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ON THE LAPLACIAN SPECTRAL CHARACTERIZATION OF Π-SHAPE TREES[1]

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A Π-shape tree is a tree with exactly two vertices having the maximum degree three. In this paper, we classify the Π-shape trees into two types, and complete the spectral characterization for one type. Exactly, we prove that all graphs of this type are determined by their Laplacian spectra with some exceptions. Moreover, we give some L-cospectral mates of some graphs for another type.

Key words : Π-shape tree; Laplacian spectrum; cospectral graphs.

1. Introduction

Throughout this paper, we are concerned only with simple undirected graphs (loops and multiple edges are not allowed). Let G be a graph with vertex set $V(G)$ and edge set $E(G)$, and graph matrix $M = M(G)$. The notation $\Phi(G;\sigma) = \det(\sigma I - M(G))$ stands for the M-*characteristic polynomials* of G. The M-*spectrum* of G consists of all the eigenvalues (together with their multiplicities) of $M(G)$, denote by $Spec_M(G)$ for short, and the M-*spectral radius* (or M-*index*) is the largest M-eigenvalue of G. Graphs with the same M-spectrum are called M-*cospectral graphs*. A graph G

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is said to be determined by its M-spectrum (DMS*-graph* for simplicity) if there is no other non-isomorphic graph with the same M-spectrum, that is, $Spec_M(H) = Spec_M(G)$ implies $H \cong G$ for any graph H; a M*-cospectral mate* of G is a graph M-cospectral but not isomorphic to G.

Let D be the diagonal degree matrix of G. The graph matrix M is called the *adjacency matrix*, the *Laplacian matrix* and the *signless Laplacian matrix* of G if M equals $A(G)$, $L(G) = D - A(G)$ and $Q(G) = D + A(G)$, respectively; and the corresponding characteristic polynomial, eigenvalues as well as DMS-graphs (i.e., DAS-graph, DLS-graph and DQS-graph) are well defined respectively. Conventionally, the *adjacency eigenvalues* and the *Laplacian eigenvalues* of a graph G are non-increased, that is, $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ and $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n = 0$.

Van Dam and Haemers in [2] asked for a problem: "*Which graphs are determined by their spectrum*?" It seems to be a difficult problem far from resolved. For a survey of the subject, one can consult [2, 3]. Many results on A-cospectral mates-trees are reported, see [5-9] for details. Schwenk [4] proved: "Almost all trees are not DAS-graphs". However, we conjecture that almost all trees are DLS-graphs although some L-cospectral mates (For instance, see G' and H' in Fig. 1) are found [2, 16]. Notice that no L-cospectral mate trees are discovered in our memory [2, 6, 10-14]. Thus, *finding* L*-cospectral mate trees is an interesting problem*.

A *double starlike* is a tree if it has exactly two adjacent vertices of degree greater than two. A Π-*shape tree* is a tree with exactly two vertices having the maximum degree three, which is precisely denoted by $\Pi = \Pi(l_1, l_2, l_3, l_4, l_5)$ (see Fig. 1 for instance) where $\sum_{i=1}^{5} l_i + 2 = n$. Without loss of generality, we assume that $l_1 \geq l_4 \geq 1$, $l_3 \geq l_5 \geq 1$ and $l_2 \geq 0$ in the sequel. In [8], Liu and Huang classified the Π-shape trees into six types according to the number of their closed walks of length 6, and proved that one such type, i.e, $\Pi(1, 0, 1, 1, 1) \cup \Pi(l_1, l_2, 1, 1, 1)$ $(l_1 > 1, l_2 \geq 1)$, are determined by their adjacency spectra but there exist many non-isomorphic A-cospectral mates in other four types (see Example 3 in [8]). In this paper, we also classify the Π-*shape trees* into two types according to the parameter l_2, and denote by $\Pi^{=0}$ the Π-shape trees $\Pi(l_1, 0, l_3, l_4, l_5)$, and $\Pi^{\geq 1}$ the Π-shape tree $\Pi(l_1, l_2, l_3, l_4, l_5)$ where $l_2 \geq 1$. Clearly, $\Pi = \Pi^{=0} \cup \Pi^{\geq 1}$. Then we complete the spectral characterization for $\Pi^{=0}$. Exactly, we prove that all graphs of $\Pi^{=0}$ are determined by their Laplacian spectra with exceptions of L-cospectral mate trees: $\Pi(k + 2m + 2, 0, k, k + m, m)$ and $\Pi(k+2m+1, 0, k+m+1, k, m)$ for $k, m \geq 1$. From this result, we confirm that the *double starlike* tree is not DLS-graph although any *starlike tree* is DLS [11]. Moreover, we give some L-cospectral mates of some graphs for another type $\Pi^{\geq 1}$.

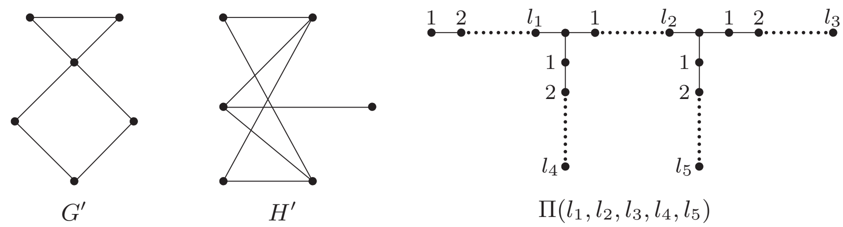


Figure 1: Graphs G', H' and $\Pi(l_1, l_2, l_3, l_4, l_5)$

2. Some Lemmas

In this section, we give some useful lemmas that are needed in the proof of our main conclusions. Firstly, we summarize some results of [2] and [15] in the following lemma.

