\mathfrak{a}}(M)$ is finite for all $i > k$. The proof is complete. □

In the reminder, by matlis duality functor, we obtain some results similar to the above results for local cohomology modules. We need the following lemmas for the proof of our results.

Lemma 2.16 — Let R be a complete local ring and M be a minimax R-module. Then $\operatorname{D}(M)$ is a minimax R-module.

PROOF : There exists a finite submodule N of M, such that M/N is artinian. Thus we have the following exact sequence:

$$0 \to N \to M \to M/N \to 0$$

and so we obtain

$$0 \to \operatorname{D}(M/N) \to \operatorname{D}(M) \to \operatorname{D}(N) \to 0.$$

But it is well known that, over a complete local ring, matlis dual of finite modules are artinian and matlis dual of artinian modules are finite. Thus by the above exact sequence we conclude that $\operatorname{D}(M)$ is a minimax R-module.

Lemma 2.17 — Let M be an R-module. Then

(i) M is artinian R-module $\Leftrightarrow \widehat{R} \otimes_R M$ is an artinian $\widehat{R}$-module.

(ii) M is minimax R-module $\Rightarrow \widehat{R} \otimes_R M$ is a minimax $\widehat{R}$-module.

PROOF : See [8, 1.14 and 1.18].

Lemma 2.18 — Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals of a local ring $(R, \mathfrak{m})$ and M a finite R-module. Let i be a natural number. Then

(i) If $\mathfrak{b} \operatorname{H}_{\mathfrak{a}}^i(M)$ is artinian R-module, then $(\mathfrak{b}\widehat{R}) \operatorname{H}_{\mathfrak{a}\widehat{R}}^i(M \otimes_R \widehat{R})$ is artinian $\widehat{R}$-module.

(ii) If $\mathfrak{b} \operatorname{H}_{\mathfrak{a}}^i(M)$ is minimax R-module, then $(\mathfrak{b}\widehat{R}) \operatorname{H}_{\mathfrak{a}\widehat{R}}^i(M \otimes_R \widehat{R})$ is minimax $\widehat{R}$-module.

PROOF : (i) By Lemma 2.17(i), $\mathfrak{b} \operatorname{H}_{\mathfrak{a}}^i(M) \otimes_R \widehat{R}$ is artinian $\widehat{R}$-module. Since $\mathfrak{b} \operatorname{H}_{\mathfrak{a}}^i(M) \otimes_R \widehat{R} \cong (\mathfrak{b}\widehat{R}) \operatorname{H}_{\mathfrak{a}\widehat{R}}^i(M \otimes_R \widehat{R})$ by [2, 4.3.2], we conclude that $(\mathfrak{b}\widehat{R}) \operatorname{H}_{\mathfrak{a}\widehat{R}}^i(M \otimes_R \widehat{R})$ is artinian $\widehat{R}$-module.

(ii) By Lemma 2.17(ii), $\mathfrak{b} \operatorname{H}_{\mathfrak{a}}^i(M) \otimes_R \widehat{R}$ is minimax $\widehat{R}$-module and by [2, 4.3.2] we have $\mathfrak{b} \operatorname{H}_{\mathfrak{a}}^i(M) \otimes_R \widehat{R} \cong (\mathfrak{b}\widehat{R}) \operatorname{H}_{\mathfrak{a}\widehat{R}}^i(M \otimes_R \widehat{R})$. Thus $(\mathfrak{b}\widehat{R}) \operatorname{H}_{\mathfrak{a}\widehat{R}}^i(M \otimes_R \widehat{R})$ is minimax $\widehat{R}$-module.

***Theorem* 2.19** — *Let $\mathfrak{a}$ be an ideal of a local ring $(R, \mathfrak{m})$ and M a finite R-module. Let i be a natural number.*

(i) If $\operatorname{H}_{\mathfrak{a}}^i(M)$ is minimax, then there exists a positive integer t such that $\mathfrak{a}^t \operatorname{H}_{\mathfrak{a}}^i(M)$ is artinian.

(ii) If $\mathrm{H}^i_{\mathfrak{a}}(M)$ is $\mathcal{C}$-minimax, then there exist positive integers s and u such that $\mathfrak{m}^s\mathfrak{a}^u\,\mathrm{H}^i_{\mathfrak{a}}(M)$ is artinian.

