Indian J. Pure Appl. Math., **49**(3): 413-430, September 2018

DOI: 10.1007/s13226-018-0277-4

GENERALIZED SUPER GABOR DUALS WITH BOUNDED INVERTIBLE OPERATORS[1]

Qiaofang Lian and Zhao Wang

Department of Mathematics, Beijing Jiaotong University,
Beijing 100044, *P. R. China*
e-mails: qflian@bjtu.edu.cn; 12121608@bjtu.edu.cn
(*Received* 11 *September* 2016; *after final revision* 26 *March* 2017;
accepted 6 *September* 2017)

In this paper, we introduce generalized super Gabor duals with bounded invertible operators by combining ideas concerning super Gabor frames with the idea of g-duals as proposed by Dehgham and Fard in 2013. Given a super Gabor frame and a bounded invertible operator A, we characterize its generalized super Gabor duals with A, and derive a parametric expression of all its generalized super Gabor duals with A. The perturbation of generalized super Gabor duals is considered as well.

Key words : Super Gabor frame; generalized super Gabor duals with A; perturbation.

1. INTRODUCTION

Frames were introduced in 1952 by Duffin and Schaeffer in the study of nonharmonic Fourier series [25]. They generalize bases and allow each element in the space to be represented as an infinite linear combination of the elements in the frame, but the expansion coefficients are not necessarily unique. This property plays a significant role in many applications, and the theory of frames has been growing rapidly in the past more than twenty years. As a generalization of frames, the concept of superframe was introduced by Balan in general Hilbert spaces, and has multiplexing applications in mobile communication network, satellite communication network and computer area network [4]. The prefix "super" is used because it is a frame for a "super space", namely the direct sum of finitely many Hilbert spaces. The idea of "multiplexing" is to encode L independent signals $f_l, l = 1, 2, ..., L$, as a single sequence that captures the time-frequency information of each f_l. Among all kinds of

[1]Supported by the National Natural Science Foundation of China (Grant No. 11271037) and the Fundamental Research Funds for the Central Universities (Grant No. 2015JBM097).

superframes, super wavelet frames and super Gabor frames are two classes of important superframes, and have interested some mathematicians and engineering specialists in recent years (see [1-7, 12, 13, 16, 18-20, 22-24, 26] and references therein).

In this paper, we are concerned with super Gabor frames on $\bigoplus_{l=1}^{L} L^2(\mathbb{R})$, which is exactly the vector-valued Hilbert space $L^2(\mathbb{R}, \mathbb{C}^L)$ endowed with the inner product

$$\langle \mathbf{f}, \mathbf{g} \rangle = \sum_{l=1}^{L} \int_{\mathbb{R}} f_l(x)\overline{g_l(x)}dx$$

for $\mathbf{f} = (f_1,\, f_2,\, \ldots,\, f_L)^t$, $\mathbf{g} = (g_1,\, g_2,\, \ldots,\, g_L)^t \in L^2(\mathbb{R}, \mathbb{C}^L)$. In 2008, Führ derived frame bound estimates for super Gabor systems in $L^2(\mathbb{R}, \mathbb{C}^L)$ with window functions belonging to Schwartz space, and obtained estimates for windows composed of the first $L+1$ Hermite functions [16]. In 2009, using growth estimates for the Weierstrass σ-function and a new type of interpolation problem for entire functions on the Bargmann-Fock space, Gröchenig and Lyubarskii characterized all lattices in $\mathbb{R}^2$ such that the Gabor system is a frame for $L^2(\mathbb{R}, \mathbb{C}^L)$ [18]. In 2010, Abreu studied the structure of super Gabor spaces $L^2(\mathbb{R}^d, \mathbb{C}^L)$ and specialized the results to the case where the spaces are generated by vectors of Hermite functions [1]. About a general super Gabor system, necessary density conditions were studied in 1999 by Balan [6]. For the rational time-frequency lattice case, Li and Han provided a sufficient and necessary density condition in 2010 [22], Li and their students in [23] and [24] using Zak transform matrix method characterized complete super Gabor systems and super Gabor frames for $L^2(\mathbb{R}, \mathbb{C}^L)$ and its subspace $L^2(\mathbb{S}, \mathbb{C}^L)$, where $\mathbb{S}$ is a periodic measurable set in $\mathbb{R}$. Less is known about a general super Gabor frame for the irrational time-frequency lattice case.

Dual frames have an essential role in reconstruction of signals from frame coefficients. When a frame admits many dual frames, we have much freedom to represent signals. Such freedom is exactly why frames attract many researchers, and has important applications in signal reconstruction and transmission [10]. In 2008, Christensen and Laugesen introduced the concept of approximately dual frames, which are easier to construct than the ordinary dual frames, and might be tailored to yield almost perfect reconstruction [9]. Furthermore, Dehghan and Fard in 2013 introduced g-dual frames for a Hilbert space H [11], which include the ordinary dual frames and approximately dual frames. In 2016, Javanshiri obtained some sufficient and necessary conditions for two frames to be g-dual frames, and provided a general method of constructing g-dual frames [21]. Given a frame $\{f_k\}_{k=1}^{\infty}$ for H. Recall that a frame $\{g_k\}_{k=1}^{\infty}$ is called a g-dual frame of $\{f_k\}_{k=1}^{\infty}$ for H if there exists a bounded invertible operator A such that

$$f=\sum_{k=1}^{\infty}\langle Af,g_k\rangle f_k$$

for all $f\in H$. In fact, a g-dual frame is an ordinary dual frame when $A=I$, and the set of approximately dual frames of a frame is a proper subset of the set of its g-dual frames (see Proposition 4.1 and Example 4.1 in [11]). Due to the bounded invertible operator A, there are many differences between a g-dual frame and an ordinary dual frame. In the ordinary case, every Riesz basis has a unique dual frame, whereas a Riesz basis can have infinitely many g-dual frames that they also are Riesz bases. More specifically, every two Riesz bases are g-dual frames (see Proposition 2.1 in [11]). Just because the properties of g-dual frames are so different from the ordinary dual frames, this inspires us to combine bounded invertible operators with super Gabor duals, which we call generalized super Gabor duals with bounded invertible operators. About the ordinary super Gabor duals, Li and Zhou in 2013 obtained an explicit expression of canonical dual and a parametrization of all its super Gabor duals for a super Gabor frame for $L^2(\mathbb{R},\mathbb{C}^L)$ [23]. In 2015, Li and Zhang obtained a necessary and sufficient condition on super Gabor duals of type I (resp. II) for a super Gabor frame for $L^2(\mathbb{S},\mathbb{C}^L)$, and establish a parametrization expression of Gabor duals of type I (resp. II) [24]. In the above two works, Zak transform matrix method is used and only applies to the rational time-frequency lattice case. In this paper, we study generalized super Gabor duals with bounded invertible operators using time-domain method instead. We first obtain some characterizations, and then derive an explicit expression of generalized super Gabor duals with bounded invertible operators. We also consider the perturbation problems. Even for the ordinary super Gabor duals, our some results are new and our method applies to the irrational time-frequency lattice case.

In what follows, for $\mathbf{f}\in L^2(\mathbb{R},\mathbb{C}^L)$ and $1\le l\le L$, we always denote by f_l the l-th component of $\mathbf{f}$. We say $\mathbf{f}$ is a bounded function in $L^2(\mathbb{R},\mathbb{C}^L)$ with compact support if each component f_l is a bounded function in $L^2(\mathbb{R})$ with compact support. For $\lambda,\,\mu\in\mathbb{R}$, we define the *vector-valued modulation operator* $\mathbf{M}_\lambda$ and the *vector-valued translation operator* $\mathbf{T}_\mu$ on $L^2(\mathbb{R},\mathbb{C}^L)$ respectively by

$$\mathbf{M}_\lambda\mathbf{f}(\cdot)=(M_\lambda f_1(\cdot),\,M_\lambda f_2(\cdot),\,\ldots,\,M_\lambda f_L(\cdot)),\tag{1.1}$$

$$\mathbf{T}_\mu\mathbf{f}(\cdot)=(T_\mu f_1(\cdot),\,T_\mu f_2(\cdot),\,\ldots,\,T_\mu f_L(\cdot))\tag{1.2}$$

for $\mathbf{f}\in L^2(\mathbb{R},\mathbb{C}^L)$, where

$$M_\lambda f_l(\cdot)=e^{2\pi i\lambda\cdot}f_l(\cdot)\text{ and }T_\mu f_l(\cdot)=f_l(\cdot-\mu)\tag{1.3}$$

for $1 \leq l \leq L$. Obviously, they are both unitary operators on $L^2(\mathbb{R}, \mathbb{C}^L)$, and

$$\mathbf{M}_\lambda \mathbf{T}_\mu \mathbf{f}(\cdot) = e^{2\pi i \lambda\mu} \mathbf{T}_\mu \mathbf{M}_\lambda \mathbf{f}(\cdot)$$

for $\mathbf{f} \in L^2(\mathbb{R}, \mathbb{C}^L)$. Let $a, b > 0$. For $\mathbf{g} \in L^2(\mathbb{R}, \mathbb{C}^L)$, define the *super Gabor system* $\mathcal{G}(\mathbf{g}, a, b)$ generated by $\mathbf{g}$ as

$$\mathcal{G}(\mathbf{g}, a, b) = \{\mathbf{M}_{mb}\mathbf{T}_{na}\mathbf{g}\}_{m,n\in\mathbb{Z}}. \tag{1.4}$$