Lemma 2.1 — Let G be a graph. For the adjacency matrix and the Laplacian matrix of G, the following invariants can be deduced from the spectrum.

(i) The number of vertices.

(ii) The number of edges.

(iii) Whether G is regular.

(iv) Whether G is regular with any fixed grith.

For the adjacency matrix, the following invariants follows from the spectrum.

(v) The number of closed walks of any length.

(vi) Whether G is bipartite.

For the Laplacian matrix, the following invariants follows from the spectrum.

(vii) The number of spanning trees.

(viii) The number of components.

(ix) The sum of the square of degrees of vertices.

Lemma 2.2 — ([1] Interlacing]). Suppose that A is a symmetric $n \times n$ matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Then the eigenvalues $\lambda'_1 \geq \lambda'_2 \geq \cdots \geq \lambda'_m$ of a principal submatrix of A of size m satisfy $\lambda_i \geq \lambda'_i \geq \lambda_{n-m+i}$ for $i = 1, 2, \ldots, m$.

For a graph G, let

$$\phi(G; \lambda) = a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n$$

be the adjacency characteristic polynomial of G with respect to $A(G)$. If there is no confusion, sometimes we write $\phi(G;\lambda)$ as $\phi(G)$. Let $G-v$ and $G-uv$ be the graph obtained from G by deleting a vertex v and an edge uv, respectively.

Lemma 2.3 — ([16] pp. 37]). *(i)* For any vertex u of the graph G,

$$\phi(G;\lambda)=\lambda\phi(G-u)-\sum_{uv\in E(G)}\phi(G-u-v)-2\sum_{Z\in\mathscr{C}(u)}\phi(G-V(Z));$$

where $\mathscr{C}(u)$ denotes the set of all cycles containing u.

(ii) For any edge uv of the graph G,

$$\phi(G;\lambda)=\phi(G-uv)-\sum_{uv\in E(G)}\phi(G-u-v)-2\sum_{Z\in\mathscr{C}(uv)}\phi(G-V(Z));$$

where $\mathscr{C}(uv)$ denotes the set of all cycles containing uv.

The *line graph* $\mathscr{L}(G)$ of a graph G is one whose vertices correspond the edges of G and two vertices in $\mathscr{L}(G)$ are adjacent if and only if the corresponding edges of G are adjacent.

Lemma 2.4 — [17]. Let G be a bipartite graph on n vertices. Then $\mu_i(G)=\lambda_i(\mathscr{L}(G))$ $+\,2$, for $i=1,2,\ldots,n-1$.

Lemma 2.5 — [18]. If $\mathscr{L}(G)\cong\mathscr{L}(H)$ with $\{G,H\}\neq\{K_3,K_{1,3}\}$, then $G\cong H$.

Lemma 2.6 — [19]. Let G be a graph with at least one edge. Then $\mu_1(G)\geq\Delta(G)+1$. Moreover, if G is connected, then the equality holds if and only if $\Delta(G)=n-1$.

Lemma 2.7 — [20, 21]. Let G be a connected graph. Then

$$\mu_1(G)\leq\max\{d(v)+m(v):v\in V\} \tag{1}$$

where $m(v)=\sum_{u\in N(v)}d(u)/d(v)$. Moreover, the equality in equation (1) holds if and only if G is a regular bipartite graph or semi-regular bipartite graph.

Lemma 2.8 — [1, 10]. Let P_n denote the path on n vertices. Then

$$\phi(P_n;\lambda)=\prod_{j=1}^{n}\big(\lambda-2\cos\frac{\pi j}{n+1}\big)=\frac{\sin\big((n+1)\arccos\lambda/2\big)}{\sin\big(\arccos\lambda/2\big)}.$$

Let $\lambda=2\cos\theta$. Set $t^{1/2}=e^{i\theta}$, it is useful to write the characteristic polynomial of P_n in the following form:

$$\phi(P_n;t^{1/2}+t^{-1/2})=t^{-n/2}(t^{n+1}-1)/(t-1). \tag{2}$$

Lemma 2.9 — [22]. If G is a graph, then its Laplacian eigenvalues and signless Laplacian eigenvalues are the same if and only if G is bipartite.

Let $N_G(H)$ be the number of subgraphs of graph G which are isomorphic to H and let $N_G(i)$ be the number of closed walks of length i in G. Let $N'_H(i)$ be the number of closed walks of H of length i which contains all the edges of H and $S_i(G)$ be the set of all the connected subgraphs H of G such that $N'_H(i) \neq 0$. Then

$$N_G(i) = \sum_{H \in S_i(G)} N_G(H) N'_H(i). \tag{3}$$

Using equation (3) one can obtain some formula for calculating the number of closed walks of length $5, 7$ for any graphs. In [23], Lemma 2.3(i) and 2.4(ii) respectively induce the formulae $N_G(5)$ and $N_G(7)$ of a graph G. Especially, we need following lemma.

Lemma 2.10 — The number of closed walks of length k $(k = 5, 7)$ of a graph G without cycles $C_i (i = 4, 5, 6, 7)$ are giving in the following.

(i) $N_G(5) = 30N_G(C_3) + 10N_G(G_1)$.
(ii) $N_G(7) = 126N_G(C_3) + 84N_G(G_1) + 14N_G(G_2) + 14N_G(G_3) + 28N_G(G_4)$ (see Fig.2).

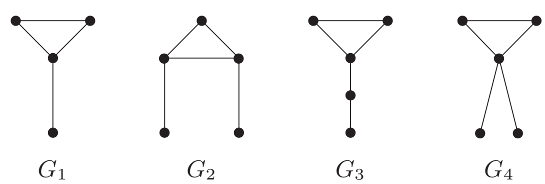


Figure 2: Some related graphs

3. Laplacian Spectral Characterization of $\Pi^{=0}$

In this section, we prove that the trees in $\Pi^{=0}$ are determined by their Laplacian spectra with exceptions. At first, we determine the rough structure of graphs L-cospectral with Π-shape tree.