PROOF : (i) By Lemma 2.18(ii) $\mathrm{H}^i_{\mathfrak{a}\widehat{R}}(M\otimes_R\widehat{R})$ is minimax $\widehat{R}$-module and so Lemma 2.16 implies that $\mathrm{D}(\mathrm{H}^i_{\mathfrak{a}\widehat{R}}(M\otimes_R\widehat{R}))$ is minimax $\widehat{R}$-module. By [3, Proposition 3.3], we have $\mathrm{D}(\mathrm{H}^i_{\mathfrak{a}\widehat{R}}(M\otimes_R\widehat{R}))\simeq \mathrm{H}_i^{\mathfrak{a}\widehat{R}}(\mathrm{D}(M\otimes_R\widehat{R}))$. Hence $\mathrm{H}_i^{\mathfrak{a}\widehat{R}}(\mathrm{D}(M\otimes_R\widehat{R}))$ is minimax $\widehat{R}$-module. By Theorem 2.6(i), there exists a positive integer t such that $(\mathfrak{a}\widehat{R})^t\,\mathrm{H}_i^{\mathfrak{a}\widehat{R}}(\mathrm{D}(M\otimes_R\widehat{R}))$ is finite $\widehat{R}$-module. But $(\mathfrak{a}\widehat{R})^t\,\mathrm{H}_i^{\mathfrak{a}\widehat{R}}(\mathrm{D}(M\otimes_R\widehat{R}))\cong \mathrm{D}((\mathfrak{a}\widehat{R})^t\,\mathrm{H}^i_{\mathfrak{a}\widehat{R}}(M\otimes_R\widehat{R}))$ and so $\mathrm{D}((\mathfrak{a}\widehat{R})^t\,\mathrm{H}^i_{\mathfrak{a}\widehat{R}}(M\otimes_R\widehat{R}))$ is finite $\widehat{R}$-module. Hence $(\mathfrak{a}\widehat{R})^t\,\mathrm{H}^i_{\mathfrak{a}\widehat{R}}(M\otimes_R\widehat{R})$ is artinian $\widehat{R}$-module. Since $(\mathfrak{a}\widehat{R})^t\,\mathrm{H}^i_{\mathfrak{a}\widehat{R}}(M\otimes_R\widehat{R})\cong \mathfrak{a}^t\,\mathrm{H}^i_{\mathfrak{a}}(M)\otimes_R\widehat{R}$ we see that $\mathfrak{a}^t\,\mathrm{H}^i_{\mathfrak{a}}(M)\otimes_R\widehat{R}$ is artinian $\widehat{R}$-module. Now Lemma 2.17(i) implies that $\mathfrak{a}^t\,\mathrm{H}^i_{\mathfrak{a}}(M)$ is artinian R-module.

(ii) Since $\mathrm{H}^i_{\mathfrak{a}}(M)$ is $\mathcal{C}$-minimax, by Theorem 2.4 there exists a positive integer t such that $\mathfrak{m}^t\,\mathrm{H}^i_{\mathfrak{a}}(M)$ is minimax. Now Lemma 2.18(ii) implies that $(\mathfrak{m}\widehat{R})^t\,\mathrm{H}^i_{\mathfrak{a}\widehat{R}}(M\otimes_R\widehat{R})$ is minimax $\widehat{R}$-module. Thus $\mathrm{D}((\mathfrak{m}\widehat{R})^t\,\mathrm{H}^i_{\mathfrak{a}\widehat{R}}(M\otimes_R\widehat{R}))$ is minimax $\widehat{R}$-module. Hence $(\mathfrak{m}\widehat{R})^t\,\mathrm{H}_i^{\mathfrak{a}\widehat{R}}(\mathrm{D}(M\otimes_R\widehat{R}))$ is minimax $\widehat{R}$-module. Now Theorem 2.4 implies that $\mathrm{H}_i^{\mathfrak{a}\widehat{R}}(\mathrm{D}(M\otimes_R\widehat{R}))$ is $\mathcal{C}$-minimax $\widehat{R}$-module and by Theorem 2.6(ii) there exists a positive integer u such that $(\mathfrak{a}\widehat{R})^u\,\mathrm{H}_i^{\mathfrak{a}\widehat{R}}(\mathrm{D}(M\otimes_R\widehat{R}))$ is coatomic $\widehat{R}$-module. Hence there exists a positive integer s such that $(\mathfrak{m}\widehat{R})^s(\mathfrak{a}\widehat{R})^u\,\mathrm{H}_i^{\mathfrak{a}\widehat{R}}(\mathrm{D}(M\otimes_R\widehat{R}))$ is finite $\widehat{R}$-module and so $\mathrm{D}((\mathfrak{m}\widehat{R})^s(\mathfrak{a}\widehat{R})^u\,\mathrm{H}^i_{\mathfrak{a}\widehat{R}}(M\otimes_R\widehat{R}))$ is finite $\widehat{R}$-module. Thus $(\mathfrak{m}\widehat{R})^s(\mathfrak{a}\widehat{R})^u\,\mathrm{H}^i_{\mathfrak{a}\widehat{R}}(M\otimes_R\widehat{R})\simeq \mathfrak{m}^s\mathfrak{a}^u\,\mathrm{H}^i_{\mathfrak{a}}(M)\otimes_R\widehat{R}$ is artinian $\widehat{R}$-module. Therefore Lemma 2.17(i) implies that $\mathfrak{m}^s\mathfrak{a}^u\,\mathrm{H}^i_{\mathfrak{a}}(M)$ is artinian R-module and the proof is complete. □