We call $\mathcal{G}(\mathbf{g}, a, b)$ a *super Gabor frame* for $L^2(\mathbb{R}, \mathbb{C}^L)$ if there exist $0 < D_1 \leq D_2 < \infty$ such that

$$D_1\|\mathbf{f}\|^2 \leq \sum_{m,n\in\mathbb{Z}} \left|\langle \mathbf{f}, \mathbf{M}_{mb}\mathbf{T}_{na}\mathbf{g}\rangle\right|^2 \leq D_2\|\mathbf{f}\|^2 \tag{1.5}$$

for $\mathbf{f} \in L^2(\mathbb{R}, \mathbb{C}^L)$, where D_1, D_2 are called *frame bounds*. In particular, $\mathcal{G}(\mathbf{g}, a, b)$ is called a *tight super Gabor frame* if $D_1 = D_2$, a *normalized tight super Gabor frame* if $D_1 = D_2 = 1$, and a *super Gabor Riesz basis* if it ceases to be a super Gabor frame whenever any one of its elements is removed. The super Gabor system $\mathcal{G}(\mathbf{g}, a, b)$ is called a *super Gabor Bessel sequence* in $L^2(\mathbb{R}, \mathbb{C}^L)$ if the right-hand side inequality in (1.5) holds. Let $\mathbf{g} \in L^2(\mathbb{R}, \mathbb{C}^L)$ be such that $\mathcal{G}(\mathbf{g}, a, b)$ is a super Gabor Bessel sequence in $L^2(\mathbb{R}, \mathbb{C}^L)$. Define the pre-frame operator $U_{\mathbf{g}}$ for $\mathcal{G}(\mathbf{g}, a, b)$ by

$$U_{\mathbf{g}} : \ell^2(\mathbb{Z}^2) \to L^2(\mathbb{R}, \mathbb{C}^L),\ U_{\mathbf{g}}\{c_{m,n}\}_{m,n\in\mathbb{Z}} = \sum_{m,n\in\mathbb{Z}} c_{m,n}\mathbf{M}_{mb}\mathbf{T}_{na}\mathbf{g}. \tag{1.6}$$

Then $U_{\mathbf{g}}$ is a bounded operator by Theorem 3.2.3 in [8]. The adjoint operator is given by

$$U^*_{\mathbf{g}} : L^2(\mathbb{R}, \mathbb{C}^L) \to \ell^2(\mathbb{Z}^2),\ U^*_{\mathbf{g}}\mathbf{f} = \{\langle \mathbf{f}, \mathbf{M}_{mb}\mathbf{T}_{na}\mathbf{g}\rangle\}_{m,n\in\mathbb{Z}}. \tag{1.7}$$

Also let $\mathbf{h} \in L^2(\mathbb{R}, \mathbb{C}^L)$ be such that $\mathcal{G}(\mathbf{h}, a, b)$ is a super Gabor Bessel sequence in $L^2(\mathbb{R}, \mathbb{C}^L)$. Denote by $U_{\mathbf{h}}$ the pre-frame operator for $\mathcal{G}(\mathbf{h}, a, b)$. By composing $U_{\mathbf{g}}$ and $U^*_{\mathbf{h}}$, we obtain the bounded operator

$$\mathcal{S}_{\mathbf{h},\mathbf{g}} :\ L^2(\mathbb{R}, \mathbb{C}^L) \to L^2(\mathbb{R}, \mathbb{C}^L),\ \mathcal{S}_{\mathbf{h},\mathbf{g}}\mathbf{f} = U_{\mathbf{g}}U^*_{\mathbf{h}}\mathbf{f} = \sum_{m,n\in\mathbb{Z}} \langle \mathbf{f}, \mathbf{M}_{mb}\mathbf{T}_{na}\mathbf{h}\rangle \mathbf{M}_{mb}\mathbf{T}_{na}\mathbf{g}. \tag{1.8}$$

We call $\mathbf{h}$ a *super Gabor dual* of $\mathbf{g}$ if $\mathcal{S}_{\mathbf{h},\mathbf{g}} = I$ on $L^2(\mathbb{R}, \mathbb{C}^L)$. In particular, when $\mathbf{g} = \mathbf{h}$ in (1.8), $\mathcal{S}_{\mathbf{g},\mathbf{g}}$ is called the frame operator for $\mathcal{G}(\mathbf{g}, a, b)$. If $\mathcal{G}(\mathbf{g}, a, b)$ is a super Gabor frame for $L^2(\mathbb{R}, \mathbb{C}^L)$ in addition, then $\mathcal{S}_{\mathbf{g},\mathbf{g}}$ is invertible by Lemma 5.1.5 in [8], and

$$\mathbf{f} = \sum_{m,n\in\mathbb{Z}} \left\langle \mathbf{f}, \mathcal{S}^{-1}_{\mathbf{g},\mathbf{g}}\mathbf{M}_{mb}\mathbf{T}_{na}\mathbf{g}\right\rangle \mathbf{M}_{mb}\mathbf{T}_{na}\mathbf{g}$$

for $\mathbf{f} \in L^2(\mathbb{R}, \mathbb{C}^L)$ by Theorem 5.1.6 in [8]. It is easy to see that $\mathcal{S}_{\mathbf{g},\mathbf{g}}$ commutes with $\mathbf{M}_{mb}$ and $\mathbf{T}_{na}$ for $m, n \in \mathbb{Z}$. So does $\mathcal{S}^{-1}_{\mathbf{g},\mathbf{g}}$. Then $\mathcal{S}^{-1}_{\mathbf{g},\mathbf{g}}\mathbf{g}$ is a super Gabor dual of $\mathbf{g}$. We call $\mathcal{S}^{-1}_{\mathbf{g},\mathbf{g}}\mathbf{g}$ the *canonical* *super Gabor dual* of $\mathbf{g}$. The fundamentals of frames and Gabor frames can be found in [8, 14, 15, 17].

Denote by $\mathcal{BI}\left(L^2(\mathbb{R})\right)$ the set of all linear, bounded and invertible operators on $L^2(\mathbb{R})$. Let $A = (A_1, A_2, \ldots, A_L)^t$ with $A_l \in \mathcal{BI}\left(L^2(\mathbb{R})\right)$, define

$$A\mathbf{f} = (A_1 f_1, A_2 f_2, \ldots, A_L f_L)^t$$

for $\mathbf{f} \in L^2(\mathbb{R}, \mathbb{C}^L)$. Then $A\mathbf{f} \in L^2(\mathbb{R}, \mathbb{C}^L)$, and

$$\|A\mathbf{f}\|^2 = \sum_{l=1}^{L} \|A_l f_l\|^2 \leq \sum_{l=1}^{L} \|A_l\|^2 \|f_l\|^2 \leq \max_{1 \leq l \leq L} \|A_l\|^2 \sum_{l=1}^{L} \|f_l\|^2 = \max_{1 \leq l \leq L} \|A_l\|^2 \|\mathbf{f}\|^2.$$

In addition, it is easy to see that A is invertible and $A^{-1} = (A_1^{-1}, A_2^{-1}, \ldots, A_L^{-1})^t$. Therefore, A is a bounded and invertible operator on $L^2(\mathbb{R}, \mathbb{C}^L)$. The adjoint operator of A can be written as $A^* = (A_1^*, A_2^*, \ldots, A_L^*)^t$, where A_l^* denotes the adjoint operator of A_l. Denote

$$\bigoplus_{l=1}^{L} \mathcal{BI}\left(L^2(\mathbb{R})\right) = \left\{A = (A_1, A_2, \ldots, A_L)^t : A_l \in \mathcal{BI}\left(L^2(\mathbb{R})\right), 1 \leq l \leq L\right\}.$$

Let $\mathbf{g}, \mathbf{h} \in L^2(\mathbb{R}, \mathbb{C}^L)$ be such that $\mathcal{G}(\mathbf{g}, a, b)$ and $\mathcal{G}(\mathbf{h}, a, b)$ are both super Gabor Bessel sequences in $L^2(\mathbb{R}, \mathbb{C}^L)$. We call $\mathbf{h}$ a *generalized super Gabor dual* of $\mathbf{g}$ if there exists an operator $A \in \bigoplus_{l=1}^{L} \mathcal{BI}\left(L^2(\mathbb{R})\right)$ such that

$$\mathbf{f} = \sum_{m,n \in \mathbb{Z}} \langle A\mathbf{f}, \mathbf{M}_{mb}\mathbf{T}_{na}\mathbf{h}\rangle \mathbf{M}_{mb}\mathbf{T}_{na}\mathbf{g} \tag{1.9}$$

for $\mathbf{f} \in L^2(\mathbb{R}, \mathbb{C}^L)$. Since the assumptions imply that $S_{\mathbf{h},\mathbf{g}} = A^{-1}$, and thus, the operator A in (1.9) is unique. Also, we say $\mathbf{h}$ is a generalized super Gabor dual of $\mathbf{g}$ with corresponding operator A. In particular, when $A = (I, I, \ldots, I)^t$, $\mathbf{h}$ is an ordinary super Gabor dual of $\mathbf{g}$. When $L = 1$, $\mathbf{h}$ is called a generalized Gabor dual of $\mathbf{g}$ with A. When $\mathcal{G}(\mathbf{g}, a, b)$ is a super Gabor frame for $L^2(\mathbb{R}, \mathbb{C}^L)$, $\mathbf{g}$ is a generalized super Gabor dual of itself with operator $\mathcal{S}_{\mathbf{g},\mathbf{g}}^{-1}$ since

$$\mathbf{f} = \sum_{m,n \in \mathbb{Z}} \left\langle \mathbf{f}, \mathcal{S}_{\mathbf{g},\mathbf{g}}^{-1}\mathbf{M}_{mb}\mathbf{T}_{na}\mathbf{g}\right\rangle \mathbf{M}_{mb}\mathbf{T}_{na}\mathbf{g} = \sum_{m,n \in \mathbb{Z}} \left\langle \mathcal{S}_{\mathbf{g},\mathbf{g}}^{-1}\mathbf{f}, \mathbf{M}_{mb}\mathbf{T}_{na}\mathbf{g}\right\rangle \mathbf{M}_{mb}\mathbf{T}_{na}\mathbf{g}$$

for $\mathbf{f} \in L^2(\mathbb{R}, \mathbb{C}^L)$. We have the following proposition.