Lemma 3.1 — Let G be a graph L-cospectral with Π. Then $\mu_1(G) < \dfrac{16}{3}$.

Proof : By Lemma 2.7, we have $\mu_1(\Pi) < 3 + \dfrac{7}{3} = \dfrac{16}{3}$. Since G and Π are L-cospectral, the result follows. □

Lemma 3.2 — Let G be a graph L-cospectral with Π. Then G is also a Π-shape tree.

PROOF : Suppose that G and Π are L-cospectral. Then by Lemma 3.1 $\mu_1(G) < \frac{16}{3}$. From Lemma 2.6 we have $\Delta(G) \leq 4$. Furthermore, by Lemma 2.1, G is also a tree, and G and Π have the same number of vertices, edges and the sum of the square of degrees of vertices. Denote by x_i and y_i the number of vertices of degree i in G and Π, respectively. Then we have the following three equations:

$$\sum_{i=1}^{4} x_i = n = \sum_{i=1}^{3} y_i, \tag{4}$$

$$\sum_{i=1}^{4} ix_i = 2n - 2 = \sum_{i=1}^{3} iy_i, \tag{5}$$

$$\sum_{i=1}^{4} i^2x_i = 4n - 2 = \sum_{i=1}^{3} i^2y_i. \tag{6}$$

By adding up these three equations with coefficients 1, $-3/2$ and $1/2$ respectively, we have $x_3 + 3x_4 = 2$.

Clearly, $x_3 = 2$ and $x_4 = 0$. From equation (4) and equation (5) we have $x_1 = 4$, and $x_2 = n-6$. Thus, G is a Π-shape tree. □

Let $\mathcal{H}(k_1, k_3, k_4, k_5)$ be the line graph of $\Pi(k_1, 0, k_3, k_4, k_5)$ shown in Fig. 3. Without loss of generality, we assume $k_1 \geq k_4 \geq 1$ and $k_3 \geq k_5 \geq 1$ in the sequel. We have the following lemma.

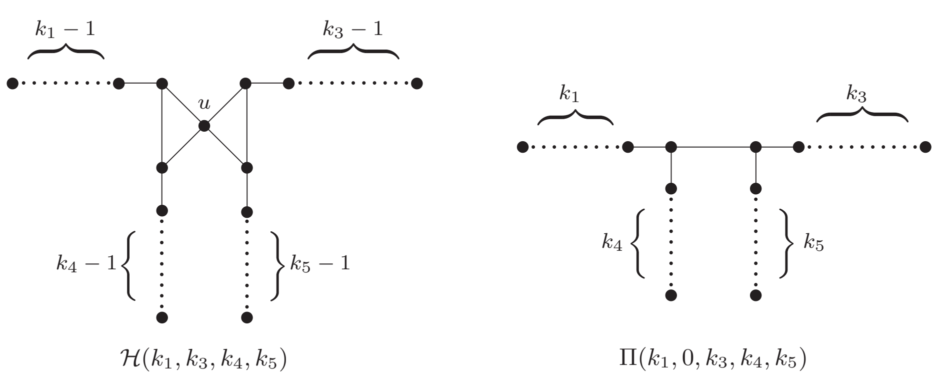


Figure 3: Graphs $\mathcal{H}(k_1, k_3, k_4, k_5)$ and $\Pi(k_1, 0, k_3, k_4, k_5)$

Lemma 3.3 — No two non-isomorphic graphs in $\mathcal{H}(k_1, k_3, k_4, k_5)$ have the same adjacency spectrum except for $(k_1, k_3, k_4, k_5) = (k + 2m + 1, k + m + 1, k, m)$ and $(k + 2m + 2, k, k + m, m)$ where $k, m \geq 1$. Moreover, $\mathcal{H}(k + 2m + 1, k + m + 1, k, m)$ and $\mathcal{H}(k + 2m + 2, k, k + m, m)$ are unique A-cospectral mate in $\mathcal{H}(k_1, k_3, k_4, k_5)$.

PROOF : Suppose that graphs $H_1 = \mathcal{H}(k_1, k_3, k_4, k_5)$ $(k_1 \geq k_4 \geq 1, k_3 \geq k_5 \geq 1)$ and $H_2 = \mathcal{H}(l_1, l_3, l_4, l_5)$ $(l_1 \geq l_4 \geq 1, l_3 \geq l_5 \geq 1)$ have the same adjacency spectrum. Applying Lemma 2.3 for the degree 4 vertex u of H_1 (see Fig. 3) we have

$$\begin{aligned}\phi(H_1;\lambda) &= \lambda\phi(P_{k_1+k_4})\phi(P_{k_3+k_5}) - \big(\phi(P_{k_1-1})\phi(P_{k_4})\phi(P_{k_3+k_5})\\ &\quad +\phi(P_{k_3-1})\phi(P_{k_5})\phi(P_{k_1+k_4}) + \phi(P_{k_4-1})\phi(P_{k_1})\phi(P_{k_3+k_5})\\ &\quad +\phi(P_{k_5-1})\phi(P_{k_3})\phi(P_{k_1+k_4})\big) - 2\big(\phi(P_{k_3-1})\phi(P_{k_5-1})\phi(P_{k_1+k_4})\\ &\quad +\phi(P_{k_1-1})\phi(P_{k_4-1})\phi(P_{k_3+k_5})\big).\end{aligned} \tag{7}$$