There are some results about the least integer i such that $\mathrm{H}^i_{\mathfrak{a}}(M)$ is not artinian (see [16] and [1]). In the next result, we give another characterization of artinian local cohomology modules.

***Theorem* 2.20** — *Let $\mathfrak{a}$ be an ideal of a local ring $(R,\mathfrak{m})$ and M a finite R-module, and let $n\in\mathbb{N}$. Then the following statements are equivalent:*

(i) $\mathrm{H}^i_{\mathfrak{a}}(M)$ is artinian for all $i<n$;

(ii) There exists a positive integer t such that $\mathfrak{m}^t\,\mathrm{H}^i_{\mathfrak{a}}(M)$ is artinian for all $i<n$.

PROOF : (i) $\Rightarrow$ (ii): It is obviously true.

(ii) $\Rightarrow$ (i): Lemma 2.18(i) implies that $(\mathfrak{m}\widehat{R})^t\,\mathrm{H}^i_{\mathfrak{a}\widehat{R}}(M\otimes_R\widehat{R})$ is artinian $\widehat{R}$-module for all $i<n$. Thus $\mathrm{D}((\mathfrak{m}\widehat{R})^t\,\mathrm{H}^i_{\mathfrak{a}\widehat{R}}(M\otimes_R\widehat{R}))\cong(\mathfrak{m}\widehat{R})^t\,\mathrm{H}_i^{\mathfrak{a}\widehat{R}}(\mathrm{D}(M\otimes_R\widehat{R}))$ is finite $\widehat{R}$-module for all $i<n$. Now Theorem 2.11 implies that $\mathrm{H}_i^{\mathfrak{a}\widehat{R}}(\mathrm{D}(M\otimes_R\widehat{R}))$ is finite $\widehat{R}$-module for all $i<n$. Hence $\mathrm{D}(\mathrm{H}^i_{\mathfrak{a}\widehat{R}}(M\otimes_R\widehat{R}))$ is finite $\widehat{R}$-module for all $i<n$ and so $\mathrm{H}^i_{\mathfrak{a}\widehat{R}}(M\otimes_R\widehat{R})$ is artinian $\widehat{R}$-module for all $i<n$. Since $\mathrm{H}^i_{\mathfrak{a}\widehat{R}}(M\otimes_R\widehat{R})\cong\mathrm{H}^i_{\mathfrak{a}}(M)\otimes_R\widehat{R}$ we conclude that $\mathrm{H}^i_{\mathfrak{a}}(M)$ is artinian R-module for all $i<n$, by

Lemma 2.17(i). □

The following result is a characterization of finite length local cohomology modules.

***Theorem* 2.21** — *Let $\mathfrak{a}$ be an ideal of a local ring $(R, \mathfrak{m})$ and M a finite R-module and let $n \in \mathbb{N}$. Then the following statements are equivalent:*

(i) $\mathrm{H}^i_{\mathfrak{a}}(M)$ *is of finite length for all* $i < n$*;*

(ii) There exists a positive integer t *such that* $\mathfrak{m}^t \mathrm{H}^i_{\mathfrak{a}}(M) = 0$ *for all* $i < n$*.*

PROOF : (i) ⇒ (ii): It follows by [2, Corollary 7.2.12].

(ii) ⇒ (i): It follows by Theorem 2.20 and [2, 9.1.2]. □

By using the above Theorem we have:

Corollary 2.22 — Let $\mathfrak{a}$ be an ideal of a local ring $(R, \mathfrak{m})$ and M a finite R-module. Then

$$\inf\left\{i \in \mathbb{N} : \mathrm{H}^i_{\mathfrak{a}}(M) \text{ is not of finite length}\right\} = \inf\{i \in \mathbb{N} : \mathfrak{m} \nsubseteq \sqrt{(0 : \mathrm{H}^i_{\mathfrak{a}}(M))}\}.$$

***Theorem* 2.23** — *Let $\mathfrak{a}$ be an ideal of a local ring $(R, \mathfrak{m})$ and M a finite R-module, and let $n \in \mathbb{N}$. Then the following statements are equivalent:*