Proposition 1.1 — Let $\mathbf{g}, \mathbf{h} \in L^2(\mathbb{R}, \mathbb{C}^L)$.

(1) $\mathcal{G}(\mathbf{g}, a, b)$ is a super Gabor Bessel sequence in $L^2(\mathbb{R}, \mathbb{C}^L)$ if and only if $\mathcal{G}(g_l, a, b)$ is a Gabor Bessel sequence in $L^2(\mathbb{R})$ for every $1 \leq l \leq L$.

(2) If $\mathcal{G}(\mathbf{g},\, a,\, b)$ is a super Gabor frame for $L^2(\mathbb{R},\mathbb{C}^L)$, then $\mathcal{G}(g_l,\, a,\, b)$ is a Gabor frame for $L^2(\mathbb{R})$ for every $1 \leq l \leq L$. However, the converse is not true.

(3) $\mathbf{h}$ is a generalized super Gabor dual of $\mathbf{g}$ with operator $A \in \bigoplus_{l=1}^{L} \mathcal{BI}\left(L^2(\mathbb{R})\right)$ if and only if h_l is a generalized Gabor dual of g_l with operator A_l for every $1 \leq l \leq L$, and

$$\sum_{m,\, n\in\mathbb{Z}} \langle A_l f, M_{mb}T_{na}h_l\rangle\, M_{mb}T_{na}g_{l'} = 0$$

for $f \in L^2(\mathbb{R})$ and $1 \leq l,\, l' \leq L$ with $l \neq l'$.

PROOF : The statement (1) is easy to be verified, and we omit the proof here. The statement (2) is a direct conclusion of Lemma 2.2 in [5]. For the proof of (3), we know from (1) that both $\mathcal{G}(\mathbf{g},\, a,\, b)$ and $\mathcal{G}(\mathbf{h},\, a,\, b)$ are super Gabor Bessel sequences in $L^2(\mathbb{R},\mathbb{C}^L)$ if and only if both $\mathcal{G}(g_l,\, a,\, b)$ and $\mathcal{G}(h_l,\, a,\, b)$ are Gabor Bessel sequences in $L^2(\mathbb{R})$ for every $1 \leq l \leq L$. Therefore, to prove (3), it suffices to prove that

$$\mathbf{f} = \sum_{m,\, n\in\mathbb{Z}} \langle A\mathbf{f}, \mathbf{M}_{mb}\mathbf{T}_{na}\mathbf{h}\rangle\, \mathbf{M}_{mb}\mathbf{T}_{na}\mathbf{g} \tag{1.10}$$

for $\mathbf{f} \in L^2(\mathbb{R},\mathbb{C}^L)$ if and only if

$$\sum_{m,\, n\in\mathbb{Z}} \langle A_l f, M_{mb}T_{na}h_l\rangle\, M_{mb}T_{na}g_{l'} = \begin{cases} f, & l = l'; \\ 0, & l \neq l' \end{cases} \tag{1.11}$$

for $f \in L^2(\mathbb{R})$ and $1 \leq l,\, l' \leq L$. We first assume that (1.10) holds. Given arbitrarily $1 \leq l \leq L$ and $f \in L^2(\mathbb{R})$, let $\mathbf{f} = (0,\, \ldots,\, 0,\, f,\, 0,\, \ldots,\, 0)^t$ with the l-th component being f and the others being zero. Then

$$\langle A\mathbf{f}, \mathbf{M}_{mb}\mathbf{T}_{na}\mathbf{h}\rangle = \langle A_l f, M_{mb}T_{na}h_l\rangle\,,$$

which together with (1.10) shows that

$$\mathbf{f} = \sum_{m,\, n\in\mathbb{Z}} \langle A_l f, M_{mb}T_{na}h_l\rangle\, \mathbf{M}_{mb}\mathbf{T}_{na}\mathbf{g}.$$

It follows that (1.11) holds. Conversely, if (1.11) holds, then, for any given $\mathbf{f} \in L^2(\mathbb{R},\mathbb{C}^L)$,

$$\sum_{m,\, n\in\mathbb{Z}} \sum_{l'=1}^{L} \langle A_{l'} f_{l'}, M_{mb}T_{na}h_{l'}\rangle\, M_{mb}T_{na}g_l = \sum_{m,\, n\in\mathbb{Z}} \langle A_l f_l, M_{mb}T_{na}h_l\rangle\, M_{mb}T_{na}g_l = f_l \tag{1.12}$$

for $1 \leq l \leq L$. Observe that

$$\begin{aligned}\sum_{m,\,n\in\mathbb{Z}} \langle A\mathbf{f}, \mathbf{M}_{mb}\mathbf{T}_{na}\mathbf{h}\rangle \mathbf{M}_{mb}\mathbf{T}_{na}\mathbf{g} &= \sum_{m,\,n\in\mathbb{Z}}\sum_{l'=1}^{L} \langle A_{l'}f_{l'}, M_{mb}T_{na}h_{l'}\rangle \mathbf{M}_{mb}\mathbf{T}_{na}\mathbf{g} \\ &= \left(\sum_{m,\,n\in\mathbb{Z}}\sum_{l'=1}^{L} \langle A_{l'}f_{l'}, M_{mb}T_{na}h_{l'}\rangle M_{mb}T_{na}g_l\right)_{1\leq l\leq L}.\end{aligned}$$

This together with (1.12) yields that

$$\sum_{m,\,n\in\mathbb{Z}} \langle A\mathbf{f}, \mathbf{M}_{mb}\mathbf{T}_{na}\mathbf{h}\rangle \mathbf{M}_{mb}\mathbf{T}_{na}\mathbf{g} = (f_1,\, f_2,\, \ldots,\, f_L)^t = \mathbf{f}.$$

So (1.10) holds. The proof is completed. □

The rest of this paper is organized as follows. Given $\mathbf{g},\,\mathbf{h} \in L^2(\mathbb{R},\mathbb{C}^L)$ and $A \in \bigoplus_{l=1}^{L} \mathcal{BI}\left(L^2(\mathbb{R})\right)$. In Section 2 we obtain a necessary and sufficient condition for $\mathbf{h}$ to be a generalized super Gabor dual of $\mathbf{g}$ with operator A. In addition, if A commutes with $\mathbf{M}_b$ and $\mathbf{T}_a$, we present a parametrization expression of all generalized super Gabor duals of $\mathbf{g}$ with operator A. In Section 3 the perturbation of generalized super Gabor duals is discussed. Let $\mathbf{h}$ be a generalized super Gabor dual of $\mathbf{g}$ with operator A. We prove that $\mathbf{h}$ is a generalized super Gabor dual of ϕ with operator $(U_\phi U_{\mathbf{h}}^*)^{-1}$ if ϕ is "close" to $\mathbf{g}$, and ϕ is a generalized super Gabor dual of $\mathbf{g}$ with operator $\left(U_{\mathbf{g}}U_\phi^*\right)^{-1}$ if ϕ is "close" to $\mathbf{h}$.

2. Characterization

Let $a,\, b > 0$. For $\mathbf{g} \in L^2(\mathbb{R},\mathbb{C}^L)$, we define the bi-infinite matrix-valued function

$$\mathcal{M}_{\mathbf{g},a,b}(x) = \left(\mathbf{g}\left(x + \frac{j}{b} - na\right)\right)_{j,\,n\in\mathbb{Z}} = \begin{pmatrix} g_1\left(x+\frac{j}{b}-na\right) \\ g_2\left(x+\frac{j}{b}-na\right) \\ \vdots \\ g_L\left(x+\frac{j}{b}-na\right)\end{pmatrix}_{j,\,n\in\mathbb{Z}}, \quad x\in\mathbb{R}. \tag{2.1}$$

Then, for $\mathbf{g},\,\mathbf{h} \in L^2(\mathbb{R},\mathbb{C}^L)$, each entry of $\mathcal{M}_{\mathbf{g},a,b}(x)\mathcal{M}^*_{\mathbf{h},a,b}(x)$ is convergent absolutely for a.e. $x \in \mathbb{R}$ by Lemma 8.2.2 in [8]. For $\mathbf{f} \in L^2(\mathbb{R},\mathbb{C}^L)$, we define the column vector-valued function

$$F_{\mathbf{f}}(x) = \begin{pmatrix} \vdots \\ \mathbf{f}\left(x + \frac{k}{b}\right) \\ \vdots \end{pmatrix}_{k\in\mathbb{Z}} = \begin{pmatrix} \vdots \\ f_1\left(x + \frac{k}{b}\right) \\ f_2\left(x + \frac{k}{b}\right) \\ \vdots \\ f_L\left(x + \frac{k}{b}\right) \\ \vdots \end{pmatrix}_{k\in\mathbb{Z}}, \quad x \in \mathbb{R}. \tag{2.2}$$