According to equation (2), it can be computed by using Maple 9.50 that

$$\phi(H_1; t^{1/2} + t^{-1/2})(t-1)^3 t^{(k_1+k_3+k_4+k_5+1)/2} = \psi_1(t) + \psi_2(t) + \psi_3(t) + \psi_4(t) + \psi_5(t) \tag{8}$$

where

$$\begin{aligned}\psi_1(t) &= -1 + 4t + 4t^{3/2} + t^2;\\ \psi_2(t) &= (t^{k_1} + t^{k_3} + t^{k_4} + t^{k_5})(-t - 2t^{3/2} - t^2);\\ \psi_3(t) &= (t^{k_1+k_4} + t^{k_3+k_5})(t + 2t^{3/2} - 2t^{5/2} - t^3);\\ \psi_4(t) &= (t^{k_1+k_3+k_4} + t^{k_1+k_3+k_5} + t^{k_3+k_4+k_5} + t^{k_1+k_4+k_5})(t^2 + 2t^{5/2} + t^3);\\ \psi_5(t) &= t^{k_1+k_3+k_4+k_5}(-t^2 - 4t^{5/2} - 4t^3 + t^4).\end{aligned} \tag{9}$$

Similarly, we obtain

$$\phi(H_2; t^{1/2} + t^{-1/2})(t-1)^3 t^{(l_1+l_3+l_4+l_5+1)/2} = \psi_1(t) + \varphi_2(t) + \varphi_3(t) + \varphi_4(t) + \varphi_5(t) \tag{10}$$

where $\varphi_2(t), \varphi_3(t), \varphi_4(t)$ and $\varphi_5(t)$ are obtained from $\psi_2(t), \psi_3(t), \psi_4(t)$ and $\psi_5(t)$ by replacing the parameters k_1, k_3, k_4 and k_5 with l_1, l_3, l_4 and l_5 respectively.

By Lemma 2.1(i) we have

$$k_1 + k_3 + k_4 + k_5 = l_1 + l_3 + l_4 + l_5. \tag{11}$$

From equation (8) and equation (10), $\phi(H_1; t^{1/2} + t^{-1/2}) = \phi(H_2; t^{1/2} + t^{-1/2})$ implies that

$$\psi_2(t) + \psi_3(t) + \psi_4(t) = \varphi_2(t) + \varphi_3(t) + \varphi_4(t). \tag{12}$$

From equation (9) we see that the leading terms of equation (12) must appear in $\psi_4(t)$ and $\varphi_4(t)$. Without loss of generality, we may assume that $k_4 \geq k_5$ and $l_4 \geq l_5$ by the symmetry. Then $t^{k_1+k_3+k_4+3}$ and $t^{l_1+l_3+l_4+3}$ are the leading terms of the both sides in equation (12). Thus, $k_1 + k_3 + k_4 = l_1 + l_3 + l_4$, combine with equation (11) we have

$$k_5 = l_5. \tag{13}$$

Remove the identical terms from the both sides of equation (12) we get

$$\eta_2(t) + \eta_3(t) + \eta_4(t) = \vartheta_2(t) + \vartheta_3(t) + \vartheta_4(t). \tag{14}$$

where

$$\begin{aligned} \eta_2(t) &= (t^{k_1} + t^{k_3} + t^{k_4})(-t - 2t^{3/2} - t^2); \\ \eta_3(t) &= (t^{k_1+k_4} + t^{k_3+k_5})(t + 2t^{3/2} - 2t^{5/2} - t^3); \\ \eta_4(t) &= (t^{k_1+k_3+k_5} + t^{k_3+k_4+k_5} + t^{k_1+k_4+k_5})(t^2 + 2t^{5/2} + t^3); \end{aligned}$$

and $\vartheta_i(t)$ $(i = 2, 3, 4)$ is obtained from $\eta_i(t)$ by replacing the parameters k_1, k_3, k_4 and k_5 with l_1, l_3, l_4 and l_5 respectively. Note that $(-t-2t^{3/2}-t^2) = -t(1+\sqrt{t})^2$, $(t^2+2t^{5/2}+t^3) = t^2(1+\sqrt{t})^2$ and $(t + 2t^{3/2} - 2t^{5/2} - t^3) = (t - t^2)(1 + \sqrt{t})^2$. Then equation (14) can be reorganized as

$$\begin{aligned} &(t^{l_1-1} + t^{l_3-1} + t^{l_4-1}) + (t^{k_1+k_4-1} + t^{k_3+k_5-1}) + (t^{l_1+l_4} + t^{l_3+l_5}) \\ &+(t^{k_1+k_3+k_5} + t^{k_3+k_4+k_5} + t^{k_1+k_4+k_5}) \\ = &(t^{k_1-1} + t^{k_3-1} + t^{k_4-1}) + (t^{l_1+l_4-1} + t^{l_3+l_5-1}) + (t^{k_1+k_4} + t^{k_3+k_5}) \\ &+(t^{l_1+l_3+l_5} + t^{l_3+l_4+l_5} + t^{l_1+l_4+l_5}). \end{aligned} \tag{15}$$

From equation (15) we see that the lowest terms of its left hand side and right hand side appears in $\{t^{l_3-1}, t^{l_4-1}\}$ and $\{t^{k_3-1}, t^{k_4-1}\}$ respectively. In fact, since $l_1 \geq l_4$, t^{l_1-1} may be excluded. If our conclusion is not true, then the lowest term in the left hand side of equation (15) appears in $\{t^{k_1+k_4-1}, t^{k_3+k_5-1}\}$. But $t^{k_1+k_4-1}$ cannot be lowered since t^{k_4-1} is a term in the right hand side of equation (15) that is lower than $t^{k_1+k_4-1}$. Similarly, since t^{k_3-1} is a term in the right hand side of equation (15), $t^{k_3+k_5-1}$ is not the lowest term. Analogously, one can verify the conclusion related to $\{t^{l_3-1}, t^{l_4-1}\}$.