(i) $\mathrm{H}^i_{\mathfrak{a}}(M)$ *is minimax for all* $i < n$*;*

(ii) There exists a positive integer t such that $\mathfrak{a}^t \mathrm{H}^i_{\mathfrak{a}}(M)$ *is artinian for all* $i < n$*;*

(iii) There exists a positive integer t such that $\mathfrak{a}^t \mathrm{H}^i_{\mathfrak{a}}(M)$ *is minimax for all* $i < n$*;*

(iv) $\mathrm{H}^i_{\mathfrak{a}}(M)$ *is* $\mathcal{C}$*-minimax for all* $i < n$*;*

(v) There exist positive integers s *and* u *such that* $\mathfrak{m}^s \mathfrak{a}^u \mathrm{H}^i_{\mathfrak{a}}(M)$ *is artinian for all* $i < n$*.*

PROOF : (i)⇒ (ii): By Theorem 2.19(i).

(ii) ⇒ (iii): Any artinian R-module is minimax.

(i) ⇒ (iv): Any minimax R-module is $\mathcal{C}$-minimax.

(iv) ⇒ (v): By Theorem 2.19(ii).

(v) ⇒ (ii): Take $t := s + u$. Thus $\mathfrak{a}^t \mathrm{H}^i_{\mathfrak{a}}(M) \leq \mathfrak{m}^s \mathfrak{a}^u \mathrm{H}^i_{\mathfrak{a}}(M)$ is artinian for all $i < n$.

(iii) ⇒ (i): By [5, Lemma 2.2]. □

***Theorem* 2.24** — *Let $\mathfrak{a}$ be an ideal of a local ring $(R, \mathfrak{m})$ and M a finite R-module, and let $k \in \mathbb{N}$. Then the following statements are equivalent:*

(i) $\mathrm{H}^i_{\mathfrak{a}}(M)$ *is artinian for all* $i > k$*;*

(ii) $\mathrm{H}^i_{\mathfrak{a}}(M)$ *is minimax for all* $i > k$*;*

(iii) $\mathrm{H}^i_{\mathfrak{a}}(M)$ *is* $\mathcal{C}$*-minimax for all* $i > k$*;*

(iv) There exist positive integers s *and* u *such that* $\mathfrak{m}^s\mathfrak{a}^u\,\mathrm{H}^i_{\mathfrak{a}}(M)$ *is artinian for all* $i > k$*;*

(v) There exists a positive integer t such that $\mathfrak{a}^t\,\mathrm{H}^i_{\mathfrak{a}}(M)$ *is artinian for all* $i > k$*.*

PROOF : (i) $\Rightarrow$ (ii) $\Rightarrow$ iii) and iv) $\Rightarrow$ v): Clear.

(iii) $\Rightarrow$ (iv): By Theorem 2.19(ii).

(v) $\Rightarrow$ (i): By Lemma 2.18(i), $(\mathfrak{a}\widehat{R})^t\,\mathrm{H}^i_{\mathfrak{a}\widehat{R}}(M\otimes_R\widehat{R})$ is artinian $\widehat{R}$-module for all $i > k$. Thus $\mathrm{D}((\mathfrak{a}\widehat{R})^t\,\mathrm{H}^i_{\mathfrak{a}\widehat{R}}(M\otimes_R\widehat{R}))\cong(\mathfrak{a}\widehat{R})^t\,\mathrm{H}_i^{\mathfrak{a}\widehat{R}}(\mathrm{D}(M\otimes_R\widehat{R}))$ is finite $\widehat{R}$-module for all $i > k$. Now Theorem 2.15, implies that $\mathrm{H}_i^{\mathfrak{a}\widehat{R}}(\mathrm{D}(M\otimes_R\widehat{R}))$ is finite $\widehat{R}$-module for all $i > k$. Hence $\mathrm{D}(\mathrm{H}^i_{\mathfrak{a}\widehat{R}}(M\otimes_R\widehat{R}))$ is finite $\widehat{R}$-module for all $i > k$ and so $\mathrm{H}^i_{\mathfrak{a}\widehat{R}}(M\otimes_R\widehat{R}))$ is artinian $\widehat{R}$-module for all $i > k$. Since $\mathrm{H}^i_{\mathfrak{a}\widehat{R}}(M\otimes_R\widehat{R})\cong\mathrm{H}^i_{\mathfrak{a}}(M)\otimes_R\widehat{R}$ by Lemma 2.17(i) we conclude that $\mathrm{H}^i_{\mathfrak{a}}(M)$ is artinian R-module for all $i > k$ and the proof is complete. □

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