Observe that

$$\int_{[0,\,\frac{1}{b}]} \sum_{k\in\mathbb{Z}} \sum_{l=1}^{L} \left| f_l\left(x + \frac{k}{b}\right)\right|^2 dx = \sum_{l=1}^{L} \sum_{k\in\mathbb{Z}} \int_{[0,\,\frac{1}{b}]} \left| f_l\left(x + \frac{k}{b}\right)\right|^2 dx = \sum_{l=1}^{L} \|f_l\|^2 = \|\mathbf{f}\|^2 < \infty.$$

It follows that

$$\sum_{k\in\mathbb{Z}} \sum_{l=1}^{L} \left| f_l\left(x + \frac{k}{b}\right)\right|^2 < \infty \tag{2.3}$$

for a.e. $x \in \left[0,\, \frac{1}{b}\right]$, and therefore, (2.3) holds for a.e. $x \in \mathbb{R}$ due to the $\frac{1}{b}$-periodicity of $\sum\limits_{k\in\mathbb{Z}} \sum\limits_{l=1}^{L} \left| f_l\left(\cdot + \frac{k}{b}\right)\right|^2$. This shows that $F_{\mathbf{f}}(x) \in l^2(\mathbb{Z})$ for a.e. $x \in \mathbb{R}$. Now we give a characterization of generalized super Gabor duals with bounded invertible operators.

***Theorem* 2.1** — *Let $A \in \bigoplus\limits_{l=1}^{L} \mathcal{BI}\left(L^2(\mathbb{R})\right)$, and $\mathbf{g}$, $\mathbf{h} \in L^2(\mathbb{R}, \mathbb{C}^L)$ be such that $\mathcal{G}(\mathbf{g},\, a,\, b)$ and $\mathcal{G}(\mathbf{h},\, a,\, b)$ are both super Gabor Bessel sequences in $L^2(\mathbb{R}, \mathbb{C}^L)$. Then $\mathbf{h}$ is a generalized super Gabor dual of $\mathbf{g}$ with A if and only if*

$$\mathcal{M}_{\mathbf{g},a,b}(\cdot)\mathcal{M}^*_{\mathbf{h},a,b}(\cdot)F_{\mathbf{f}}(\cdot) = bF_{A^{-1}\mathbf{f}}(\cdot) \tag{2.4}$$

a.e. on $\mathbb{R}$ for all bounded functions $\mathbf{f}$ with compact support.

PROOF : By (1.9), $\mathbf{h}$ is a generalized super Gabor dual of $\mathbf{g}$ with A if and only if

$$\mathbf{f} = \sum_{m,\,n\in\mathbb{Z}} \langle A\mathbf{f}, \mathbf{M}_{mb}\mathbf{T}_{na}\mathbf{h}\rangle\, \mathbf{M}_{mb}\mathbf{T}_{na}\mathbf{g}$$

for $\mathbf{f} \in L^2(\mathbb{R}, \mathbb{C}^L)$, or equivalently,

$$A^{-1}\mathbf{f} = \sum_{m,\,n\in\mathbb{Z}} \langle \mathbf{f}, \mathbf{M}_{mb}\mathbf{T}_{na}\mathbf{h}\rangle\, \mathbf{M}_{mb}\mathbf{T}_{na}\mathbf{g} = \mathcal{S}_{\mathbf{h},\mathbf{g}}\mathbf{f} \tag{2.5}$$

for $\mathbf{f} \in L^2(\mathbb{R}, \mathbb{C}^L)$. Note that $\mathcal{S}_{\mathbf{h},\mathbf{g}}$ is a bounded operator since $\mathcal{G}(\mathbf{g},\, a,\, b)$ and $\mathcal{G}(\mathbf{h},\, a,\, b)$ are both super Gabor Bessel sequences. Therefore, (2.5) holds for all $\mathbf{f} \in L^2(\mathbb{R}, \mathbb{C}^L)$ if and only if it holds for all bounded functions $\mathbf{f} \in L^2(\mathbb{R}, \mathbb{C}^L)$ with compact support. Now let $\mathbf{f}$ be a bounded function with compact support. For $1 \leq l \leq L$ and $n \in \mathbb{Z}$, consider the $\frac{1}{b}$-periodic function

$$H_{l,\,n}(x) = \sum_{k \in \mathbb{Z}} f_l\left(x + \frac{k}{b}\right)\overline{h_l\left(x + \frac{k}{b} - na\right)},$$

which is well defined for a.e. $x \in \mathbb{R}$ by Lemma 8.2.2 in [8]. Observe that, for a given $x \in \mathbb{R}$, the compact support of $\mathbf{f}$ implies that, for each $1 \leq l \leq L$, $f_l\left(x + \frac{k}{b}\right)$ can be non-zero only for finitely k-values and the number of k-values for which $f_l\left(x + \frac{k}{b}\right) \neq 0$ is uniformly bounded. It follows that there exists a positive constant K_l such that

$$\int_{[0,\,\frac{1}{b}]} |H_{l,\,n}(x)|^2 dx \leq K_l \sum_{k \in \mathbb{Z}} \int_{[0,\,\frac{1}{b}]} \left| f_l\left(x + \frac{k}{b}\right)\right|^2 \left| h_l\left(x + \frac{k}{b} - na\right)\right|^2 dx$$

$$= K_l \int_{\mathbb{R}} |f_l(x)|^2 |h_l(x - na)|^2 dx,$$

which combined with the boundedness of f_l yields that $H_{l,\,n} \in L^2([0,\,\frac{1}{b}])$. By Lemma 8.4.2 in [8],

$$\langle \mathbf{f}, \mathbf{M}_{mb}\mathbf{T}_{na}\mathbf{h}\rangle = \sum_{l=1}^{L} \langle f_l, M_{mb}T_{na}h_l\rangle = \sum_{l=1}^{L} \int_{[0,\,\frac{1}{b}]} H_{l,\,n}(y) e^{-2\pi i m b y} dy$$

for $m,\, n \in \mathbb{Z}$. Then (2.5) can be rewritten as

$$A^{-1}\mathbf{f}(x) = \sum_{m,\,n \in \mathbb{Z}} \left(\sum_{l=1}^{L} \int_{[0,\,\frac{1}{b}]} H_{l,\,n}(y) e^{-2\pi i m b y} dy\right) e^{2\pi i m b x}\mathbf{g}(x - na) = \frac{1}{b}\sum_{n \in \mathbb{Z}}\sum_{l=1}^{L} H_{l,\,n}(x)\mathbf{g}(x - na)$$

for a.e. $x \in \mathbb{R}$, which together with the $\frac{1}{b}$-periodicity of $H_{l,\,n}$ shows that

$$A^{-1}\mathbf{f}\left(x + \frac{j}{b}\right) = \frac{1}{b}\sum_{n \in \mathbb{Z}}\sum_{l=1}^{L} H_{l,\,n}(x)\mathbf{g}\left(x + \frac{j}{b} - na\right)$$

for a.e. $x \in \mathbb{R}$, or equivalently, (2.4) holds. The proof is completed. □

In particular, by letting $A = (I,\, I,\, \ldots,\, I)^t$ in Theorem 2.1, we obtain a characterization of super Gabor duals.

Corollary 2.1 — Let $\mathbf{g},\, \mathbf{h} \in L^2(\mathbb{R}, \mathbb{C}^L)$ be such that $\mathcal{G}(\mathbf{g},\, a,\, b)$ and $\mathcal{G}(\mathbf{h},\, a,\, b)$ are both super Gabor Bessel sequences in $L^2(\mathbb{R}, \mathbb{C}^L)$. Then $\mathbf{h}$ is a super Gabor dual of $\mathbf{g}$ if and only if

$$\mathcal{M}_{\mathbf{g},a,b}(\cdot)\mathcal{M}^*_{\mathbf{h},a,b}(\cdot)C = bC \tag{2.6}$$

a.e. on $\mathbb{R}$ for all finite sequences C.

PROOF : Let $A = (I,\, I,\, \ldots,\, I)^t$ in Theorem 2.1. Then $\mathbf{h}$ is a super Gabor dual of $\mathbf{g}$ if and only if

$$\mathcal{M}_{\mathbf{g},a,b}(\cdot)\mathcal{M}^*_{\mathbf{h},a,b}(\cdot)F_{\mathbf{f}}(\cdot) = bF_{\mathbf{f}}(\cdot) \tag{2.7}$$

a.e. on $\mathbb{R}$ for all bounded functions $\mathbf{f}$ with compact support. So, to finish the proof, we only need to prove the equivalence between (2.6) and (2.7). Obviously, (2.6) implies (2.7) since for a.e $x \in \mathbb{R}$, $F_{\mathbf{f}}(x)$ is a finite sequence if $\mathbf{f}$ is a bounded function with compact support. Now suppose (2.7) holds. Let C be a finite sequence. We can write C as $C = (\ldots,\, 0,\, C_1,\, C_2,\, \ldots,\, C_K,\, 0,\, \ldots)^t$, where $C_k = (c_{k,1},\, c_{k,\,2},\, \ldots,\, c_{k,\,L})^t$ with $1 \le k \le K$. Given arbitrarily $\ell \in \mathbb{Z}$. For $1 \le l \le L$, we define f_l by

$$f_l\left(\cdot + \frac{k}{b}\right) = \begin{cases} c_{k,\,l} & \text{if } 1 \le k \le K; \\ 0 & \text{otherwise} \end{cases}$$

on $\left[\frac{\ell}{b},\, \frac{\ell+1}{b}\right]$. Then f_l is a measurable and bounded function on $\mathbb{R}$ with compact support, and so is $\mathbf{f}$. Observe that $F_{\mathbf{f}}(x) = C$ for $x \in \left[\frac{\ell}{b},\, \frac{\ell+1}{b}\right]$. By (2.7), we have

$$\mathcal{M}_{\mathbf{g},a,b}(\cdot)\mathcal{M}^*_{\mathbf{h},a,b}(\cdot)C = bC$$

a.e. on $\left[\frac{\ell}{b},\, \frac{\ell+1}{b}\right]$, and therefore, it holds a.e. on $\mathbb{R}$ by the arbitrariness of ℓ. The proof is completed. □