If $l_3 = k_3$ then $l_4 = k_4$ by the above, and so $k_1 = l_1$. Thus $H_1 \cong H_2$. Similarly, $H_1 \cong H_2$ since $l_4 = k_4$ implies $l_3 = k_3$. Therefore, we need only to consider that $l_3 = k_4$ or $l_4 = k_3$. Without loss of generality we always assume that

$$k_1,\ k_3 \geq k_4 = l_3 \geq k_5 = l_5 \quad \text{and} \quad l_1,\ l_4 \geq l_3 = k_4 \geq l_5 = k_5. \tag{16}$$

It immediately follows

$$k_1 + k_3 = l_1 + l_4 \tag{17}$$

from equation (11) and equation (16).

According to equation (17), equation (15) becomes

$$\begin{aligned}&(t^{l_1-1}+t^{l_4-1})+(t^{k_1+k_4-1}+t^{k_3+k_5-1})+(t^{l_1+l_4}+t^{l_3+l_5})+(t^{k_3+k_4+k_5}+t^{k_1+k_4+k_5})\\=&(t^{k_1-1}+t^{k_3-1})+(t^{l_1+l_4-1}+t^{l_3+l_5-1})+(t^{k_1+k_4}+t^{k_3+k_5})+(t^{l_1+l_3+l_5}+t^{l_3+l_4+l_5})\end{aligned} \tag{18}$$

If $l_1 = k_1$ then $l_4 = k_3$ by equation (17). Then equation (18) becomes

$$(t^{k_1+k_4-1}+t^{k_3+k_5-1})+(t^{k_3+k_1}+t^{k_4+k_5}) = (t^{k_1+k_3-1}+t^{k_4+k_5-1})+(t^{k_1+k_4}+t^{k_3+k_5}) \tag{19}$$

which gives that $k_3 = k_4$ or $k_1 = k_5$. Both cases imply $H_1 \cong H_2$. If $l_1 = k_3$ then $l_4 = k_1$ by equation (17), we again obtain equation (19) from equation (18), and then $H_1 \cong H_2$. By the above arguments, we may further assume that

$$l_1 > k_1 \geq k_3 > l_4 \geq l_3 = k_4 \geq l_5 = k_5.$$

Clearly, the leading term in the right hand side of equation (18) is $t^{l_1+l_3+l_5}$, and the left hand side is in $\{t^{k_1+k_4+k_5}, t^{l_1+l_4}\}$. If $t^{l_1+l_3+l_5} = t^{k_1+k_4+k_5}$ then $l_1 = k_1$, it contradicts our assumption. If $t^{l_1+l_3+l_5} = t^{l_1+l_4}$ then $l_1 + l_3 + l_5 = l_1 + l_4$. We obtain that

$$l_1 = k_1 + k_3 - k_4 - k_5, \quad l_4 = k_4 + k_5, \quad l_3 = k_4, \quad l_5 = k_5. \tag{20}$$

Then (18) becomes

$$\begin{aligned}&(t^{k_1+k_3-k_4-k_5-1})+(t^{k_1+k_4-1}+t^{k_3+k_5-1})+(t^{k_4+k_5})+(t^{k_3+k_4+k_5}+t^{k_1+k_4+k_5})\\=&(t^{k_1-1}+t^{k_3-1})+(t^{k_1+k_3-1})+(t^{k_1+k_4}+t^{k_3+k_5})+(t^{2k_4+2k_5})\end{aligned} \tag{21}$$

It is easy to see that the leading term in the left hand side of equation (21) appears in $\{t^{k_1+k_3-k_4-k_5-1}, t^{k_1+k_4+k_5}\}$, and the leading term in the right hand side of equation (21) appears in $\{t^{k_1+k_3-1}, t^{2k_4+2k_5}\}$. Since $l_1 \geq l_4$, we have $k_1 + k_3 > 2k_4 + 2k_5$ by equation (20). Thus $t^{k_1+k_3-1} = t^{k_1+k_3-k_4-k_5-1}$ or $t^{k_1+k_4+k_5}$. The former is impossible, thus we have $k_1 + k_3 - 1 = k_1 + k_4 + k_5$. Then equation (21) becomes

$$t^{k_1} + t^{k_1+k_4-1} + t^{k_3+k_5-1} + t^{k_3+k_4+k_5} = t^{k_1-1} + t^{k_1+k_4} + t^{k_3+k_5} + t^{2k_4+2k_5} \tag{22}$$

By observation, we known that $t^{k_3+k_4+k_5} = t^{k_1+k_4}$ is the leading terms in two sides of equation (22), and so $k_3 + k_4 + k_5 = k_1 + k_4$. Thus

$$k_1 = k_3 + k_5, \quad k_3 = k_4 + k_5 + 1. \tag{23}$$

Set $k_4 = k \geq m = k_5$. We obtain $(k_1, k_3, k_4, k_5) = (k+2m+1, k+m+1, k, m)$, which returns to equation (20), we have $(l_1, l_3, l_4, l_5) = (k+2m+2, k, k+m, m)$.

Clearly, from Fig. 3 we detect that $\mathcal{H}(k+2m+1, k+m+1, k, m)$ and $\mathcal{H}(k+2m+2, k, k+m, m)$ are not isomorphic. According to equation (20) and equation (23) we further know that $\mathcal{H}(k+2m+1, k+m+1, k, m)$ and $\mathcal{H}(k+2m+2, k, k+m, m)$ are uniquely A-cospectral mate. Conversely, it is easy to verify that $\mathcal{H}(k+2m+1, k+m+1, k, m)$ and $\mathcal{H}(k+2m+2, k, k+m, m)$ are A-cospectral.

Hence, $H_1 \cong H_2$ except for $H_1 = \mathcal{H}(k+2m+1, k+m+1, k, m)$ and $H_2 = \mathcal{H}(k+2m+2, k, k+m, m)$ where $k, m \geq 1$. Moreover, $\mathcal{H}(k+2m+1, k+m+1, k, m)$ has the unique A-cospectral mate $\mathcal{H}(k+2m+2, k, k+m, m)$.

We complete this proof. □

Let $\Pi(k_1, 0, k_3, k_4, k_5)$ be a graph in $\Pi^{=0}$ on n vertices, and $\mathcal{H}(k_1, k_3, k_4, k_5)$ its line graph (see Fig. 3). Let $\Pi(l_1, l_2, l_3, l_4, l_5)$ be a graph of $\Pi^{\geq 1}$ on n vertices, and $\mathcal{D}(l_1, l_2, l_3, l_4, l_5)$ its line graph (see Fig. 4). The following lemma will show that $\mathcal{H}(k_1, k_3, k_4, k_5)$ and $\mathcal{D}(l_1, l_2, l_3, l_4, l_5)$ are not A-cospectral.

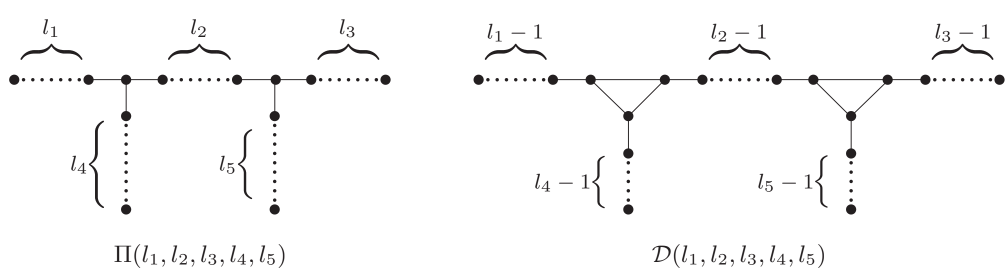


Figure 4: Graphs $\Pi(l_1, l_2, l_3, l_4, l_5)$ and $\mathcal{D}(l_1, l_2, l_3, l_4, l_5)$ $(l_2 \geq 1)$

Lemma 3.4 — No graph $\mathcal{D}(l_1, l_2, l_3, l_4, l_5)(l_2 \geq 1)$ has the same adjacency spectrum with $\mathcal{H}(k_1, k_3, k_4, k_5)$ where $\sum_{i=1}^{5} l_i = \sum_{i=1, i\neq 2}^{5} k_i = n - 2$.

PROOF : To the contrary, assume that the graphs $H_1 = \mathcal{D}(l_1, l_2, l_3, l_4, l_5)$ and $H_2 = \mathcal{H}(k_1, k_3, k_4, k_5)$ have the same adjacency spectrum. Furthermore, by Lemma 2.1 we have $N_{H_1}(k) = N_{H_2}(k)$ for $k \geq 1$. We consider two situations as follows.

First assume that $l_2 \geq 2$. Then H_1 has $2C_3$ as its induced subgraph where $2C_3$ denotes *disjoint union* of two graphs C_3. From Lemma 2.2 we have $\lambda_2(H_1) \geq \lambda_1(C_3) = 2$. For the graph H_2, let v be the vertex of degree 4 in H_2 (see Fig. 3). Clearly, removing u leaves two disjoint paths. By Lemma 2.2 again we get $\lambda_2(H_2) \leq \lambda_1(H_2 - u) < 2$. Thus, it contradicts $\lambda_2(H_1) = \lambda_2(H_2)$.

Next suppose that $l_2 = 1$. By Lemma 2.10,

$$30N_{H_1}(C_3) + 10N_{H_1}(G_1) = N_{H_1}(5) = N_{H_2}(5) = 30N_{H_2}(C_3) + 10N_{H_2}(G_1),$$

which gives $N_{H_1}(G_1) = N_{H_2}(G_1)$ since $N_{H_1}(C_3) = N_{H_2}(C_3)$. By simple observation, $2 \leq N_{H_1}(G_1) \leq 6$, and $4 \leq N_{H_2}(G_1) \leq 8$. It follows that $4 \leq N_{H_1}(G_1) = N_{H_2}(G_1) \leq 6$. It suffices to distinct the following three cases.

If $N_{H_1}(G_1) = 4$, then $|V(H_2)| = 5$ (see Fig. 3), but $|V(H_1)| \geq 8$ (see Fig. 4), it is impossible.

If $N_{H_1}(G_1) = 5$, then H_1 has just one l_i equal to 1, say $l_5 = 1$ and $l_i \geq 2$ $(i \neq 2, 5)$; for H_2 we may assume that $k_1 \geq 2$ and $k_i = 1$ $(i \neq 1)$ (see Fig. 5). By Lemma 2.10, $N_{H_1}(7) = N_{H_2}(7)$ implies that

$$N_{H_1}(G_2) + N_{H_1}(G_3) + 2N_{H_1}(G_4) = N_{H_2}(G_2) + N_{H_2}(G_3) + 2N_{H_2}(G_4). \tag{24}$$

From Fig. 5(a), $N_{H_1}(G_2) = 4$, $N_{H_1}(G_4) = 0$ and $4 \leq N_{H_1}(G_3) \leq 7$. From Fig. 5(b), $N_{H_2}(G_2) = 2$, $N_{H_2}(G_4) = 2$ and $5 \leq N_{H_2}(G_3) \leq 6$. By equation (24) we obtain that $N_{H_1}(G_3) = 7$ and $N_{H_2}(G_3) = 5$. However, $N_{H_1}(G_3) = 7$ implies that $|V(H_1)| \geq 12$, and $N_{H_2}(G_3) = 5$ implies that $|V(H_2)| = 6$, it contradicts to $|V(H_1)| = |V(H_2)|$.