Remark 2.1 : Let $L = 1$, then Corollary 2.1 is reduced to the duality condition for a pair of Gabor systems, that is, two Gabor Bessel systems $\mathcal{G}(g,\, a,\, b)$ and $\mathcal{G}(h,\, a,\, b)$ in $L^2(\mathbb{R})$ form dual frames for $L^2(\mathbb{R})$ if and only if

$$\sum_{n\in\mathbb{Z}} g\left(\cdot + \frac{j}{b} - na\right)\overline{h(\cdot - na)} = b\delta_{j,\,0}$$

a.e. on $\mathbb{R}$ for $j \in \mathbb{Z}$, which is presented in [14] and [17].

When $A \in \bigoplus_{l=1}^{L} \mathcal{BI}\left(L^2(\mathbb{R})\right)$ commutes with $\mathbf{M}_b$ and $\mathbf{T}_a$, we obtain a characterization of generalized super Gabor dual of $\mathbf{g}$ with A by applying Corollary 2.1.

***Theorem* 2.2** — *Let* $\mathbf{g},\, \mathbf{h} \in L^2(\mathbb{R}, \mathbb{C}^L)$, *and let* $A \in \bigoplus_{l=1}^{L} \mathcal{BI}\left(L^2(\mathbb{R})\right)$ *commute with* $\mathbf{M}_b$ *and* $\mathbf{T}_a$. *Suppose* $\mathcal{G}(\mathbf{g},\, a,\, b)$ *and* $\mathcal{G}(\mathbf{h},\, a,\, b)$ *are both super Gabor Bessel sequences in* $L^2(\mathbb{R}, \mathbb{C}^L)$. *Then* $\mathbf{h}$ *is a generalized super Gabor dual of* $\mathbf{g}$ *with* A *if and only if*

$$\mathcal{M}_{\mathbf{g},a,b}(\cdot)\mathcal{M}^*_{A^*\mathbf{h},a,b}(\cdot)C = bC \tag{2.8}$$

a.e. on $\mathbb{R}$ *for all finite sequences* C.

PROOF : Observe that $\mathbf{M}_{-b}$ is the inverse of $\mathbf{M}_b$, and $\mathbf{T}_{-a}$ is the inverse of $\mathbf{T}_a$. We conclude that A commutes with $\mathbf{M}_{\pm b}$ and $\mathbf{T}_{\pm a}$, so does A^*. Then A^* commutes with $\mathbf{M}_{mb}$ and $\mathbf{T}_{na}$ for $m,\, n \in \mathbb{Z}$.

It follows that $\mathbf{h}$ is a generalized super Gabor dual of $\mathbf{g}$ with A if and only if

$$\mathbf{f} = \sum_{m,\, n\in\mathbb{Z}} \langle \mathbf{f}, \mathbf{M}_{mb}\mathbf{T}_{na}A^*\mathbf{h}\rangle\, \mathbf{M}_{mb}\mathbf{T}_{na}\mathbf{g} \tag{2.9}$$

for $\mathbf{f} \in L^2(\mathbb{R},\, \mathbb{C}^L)$. Recall that A is a bounded operator. So $\mathcal{G}(A^*\mathbf{h},\, a,\, b)$ is a super Gabor Bessel sequence in $L^2(\mathbb{R}, \mathbb{C}^L)$. Then (2.9) holds if and only if $A^*\mathbf{h}$ is a super Gabor dual of $\mathbf{g}$, which is equivalent to that (2.8) holds by Corollary 2.1. □

Remark 2.2 : Let $p \in \mathbb{Z}$ and $\lambda \in \mathbb{C}\backslash\{0\}$. Define the operator $\mathbf{N}_\lambda$ on $L^2(\mathbb{R},\, \mathbb{C}^L)$ by $\mathbf{N}_\lambda\mathbf{f} = \lambda\mathbf{f}$ for $\mathbf{f} \in L^2(\mathbb{R},\, \mathbb{C}^L)$. Then $\mathbf{T}_{\frac{p}{b}}$, $\mathbf{M}_{\frac{p}{a}}$, $\mathbf{N}_\lambda \in \bigoplus_{l=1}^{L} \mathcal{BI}\left(L^2(\mathbb{R})\right)$. Observe that three operators $\mathbf{T}_{\frac{p}{b}}$, $\mathbf{M}_{\frac{p}{a}}$ and $\mathbf{N}_\lambda$ commute with the operators $\mathbf{M}_b$ and $\mathbf{T}_a$, and

$$\mathbf{T}^*_{\frac{p}{b}} = \mathbf{T}_{-\frac{p}{b}}, \quad \mathbf{M}^*_{\frac{p}{a}} = \mathbf{M}_{-\frac{p}{a}}, \quad \mathbf{N}^*_\lambda = \mathbf{N}_{\overline{\lambda}}$$

In Theorem 2.2, let A be $\mathbf{T}_{\frac{p}{b}}$, $\mathbf{M}_{\frac{p}{a}}$ and $\mathbf{N}_\lambda$, respectively. Then by Theorem 2.2 the following hold:

(1) $\mathbf{h}$ is a generalized super Gabor dual of $\mathbf{g}$ with $\mathbf{T}_{\frac{p}{b}}$ if and only if

$$\mathcal{M}_{\mathbf{g},a,b}(\cdot)\mathcal{M}^*_{\mathbf{h},a,b}\left(\cdot + \frac{p}{b}\right) C = bC$$

a.e. on $\mathbb{R}$ for all finite sequences C.

(2) $\mathbf{h}$ is a generalized super Gabor dual of $\mathbf{g}$ with $\mathbf{M}_{\frac{p}{a}}$ if and only if

$$\mathcal{M}_{\mathbf{g},a,b}(\cdot)\mathcal{M}^*_{\mathbf{M}_{-\frac{p}{a}}\mathbf{h},a,b}(\cdot)\, C = bC$$

a.e. on $\mathbb{R}$ for all finite sequences C.

(3) $\mathbf{h}$ is a generalized super Gabor dual of $\mathbf{g}$ with $\mathbf{N}_\lambda$ if and only if

$$\mathcal{M}_{\mathbf{g},a,b}(\cdot)\mathcal{M}^*_{\mathbf{h},a,b}(\cdot)\, C = \frac{b}{\lambda}C$$

a.e. on $\mathbb{R}$ for all finite sequences C.

In the following theorem we derive an explicit expression of all generalized super Gabor duals of $\mathbf{g}$ with A, which commutes with $\mathbf{M}_b$ and $\mathbf{T}_a$.

***Theorem* 2.3** — *Suppose that $\mathcal{G}(\mathbf{g},\, a,\, b)$ is a super Gabor frame for $L^2(\mathbb{R},\, \mathbb{C}^L)$ with frame operator $S_{\mathbf{g},\,\mathbf{g}}$. Let $A \in \bigoplus_{l=1}^{L} \mathcal{BI}\left(L^2(\mathbb{R})\right)$ commute with $\mathbf{M}_b$ and $\mathbf{T}_a$. Then generalized super Gabor duals of $\mathbf{g}$ with A are precisely of the form*

$$\mathbf{h} = (A^*)^{-1}\, S^{-1}_{\mathbf{g},\,\mathbf{g}}\mathbf{g} + \phi - \sum_{m,\, n\in\mathbb{Z}} \left\langle S^{-1}_{\mathbf{g},\,\mathbf{g}}\mathbf{g},\, \mathbf{M}_{mb}\mathbf{T}_{na}\mathbf{g}\right\rangle \mathbf{M}_{mb}\mathbf{T}_{na}\phi, \tag{2.10}$$

where $\phi \in L^2(\mathbb{R}, \mathbb{C}^L)$ *with* $\mathcal{G}(\phi, a, b)$ *being a super Gabor Bessel sequence in* $L^2(\mathbb{R}, \mathbb{C}^L)$.

PROOF : We know from the proof of Theorem 2.2 that A^* commutes with $\mathbf{M}_{mb}$ and $\mathbf{T}_{na}$ for $m, n \in \mathbb{Z}$, so does $(A^*)^{-1}$. Assume that $\mathbf{h}$ is a generalized super Gabor dual of $\mathbf{g}$ with A. Then, by Lemma 2.1 in [11], $\mathbf{g}$ is a generalized super Gabor dual of $\mathbf{h}$ with A^*, and thus,

$$\mathbf{f} = \sum_{m,\, n\in\mathbb{Z}} \langle A^*\mathbf{f}, \mathbf{M}_{mb}\mathbf{T}_{na}\mathbf{g}\rangle\, \mathbf{M}_{mb}\mathbf{T}_{na}\mathbf{h} \tag{2.11}$$

for $\mathbf{f} \in L^2(\mathbb{R}, \mathbb{C}^L)$. Let $\phi = \mathbf{h} - (A^*)^{-1} S^{-1}_{\mathbf{g},\mathbf{g}}\mathbf{g}$, then $\phi \in L^2(\mathbb{R}, \mathbb{C}^L)$, $\mathcal{G}(\phi, a, b)$ is a super Gabor Bessel sequence in $L^2(\mathbb{R}, \mathbb{C}^L)$, and

$$\begin{aligned}
&\sum_{m,\, n\in\mathbb{Z}} \left\langle S^{-1}_{\mathbf{g},\mathbf{g}}\mathbf{g},\, \mathbf{M}_{mb}\mathbf{T}_{na}\mathbf{g}\right\rangle \mathbf{M}_{mb}\mathbf{T}_{na}\phi \\
=\ & \sum_{m,\, n\in\mathbb{Z}} \left\langle S^{-1}_{\mathbf{g},\mathbf{g}}\mathbf{g},\, \mathbf{M}_{mb}\mathbf{T}_{na}\mathbf{g}\right\rangle \mathbf{M}_{mb}\mathbf{T}_{na}\mathbf{h} - \sum_{m,\, n\in\mathbb{Z}} \left\langle S^{-1}_{\mathbf{g},\mathbf{g}}\mathbf{g},\, \mathbf{M}_{mb}\mathbf{T}_{na}\mathbf{g}\right\rangle \mathbf{M}_{mb}\mathbf{T}_{na} (A^*)^{-1} S^{-1}_{\mathbf{g},\mathbf{g}}\mathbf{g} \\
=\ & (A^*)^{-1} S^{-1}_{\mathbf{g},\mathbf{g}}\mathbf{g} - (A^*)^{-1} \sum_{m,\, n\in\mathbb{Z}} \left\langle S^{-1}_{\mathbf{g},\mathbf{g}}\mathbf{g},\, \mathbf{M}_{mb}\mathbf{T}_{na}\mathbf{g}\right\rangle \mathbf{M}_{mb}\mathbf{T}_{na} S^{-1}_{\mathbf{g},\mathbf{g}}\mathbf{g},
\end{aligned}$$

where we use (2.11) in the last equality. Recall that $S^{-1}_{\mathbf{g},\mathbf{g}}\mathbf{g}$ is a super Gabor dual of $\mathbf{g}$. It follows that

$$\sum_{m,\, n\in\mathbb{Z}} \left\langle S^{-1}_{\mathbf{g},\mathbf{g}}\mathbf{g},\, \mathbf{M}_{mb}\mathbf{T}_{na}\mathbf{g}\right\rangle \mathbf{M}_{mb}\mathbf{T}_{na}\phi = (A^*)^{-1} S^{-1}_{\mathbf{g},\mathbf{g}}\mathbf{g} - (A^*)^{-1} S^{-1}_{\mathbf{g},\mathbf{g}}\mathbf{g} = 0,$$

and therefore, (2.10) holds. For the other implication, assume that $\mathbf{h}$ is of the form (2.10). Then $\mathbf{h} \in L^2(\mathbb{R}, \mathbb{C}^L)$ and $\mathcal{G}(\mathbf{h}, a, b)$ is a super Gabor Bessel sequence in $L^2(\mathbb{R}, \mathbb{C}^L)$. To prove that $\mathbf{h}$ is a generalized super Gabor dual of $\mathbf{g}$ with A, by Theorem 2.2 it suffices to prove that

$$\mathcal{M}_{\mathbf{g},a,b}(\cdot)\mathcal{M}^*_{A^*\mathbf{h},a,b}(\cdot)C = bC \tag{2.12}$$

a.e. on $\mathbb{R}$ for all finite sequences C. Now we prove (2.12) holds. By (2.10), we have

$$A^*\mathbf{h} = S^{-1}_{\mathbf{g},\mathbf{g}}\mathbf{g} + A^*\phi - \sum_{m',\, n'\in\mathbb{Z}} \left\langle S^{-1}_{\mathbf{g},\mathbf{g}}\mathbf{g},\, \mathbf{M}_{m'b}\mathbf{T}_{n'a}\mathbf{g}\right\rangle \mathbf{M}_{m'b}\mathbf{T}_{n'a}A^*\phi,$$

and thus,

$$\begin{aligned}
&\sum_{n\in\mathbb{Z}} \mathbf{g}\left(x+\frac{j}{b}-na\right)(A^*\mathbf{h})^*\left(x+\frac{k}{b}-na\right) \\
=\ & \sum_{n\in\mathbb{Z}} \mathbf{g}\left(x+\frac{j}{b}-na\right)(S^{-1}_{\mathbf{g},\mathbf{g}}\mathbf{g})^*\left(x+\frac{k}{b}-na\right) \\
& + \sum_{n\in\mathbb{Z}} \mathbf{g}\left(x+\frac{j}{b}-na\right)(A^*\phi)^*\left(x+\frac{k}{b}-na\right) - \prod
\end{aligned} \tag{2.13}$$

for a.e. $x \in \mathbb{R}$ and $j,\, k \in \mathbb{Z}$, where

$$\prod = \sum_{n\in\mathbb{Z}} \mathbf{g}\left(x+\frac{j}{b}-na\right) \sum_{m',\,n'\in\mathbb{Z}} \overline{\langle S_{\mathbf{g},\mathbf{g}}^{-1}\mathbf{g},\, \mathbf{M}_{m'b}\mathbf{T}_{n'a}\mathbf{g}\rangle}\, (\mathbf{M}_{m'b}\mathbf{T}_{n'a}A^*\phi)^* \left(x+\frac{k}{b}-na\right).$$

Observing that

$$(\mathbf{M}_{m'b}\mathbf{T}_{n'a}A^*\phi)^* \left(x+\frac{k}{b}-na\right) = e^{-2\pi i m' b(x-na)}\, (A^*\phi)^* \left(x+\frac{k}{b}-(n+n')a\right),$$

and by making the change of variables $n+n' \to n'$ and $m' \to -m'$, we arrive at

$$\begin{aligned}\prod &= \sum_{n\in\mathbb{Z}} \mathbf{g}\left(x+\frac{j}{b}-na\right) \sum_{m',\,n'\in\mathbb{Z}} \langle \mathbf{T}_{n'a}\mathbf{g},\, \mathbf{M}_{m'b}\mathbf{T}_{na}S_{\mathbf{g},\mathbf{g}}^{-1}\mathbf{g}\rangle e^{2\pi i m' bx}\, (A^*\phi)^* \left(x+\frac{k}{b}-n'a\right)\\ &= \sum_{n'\in\mathbb{Z}} \left[\sum_{m',\,n\in\mathbb{Z}} \langle \mathbf{T}_{n'a}\mathbf{g},\, \mathbf{M}_{m'b}\mathbf{T}_{na}S_{\mathbf{g},\mathbf{g}}^{-1}\mathbf{g}\rangle \mathbf{M}_{m'b}\mathbf{T}_{na}\mathbf{g}\left(x+\frac{j}{b}\right)\right] (A^*\phi)^* \left(x+\frac{k}{b}-n'a\right)\\ &= \sum_{n'\in\mathbb{Z}} \mathbf{g}\left(x+\frac{j}{b}-n'a\right) (A^*\phi)^* \left(x+\frac{k}{b}-n'a\right)\end{aligned}$$

for a.e. $x \in \mathbb{R}$ and $j,\, k \in \mathbb{Z}$, where we use that fact that $S_{\mathbf{g},\mathbf{g}}^{-1}\mathbf{g}$ is a super Gabor dual of $\mathbf{g}$ in the last equality. Then (2.13) can be written as

$$\sum_{n\in\mathbb{Z}} \mathbf{g}\left(x+\frac{j}{b}-na\right) (A^*\mathbf{h})^* \left(x+\frac{k}{b}-na\right) = \sum_{n\in\mathbb{Z}} \mathbf{g}\left(x+\frac{j}{b}-na\right) (S_{\mathbf{g},\mathbf{g}}^{-1}\mathbf{g})^* \left(x+\frac{k}{b}-na\right) \tag{2.14}$$

for a.e. $x \in \mathbb{R}$ and $j,\, k \in \mathbb{Z}$. Also since $S_{\mathbf{g},\mathbf{g}}^{-1}\mathbf{g}$ is a super Gabor dual of $\mathbf{g}$, by Corollary 2.1 we have

$$\mathcal{M}_{\mathbf{g},a,b}(\cdot)\mathcal{M}^*_{S_{\mathbf{g},\mathbf{g}}^{-1}\mathbf{g},a,b}(\cdot)C = bC$$

a.e. on $\mathbb{R}$ for all finite sequences C. Write C as $C = (C_k)_{k\in\mathbb{Z}}$, where $C_k = (c_{k,1},\, c_{k,2},\, \ldots,\, c_{k,L})^t$. Then

$$\sum_{k\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} \mathbf{g}\left(x+\frac{j}{b}-na\right) (S_{\mathbf{g},\mathbf{g}}^{-1}\mathbf{g})^* \left(x+\frac{k}{b}-na\right) C_k = bC_j$$

for a.e. $x \in \mathbb{R}$, $j \in \mathbb{Z}$ and all finite sequences C. This combined with (2.14) shows that

$$\sum_{k\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} \mathbf{g}\left(x+\frac{j}{b}-na\right) (A^*\mathbf{h})^* \left(x+\frac{k}{b}-na\right) C_k = bC_j,$$

for a.e. $x \in \mathbb{R}$, $j \in \mathbb{Z}$ and all finite sequences C, which implies that (2.12) holds. The proof is completed. □