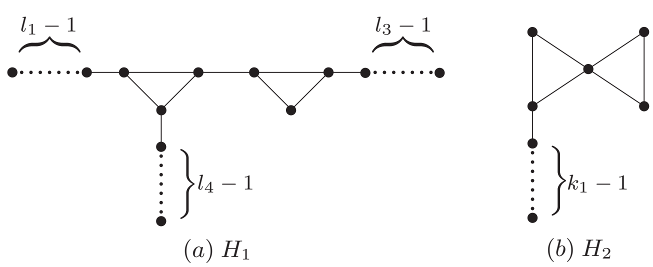


Figure 5: Graphs H_1 and H_2

If $N_{H_1}(G_1) = 6$, then all the parameters l_1, l_3, l_4, l_5 of H_1 are no less than 2, and H_2 has just two parameters are equal to 1, the others are greater than 1. By the symmetry, we may assume that H_2 has two forms $H_2^1 = \mathcal{H}(k_1, k_3, 1, 1)$ $(k_1, k_3 \geq 2)$ and $H_2^2 = \mathcal{H}(k_1, 1, k_4, 1)$ $(k_1, k_4 \geq 1)$ which are shown in Fig. 6. Clearly, equation (24) also holds for H_1 and H_2^i for $i = 1, 2$.

From Fig. 4, it is easy to see that $N_{H_1}(G_2) = 6$, $4 \leq N_{H_1}(G_3) \leq 8$ and $N_{H_1}(G_4) = 0$. Now, if H_2 has the form of H_2^1. Then by Fig. 6(a) we get $N_{H_2^1}(G_2) = 4$, $6 \leq N_{H_2^1}(G_3) \leq 8$ and $N_{H_2^1}(G_4) = 2$. Thus, by (24) we have $N_{H_1}(G_3) = 8$ and $N_{H_2^1}(G_3) = 6$, which imply that

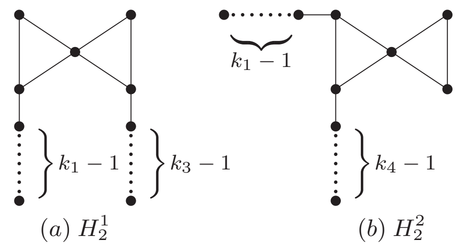


Figure 6: Graphs H_1^1 and H_2^2

$l_i \geq 3$ $(i = 1, 3, 4, 5)$ and $k_1 = k_3 = 2$, respectively. Therefore, $|V(H_1)| \geq 14$ and $|V(H_2^1)| = 7$, which is a contradiction; if H_2 has the form of H_2^2. Then by Fig. 6(b) we get $N_{H_2^2}(G_2) = 5$, $6 \leq N_{H_2^2}(G_3) \leq 8$ and $N_{H_2^2}(G_4) = 2$. Also by equation (24) we find that it is impossible.

We complete the proof. □

Immediately, by Lemmas 3.4 and 2.4, we have the following lemma:

Lemma 3.5 — The Π-shape tree $\Pi(k_1, 0, k_3, k_4, k_5)$ is not L-cospectral with $\Pi(l_1, l_2, l_3, l_4, l_5)$ where $\sum_{i=1, i\neq 2}^{5} k_i = \sum_{i=1}^{5} l_i = n - 2$.

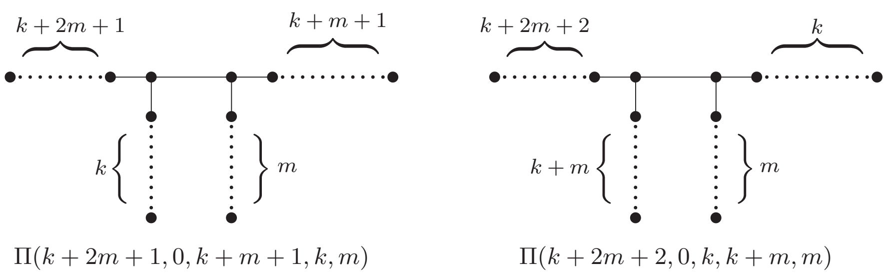


Figure 7: Laplacian cospectral trees

Now, we can characterize any Π-shape tree of $\Pi^{=0}$ by its Laplacian spectrum in the following.

***Theorem* 3.1** — *All* Π*-shape trees in* $\Pi^{=0}$ *are determined by their Laplacian spectra if and only if* $(l_1, l_2, l_3, l_4, l_5) \neq (k+2m+1, 0, k+m+1, k, m)$*, and* $(k+2m+2, 0, k, k+m, m)$*. Moreover, if* $(l_1, l_2, l_3, l_4, l_5) = (k+2m+1, 0, k+m+1, k, m)$*, then* $\Pi(k+2m+1, 0, k+m+1, k, m)$ *has the unique* L*-cospectral mate* $\Pi(k+2m+2, 0, k, k+m, m)$ *for* $k, m \geq 1$ *(see Fig 7 for instance).*

PROOF : Let $\Pi(l_1, 0, l_3, l_4, l_5) \in \Pi^{=0}$, and assume that H is any graph L-cospectral with $\Pi(l_1, 0, l_3, l_4, l_5)$. Then H is a Π-shape tree by Lemma 3.2. Furthermore, Lemma 3.5 implies that H

belongs to $\Pi^{=0}$.

If $(l_1, 0, l_3, l_4, l_5) \neq (k+2m+2, 0, k, k+m, m)$ and $(k+2m+1, 0, k+m+1, k, m)$, by Lemma 2.4 we obtain that their line graphs $\mathcal{L}(H)$ and $\mathcal{H}(l_1, l_3, l_4, l_5)$ are A-cospectral. Thus, by Lemma 3.3, $\mathcal{L}(H) \cong \mathcal{H}(l_1, l_3, l_4, l_5)$. It follows from Lemma 2.5 that $H \cong \Pi(l_1, 0, l_3, l_4, l_5)$.