***Theorem* 2.4** — *Suppose that $\mathcal{G}(\mathbf{g},\, a,\, b)$ is a super Gabor frame for $L^2(\mathbb{R},\, \mathbb{C}^L)$ with frame operator $S_{\mathbf{g},\,\mathbf{g}}$. Let $A \in \bigoplus_{l=1}^{L} \mathcal{BI}\left(L^2(\mathbb{R})\right)$ commute with $\mathbf{M}_b$ and $\mathbf{T}_a$. Then $\mathbf{g}$ has a unique generalized super Gabor dual with A if and only if $\mathcal{G}(\mathbf{g},\, a,\, b)$ is a super Gabor Riesz basis for $L^2(\mathbb{R},\, \mathbb{C}^L)$.*

PROOF : First we prove the sufficiency. Suppose that $\mathcal{G}(\mathbf{g},\, a,\, b)$ is a super Gabor Riesz basis for $L^2(\mathbb{R},\, \mathbb{C}^L)$. By Theorem 6.1.1 in [8], $\mathcal{G}(\mathbf{g},\, a,\, b)$ and $\mathcal{G}(S^{-1}_{\mathbf{g},\,\mathbf{g}}\mathbf{g},\, a,\, b)$ are biorthogonal. Then

$$\left\langle S^{-1}_{\mathbf{g},\,\mathbf{g}}\mathbf{g},\, \mathbf{M}_{mb}\mathbf{T}_{na}\mathbf{g}\right\rangle = \delta_{m,\,0}\delta_{n,\,0}$$

for $m,\, n \in \mathbb{Z}$, and therefore, for any $\phi \in L^2(\mathbb{R},\, \mathbb{C}^L)$ with $\mathcal{G}(\phi,\, a,\, b)$ being a super Gabor Bessel sequence in $L^2(\mathbb{R},\, \mathbb{C}^L)$, we have

$$\sum_{m,\,n\in\mathbb{Z}} \langle S^{-1}_{\mathbf{g},\,\mathbf{g}}\mathbf{g},\, \mathbf{M}_{mb}\mathbf{T}_{na}\mathbf{g}\rangle \mathbf{M}_{mb}\mathbf{T}_{na}\phi = \phi.$$

This combined with Theorem 2.3 shows that $(A^*)^{-1}\, S^{-1}_{\mathbf{g},\,\mathbf{g}}\mathbf{g}$ is the unique generalized super Gabor dual of $\mathbf{g}$ with A. Next we prove the necessity by contradiction. Assume that $\mathbf{g}$ has a unique generalized super Gabor dual with A, but $\mathcal{G}(\mathbf{g},\, a,\, b)$ is not a super Gabor Riesz basis for $L^2(\mathbb{R},\, \mathbb{C}^L)$. Then, by Theorem 4.4 in [23], there exist $\mathbf{h},\, \phi \in L^2(\mathbb{R},\, \mathbb{C}^L)$ such that they are two different super Gabor duals of $\mathbf{g}$. Also observing that $(A^*)^{-1}$ commutes with $\mathbf{M}_b$ and $\mathbf{T}_a$, we have that both $(A^*)^{-1}\, \mathbf{h}$ and $(A^*)^{-1}\, \phi$ are generalized super Gabor duals of $\mathbf{g}$ with A. This contradicts with the fact that $\mathbf{g}$ has a unique generalized super Gabor dual with A. The proof is completed. □

3. PERTURBATION

Let $\mathbf{g},\, \mathbf{h} \in L^2(\mathbb{R}, \mathbb{C}^L)$ be such that $\mathbf{h}$ is a generalized super Gabor dual of $\mathbf{g}$ with $A \in \bigoplus_{l=1}^{L} \mathcal{BI}\left(L^2(\mathbb{R})\right)$, and let $\phi \in L^2(\mathbb{R}, \mathbb{C}^L)$. In this section we consider the following two perturbation questions:

Question 1 : If ϕ is close to $\mathbf{g}$, does it follows that $\mathbf{h}$ is a generalized super Gabor dual of ϕ with some bounded invertible operator?

Question 2 : If ϕ is close to $\mathbf{h}$, does it follows that ϕ is a generalized super Gabor dual of $\mathbf{g}$ with some bounded invertible operator?

To answer Question 1, we in the following theorem provide a sufficient condition for ϕ such that $\mathbf{h}$ is a generalized super Gabor dual of ϕ with $(U_\phi U^*_{\mathbf{h}})^{-1}$.

***Theorem* 3.1** — *Let $\mathbf{g},\, \mathbf{h} \in L^2(\mathbb{R}, \mathbb{C}^L)$ be such that $\mathbf{h}$ is a generalized super Gabor dual of $\mathbf{g}$*

with $A \in \bigoplus_{l=1}^{L} \mathcal{BI}\left(L^2(\mathbb{R})\right)$*, and let* $\phi = (\phi_1,\, \phi_2,\, \ldots, \phi_L) \in L^2(\mathbb{R}, \mathbb{C}^L)$ *satisfy*

$$R_l := \frac{1}{b} \sup_{x\in[0,\,a]} \sum_{k\in\mathbb{Z}} \left| \sum_{n\in\mathbb{Z}} \mathbf{T}_{na}(\phi_l(x) - g_l(x))\mathbf{T}_{na+\frac{k}{b}}(\phi_l(x) - g_l(x)) \right| < \infty \tag{3.1}$$

for $1 \le l \le L$*. Denote by* $U_{\mathbf{h}}$ *and* U_ϕ *the pre-frame operator for* $\mathcal{G}(\mathbf{h},\, a,\, b)$ *and* $\mathcal{G}(\phi,\, a,\, b)$*, respectively. If* $\max_{1\le l\le L} R_l < \frac{1}{LD\|A\|^2}$*, then* $\mathbf{h}$ *is a generalized super Gabor dual of* ϕ *with* $(U_\phi U_{\mathbf{h}}^*)^{-1}$*, where* D *is the upper frame bound for* $\mathcal{G}(\mathbf{h},\, a,\, b)$*.*

PROOF : By Theorem 8.4.4 in [8], $\mathcal{G}(\phi_l - g_l,\, a,\, b)$ is a Gabor Bessel sequence in $L^2(\mathbb{R})$ with bound R_l for each $1 \le l \le L$, and therefore, $\mathcal{G}(\phi - \mathbf{g},\, a,\, b)$ is a super Gabor Bessel sequence in $L^2(\mathbb{R},\, \mathbb{C}^L)$ with bound $L \max_{1\le l\le L} R_l$ by Proposition 1.1. Also since $\mathcal{G}(\mathbf{g},\, a,\, b)$ is a super Gabor Bessel sequence in $L^2(\mathbb{R},\, \mathbb{C}^L)$, so is $\mathcal{G}(\phi,\, a,\, b)$. Then

$$(I - U_\phi U_{\mathbf{h}}^* A)\,\mathbf{f} = \mathbf{f} - \sum_{m,n\in\mathbb{Z}} \langle A\mathbf{f},\, \mathbf{M}_{mb}\mathbf{T}_{na}\mathbf{h}\rangle\, \mathbf{M}_{mb}\mathbf{T}_{na}\phi \tag{3.2}$$

for $\mathbf{f} \in L^2(\mathbb{R},\, \mathbb{C}^L)$. Since $\mathbf{h}$ is a generalized super Gabor dual of $\mathbf{g}$ with A, we have

$$\mathbf{f} = \sum_{m,n\in\mathbb{Z}} \langle A\mathbf{f},\, \mathbf{M}_{mb}\mathbf{T}_{na}\mathbf{h}\rangle\, \mathbf{M}_{mb}\mathbf{T}_{na}\mathbf{g}$$

for $\mathbf{f} \in L^2(\mathbb{R},\, \mathbb{C}^L)$, and thus (3.2) can be rewritten as

$$(I - U_\phi U_{\mathbf{h}}^* A)\,\mathbf{f} = \sum_{m,n\in\mathbb{Z}} \langle A\mathbf{f},\, \mathbf{M}_{mb}\mathbf{T}_{na}\mathbf{h}\rangle\, \mathbf{M}_{mb}\mathbf{T}_{na}\,(\mathbf{g} - \phi)$$

for $\mathbf{f} \in L^2(\mathbb{R},\, \mathbb{C}^L)$. Recall that $\mathcal{G}(\phi - \mathbf{g},\, a,\, b)$ is a super Gabor Bessel sequence in $L^2(\mathbb{R},\, \mathbb{C}^L)$ with bound $L \max_{1\le l\le L} R_l$. By Theorem 3.2.3 in [8], we attain

$$\begin{aligned} \|(I - U_\phi U_{\mathbf{h}}^* A)\,\mathbf{f}\|^2 &\le L \max_{1\le l\le L} R_l \sum_{m,n\in\mathbb{Z}} |\langle A\mathbf{f},\, \mathbf{M}_{mb}\mathbf{T}_{na}\mathbf{h}\rangle|^2 \\ &\le LD \max_{1\le l\le L} R_l \|A\mathbf{f}\|^2 \\ &\le LD \max_{1\le l\le L} R_l \|A\|^2 \|\mathbf{f}\|^2 \end{aligned}$$

for $\mathbf{f} \in L^2(\mathbb{R},\, \mathbb{C}^L)$. It follows that

$$\|(I - U_\phi U_{\mathbf{h}}^* A)\| \le \sqrt{LD \max_{1\le l\le L} R_l} \|A\| < 1.$$

By Theorem A.5.3 in [8], $U_\phi U_{\mathbf{h}}^* A$ is invertible, and so is $U_\phi U_{\mathbf{h}}^*$. Then

$$\mathbf{f} = (U_\phi U_{\mathbf{h}}^*)\,(U_\phi U_{\mathbf{h}}^*)^{-1}\,\mathbf{f} = \sum_{m,\,n\in\mathbb{Z}} \left\langle (U_\phi U_{\mathbf{h}}^*)^{-1}\,\mathbf{f},\, \mathbf{M}_{mb}\mathbf{T}_{na}\mathbf{h}\right\rangle \mathbf{M}_{mb}\mathbf{T}_{na}\phi$$

for $\mathbf{f} \in L^2(\mathbb{R},\,\mathbb{C}^L)$. This shows that $\mathbf{h}$ is a generalized super Gabor dual of ϕ with $(U_\phi U_{\mathbf{h}}^*)^{-1}$. □

Corollary 3.1 — Suppose that $\mathcal{G}(\mathbf{g},\,a,\,b)$ is a super Gabor frame for $L^2(\mathbb{R},\,\mathbb{C}^L)$ with upper frame bound D. Let $\phi = (\phi_1,\,\phi_2,\,\ldots,\phi_L) \in L^2(\mathbb{R},\mathbb{C}^L)$ satisfy

$$R_l := \frac{1}{b}\sup_{x\in[0,\,a]}\sum_{k\in\mathbb{Z}}\left|\sum_{n\in\mathbb{Z}}\mathbf{T}_{na}(\phi_l(x)-g_l(x))\mathbf{T}_{na+\frac{k}{b}}(\phi_l(x)-g_l(x))\right| < \infty \tag{3.3}$$

for $1 \le l \le L$. Denote by $U_{\mathbf{g}}$ and U_ϕ the pre-frame operator for $\mathcal{G}(\mathbf{g},\,a,\,b)$, and $\mathcal{G}(\phi,\,a,\,b)$, respectively. If $\max\limits_{1\le l\le L} R_l < \frac{1}{LD\|\left(U_{\mathbf{g}}U_{\mathbf{g}}^*\right)^{-1}\|^2}$, then $\mathbf{g}$ is a generalized super Gabor dual of ϕ with $\left(U_\phi U_{\mathbf{g}}^*\right)^{-1}$.

PROOF : Recall that $\mathbf{g}$ is a generalized super Gabor dual of itself with $\left(U_{\mathbf{g}}U_{\mathbf{g}}^*\right)^{-1}$. In Theorem 3.1 let $\mathbf{h} = \mathbf{g}$ and $A = \left(U_{\mathbf{g}}U_{\mathbf{g}}^*\right)^{-1}$, then the desired result follows from Theorem 3.1. □

In the following theorem, a sufficient condition for ϕ is given such that ϕ is a generalized super Gabor dual of $\mathbf{g}$ with $\left(U_{\mathbf{g}}U_\phi^*\right)^{-1}$, which answers Question 2.

***Theorem* 3.2** — *Let* $\mathbf{g},\,\mathbf{h} \in L^2(\mathbb{R},\mathbb{C}^L)$ *be such that* $\mathbf{h}$ *is a generalized super Gabor dual of* $\mathbf{g}$ *with* $A \in \bigoplus_{l=1}^{L} \mathcal{BI}\left(L^2(\mathbb{R})\right)$, *and let* $\phi = (\phi_1,\,\phi_2,\,\ldots,\phi_L) \in L^2(\mathbb{R},\mathbb{C}^L)$ *satisfy*

$$R_l := \frac{1}{b}\sup_{x\in[0,\,a]}\sum_{k\in\mathbb{Z}}\left|\sum_{n\in\mathbb{Z}}\mathbf{T}_{na}(\phi_l(x)-h_l(x))\mathbf{T}_{na+\frac{k}{b}}(\phi_l(x)-h_l(x))\right| < \infty \tag{3.4}$$

for $1 \le l \le L$. *Denote by* $U_{\mathbf{g}}$ *and* U_ϕ *the pre-frame operator for* $\mathcal{G}(\mathbf{g},\,a,\,b)$ *and* $\mathcal{G}(\phi,\,a,\,b)$, *respectively. If* $\max\limits_{1\le l\le L} R_l < \frac{1}{LD\|A\|^2}$, *then* ϕ *is a generalized super Gabor dual of* $\mathbf{g}$ *with* $\left(U_{\mathbf{g}}U_\phi^*\right)^{-1}$, *where* D *is the upper frame bound for* $\mathcal{G}(\mathbf{g},\,a,\,b)$.

PROOF : By Lemma 2.1 in [11], $\mathbf{g}$ is a generalized super Gabor dual of $\mathbf{h}$ with A^*, which together with Theorem 3.1 shows that $\mathbf{g}$ is a generalized super Gabor dual of ϕ with $\left(U_\phi U_{\mathbf{g}}^*\right)^{-1}$. Also by Lemma 2.1 in [11], ϕ is a generalized super Gabor dual of $\mathbf{g}$ with $\left(U_{\mathbf{g}}U_\phi^*\right)^{-1}$. □

ACKNOWLEDGEMENT

The authors would like to thank the referees for their valuable suggestions, which greatly improve the quality of this article.

REFERENCES

1. L. D. Abreu, On the structure of Gabor and super Gabor spaces, *Monatsh. Math.*, **161** (2010), 237-253.
2. L. D. Abreu, Super-wavelets versus poly-Bergman spaces, *Integr. Equat. Oper. Theory*, **73** (2012), 177-193.
3. L. D. Abreu, Superframes and polyanalytic wavelets, *J. Fourier Anal. Appl.*, **23** (2017), 1-20.
4. R. Balan, *Multiplexing of signals using superframes*, Wavelets and Applications in Signal and Image Processing VIII, A. Aldroubi, A. Laine (Eds.), SPIE Proceedings, 4119 (2000), 118-130.
5. R. Balan, Extensions of no-go theorems to many signal systems, Wavelets, multiwavelets, and their applications (San Diego, CA, 1997), 3-14, *Contemp. Math.*, 216, *Amer. Math. Soc.*, Providence, RI, 1998.
6. R. Balan, Density and redundancy of the noncoherent Weyl-Heisenberg superframes, *Contemp. Math.*, **247** (1999), 29-41.
7. S. Bildea, D. E. Dutkay and G. Picioroaga, MRA super-wavelets, *New York J. Math.*, **11** (2005), 1-19.
8. O. Christensen, *An introduction to frames and Riesz bases*, Birkhäuser, Boston, 2003.
9. O. Christensen and R. S. Laugesen, Approximately dual frames in Hilbert spaces and application to Gabor frames, *Sampl. Theory Signal Image Process*, **9** (2008), 77-89.
10. I. Daubechies, The wavelet transform, time-frequency localization and signal analysis, *IEEE Trans. Inform. Theory*, **36** (1990), 961-1005.
11. M. A. Dehghan, M. A. H. Fard, G-dual frames in Hilbert spaces, *U. P. B. Sci. Bull., Series A*, **75** (2013), 129-140.
12. D. E. Dutkay, The local trace function for super-wavelets, Wavelets, frames and operator theory, *Contem. Math. Amer. Math. Soc.*, **345** (2004), 115-136.
13. D. E. Dutkay and P. Jorgensen, Oversampling generates super-wavelets, *Proc. Amer. Math. Soc.*, **135** (2007), 2219-2227.
14. H. G. Feichtinger and T. Strohmer, eds., *Gabor analysis and algorithms, theory and applications*, in Applied and Numerical Harmonic Analysis, Birkhäuser, Boston, 1998.
15. H. G. Feichtinger and T. Strohmer, eds., *Advances in Gabor analysis*, in Applied and Numerical Harmonic Analysis, Birkhäuser, Boston, 2003.
16. H. Führ, Simultaneous estimates for vector-valued Gabor frames of Hermite functions, *Adv. Comput. Math.*, **29** (2008), 357-373.
17. K. Gröchenig, *Foundations of time-frequency analysis*, Birkhäuser, Boston, 2001.

18. K. Gröchenig and Y. Lyubarskii, Gabor (super)frames with Hermite functions, *Math. Ann.*, **345** (2009), 267-286.

19. Q. Gu and D. Han, Super-wavelets and decomposable wavelet frames, *J. Fourier Anal. Appl.*, **11** (2005), 683-696.

20. D. Han and D. Larson, Frames, bases and group representations, *Mem. Amer. Math. Soc.*, **147**(697) (2000), 94.

21. H. Javanshiri, Some properties of generalized and approximately dual frames in Hilbert spaces, *Results Math.*, **70** (2016), 475-485.

22. Z.-Y. Li and D. Han, Constructing super Gabor frames: the rational time-frequency lattice case, *Sci. China Math.*, **53** (2010), 3179-3186.

23. Y.-Z. Li and F.-Y. Zhou, Rational time-frequency super Gabor frames and their duals, *J. Math. Anal. Appl.*, **403** (2013), 619-632.

24. Y.-Z. Li and Y. Zhang, Vector-valued Gabor frames associated with periodic subsets of the real line, *Appl. Math. Comput.*, **253** (2015), 102-115.

25. R. M. Young, *An introduction to nonharmonic Fourier series*, Academic Press, New York, 1980.

26. Z. Zhao and W. Sun, Sufficient conditions and stability of wavelet superframes, *Appl. Anal.*, **91** (2012), 1393-1406.