If $(l_1, l_2, l_3, l_4, l_5) = (k+2m+1, 0, k+m+1, k, m)$, then by Lemma 3.3, $\mathcal{H}(k+2m+1, 0, k+m+1, k, m)$ is uniquely A-cospectral with $\mathcal{H}(k+2m+2, 0, k, k+m, m)$. Consequently, Lemma 2.4 implies that $\Pi(k+2m+1, 0, k+m+1, k, m)$ is L-cospectral with $\Pi(k+2m+2, 0, k, k+m, m)$. From Fig. 7, it is easy to see that those graphs are not isomorphic. Thus, Π-shape tree $\Pi(k+2m+1, 0, k+m+1, k, m)$ has the unique L-cospectral mate $\Pi(k+2m+2, 0, k, k+m, m)$ for $k, m \geq 1$.

The proof is completed. $\square$

Remark : Let $\Pi_1(k, m) = \Pi(k+2m+1, 0, k+m+1, k, m)$, $\Pi_2(k, m) = \Pi(k+2m+2, 0, k, k+m, m)$ for $k, m \geq 1$. We depicted the L-cospectral mates of $\Pi_1(k, m)$ and $\Pi_2(k, m)$ for $(k, m) = (1, 1), (2, 1), (2, 2)$ in Table 1. Clearly, by Lemma 2.9, those graphs are also Q-cospectral.

4. Concluding Remarks

By Lemma 32., a graph L-cospectral with Π-shape tree must be a Π-shape tree. Thus, if we can prove that no two non-isomorphic Π-shape trees are L-cospectral, then the Π-shape tree is determined by its Laplacian spectrum. Unfortunately, we find that there exist many L-cospectral mates in Π-shape tree. For convenience we classify the Π-shape trees into two types according to l_2, i.e., $\Pi^{=0}$ and $\Pi^{\geq 1}$.

For the Π-shape trees in $\Pi^{=0}$, we show that all graphs are determined by their Laplacian spectra with exceptions of L-cospectral mate trees: $\Pi(k+2m+2, 0, k, k+m, m)$ and $\Pi(k+2m+1, 0, k+m+1, k, m)$ for $k, m \geq 1$. Although any *starlike tree* is determined by its Laplacian spectrum, our result implies that *double-starlike tree* is not determined by its Laplacian spectrum.

For the Π-shape trees in $\Pi^{\geq 1}$, we also find many L-cospectral mates in itself, such as $\Pi(2, 3, 1, 2, 1)$ and $\Pi(2, 2, 3, 1, 1)$ are L-cospectral mates, also $\Pi(3, 4, 1, 2, 1)$ and $\Pi(3, 2, 4, 1, 1)$, $\Pi(3, 4, 2, 3, 1)$ and $\Pi(3, 3, 4, 1, 2)$, $\Pi(4, 5, 1, 2, 1)$ and $\Pi(4, 2, 5, 1, 1)$, $\Pi(4, 6, 2, 3, 1)$ and $\Pi(4, 3, 6, 2, 1)$, and so on (see Table. 2). But our further spectral characterizations for $\Pi^{\geq 1}$ encounter difficulty. However there are some clues to be mentioned below:

- Each L-cospectral mate in $\Pi^{\geq 1}$ has the same parameter set $\{l_1, l_2, l_3, l_4, l_5\}$ but different permutation such as $\Pi(2, 3, 1, 2, 1)$ and $\Pi(2, 2, 3, 1, 1)$. This reminds us if all L-cospectral mates in $\Pi^{\geq 1}$ have the same parameter set ?

Table 1: Some L-cospectral mates of $\Pi^{=0}$

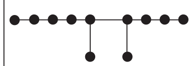
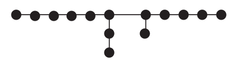
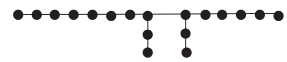
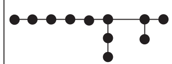
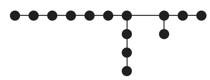
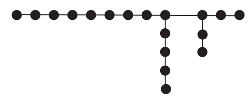
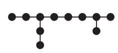
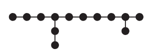
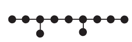
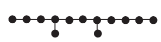
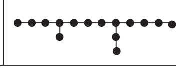
	$k=1, m=1$	$k=2, m=1$	$k=2, m=2$	$\cdots$
$\Pi_1(k,m)$				$\cdots$
$\Pi_2(k,m)$				$\cdots$

Table 2: Some L-cospectral mates of $\Pi^{\geq 1}$

$\Pi(2,3,1,2,1)$	$\Pi(3,4,1,2,1)$	$\Pi(3,4,2,3,1)$	$\Pi(4,5,1,2,1)$	$\Pi(4,6,2,3,1)$	$\cdots$
					$\cdots$
$\Pi(2,2,3,1,1)$	$\Pi(3,2,4,1,1)$	$\Pi(3,3,4,1,2)$	$\Pi(4,2,5,1,1)$	$\Pi(4,3,6,2,1)$	$\cdots$
					$\cdots$

- By the computer we can find many pairs of L-cospectral mates $\Pi(l_1, l_2, l_3, l_4, l_5)$ and $\Pi(l'_1, l'_2, l'_3, l'_4, l'_5)$ in $\Pi^{\geq 1}$, where $(l'_1, l'_2, l'_3, l'_4, l'_5)$ is the permutation of $(l_1, l_2, l_3, l_4, l_5)$, but we do not know if such pair of L-cospectral mates is infinite ?

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