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RIGIDITY OF τ-QUASI RICCI-HARMONIC METRICS[1]

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We study τ-quasi Ricci-harmonic metrics. First, we shall derive some formulae which will give some integral formulae for such a class of compact manifolds that permit to obtain some rigidity results. Second, and particularly, if a τ-quasi Ricci-harmonic metric possesses constant generalized scalar curvature then we determine the generalized scalar curvature in explicit form. These results are generalizations of ones found in [1-3, 5, 10].

Key words : Gradient Ricci-harmonic soliton metric; quasi Ricci-harmonic metric; harmonic Einstein metric; rigid property; scalar curvature.

1. INTRODUCTION

Geometric heat flows are one of the most powerful tools in mathematics and have been studied extensively. In [6], Eells and Sampson introduced the harmonic map heat flow

$$\frac{\partial \phi}{\partial t}(t) = \tau_g \phi(t), \quad \phi(0) = \phi_0$$

to find a harmonic map homotopic to a given map $\phi_0 : (M, g) \to (N, h)$, where $\phi_t : (M, g) \to (N, h)$ is a family of smooth maps between two Riemannian manifolds and $\tau_g \phi_t := trace \nabla d\phi(t)$ denotes the tension field of $\phi(t)$ with respect to g. Inspired by the work of Eells and Sampson [6], Hamilton [8] introduced the Ricci flow

$$\frac{\partial g}{\partial t}(t) = -2Ric_{g(t)}, \quad g(0) = g_0$$

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to find a canonical metric on given Riemannian manifolds, where $Ric_{g(t)}$ denotes the Ricci tensor with respect to the metric $g(t)$ and g_0 is an initial Riemannian metric on M. In order to find a harmonic map between two Riemannian manifolds, the above two flows were combined by Müller [14] to introduce the following Ricci-harmonic flow.

Let $(M, g(t))$ be a family of complete Riemannian manifolds with Riemannian metrics $g(t)$ evolving by the Ricci-harmonic flow

$$\begin{cases} \frac{\partial g}{\partial t}(t) = -2Ric_{g(t)} + 2\alpha(t)\nabla\phi(t) \otimes \nabla\phi(t), \;\; g(0) = g_0, \\ \frac{\partial \phi}{\partial t}(t) = \tau_{g(t)}\phi(t), \;\; \phi(0) = \phi_0, \end{cases} \tag{1.1}$$

where $\alpha(t) \geq 0$ is a non-negative time-dependent coupling constant, $\phi(t) : (M, g(t)) \to (N, h)$ is a family of smooth maps between $(M, g(t))$ and a fixed complete Riemannian manifold (N, h) and $\nabla\phi(t) \otimes \nabla\phi(t) := \phi(t)^*h$ is the pull-back of the metric h via $\phi(t)$. In addition to sharing many good properties with the Ricci flow, the Ricci-harmonic flow is less singular than the Ricci flow or the harmonic map flow alone (cf. [13, 14]).

In this paper, we shall focus our attention on soliton-type solutions of the flow (1.1). Müller [12] introduced the following gradient Ricci-harmonic soliton metric.

Definition 1.1 — [12]. Let (N^n, h) be a fixed Riemannian manifold and (M, g) be a complete Riemannian manifold. A metric g of M^m is a **gradient Ricci-harmonic** (with respect to h) **soliton metric**, if for some map $\phi : (M, g) \to (N, h)$, some potential function $f : M \to \mathbb{R}$ and some constant λ, g satisfies the following coupled system

$$\begin{cases} Ric_f - \alpha\nabla\phi \otimes \nabla\phi = \lambda g, \\ \tau_g\phi = \langle\nabla\phi, \nabla f\rangle, \end{cases} \tag{1.2}$$

where (∞-)Bakry-Emery curvature $Ric_f = Ric + \nabla^2 f$, $\tau_g\phi = trace\nabla d\phi$ denotes the tension field of ϕ and α is a time-dependent coupling constant. We call a gradient Ricci-harmonic soliton metric shrinking, steady or expanding, if $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$, respectively.

The solitons (1.2) above correspond to self-similar solutions for the coupled system (1.1). Note that, if $\phi : (M, g) \to (\mathbb{R}, dr^2)$ is a constant function in (1.2), then the soliton is exactly a gradient Ricci soliton, which plays a very important role in Hamilton's Ricci flow, as they correspond to self-similar solutions and often arise as singularity models. We highlight that some explicit examples of gradient Ricci-harmonic soliton metric (cf. [7]). When the potential function f is a constant function, the gradient Ricci-harmonic soliton metric is called **harmonic Einstein**, which satisfies the following coupled system

$$\begin{cases} Ric - \alpha\nabla\phi \otimes \nabla\phi = \lambda g, \\ \tau_g\phi = 0. \end{cases} \tag{1.3}$$

Harmonic Einstein manifolds are trivial examples of gradient Ricci-harmonic soliton metric with constant potential function and thus they are called **trivial gradient Ricci-harmonic soliton metric**. The second equation $\tau_g\phi = 0$ implies ϕ is harmonic. Obviously, harmonic Einstein metrics are natural generalizations of Einstein metrics.

For simplicity, we put

$$Ric_\phi := Ric - \alpha\nabla\phi \otimes \nabla\phi$$

and call it a **generalized Ricci curvature**. We denote by

$$R_\phi := trace(Ric_\phi) = R - \alpha|\nabla\phi|^2$$

and call it a **generalized scalar curvature**.

Because of their importance in both mathematics and physics, the study of the Einstein manifolds and their various generalizations is always an attractive topic in modern Riemannian geometry. In recent years, there has been increasing interest on so-called quasi-Einstein manifolds. Recall that, for a positive integer τ, a complete Riemannian manifold (M^m, g) with a potential function f is called τ-quasi-Einstein if its associated τ-Bakry-Emery Ricci tensor

$$Ric_{f,\tau} = Ric + \nabla^2 f - \frac{1}{\tau} df \otimes df$$

is a constant multiple of the metric g (cf. [3, 9]). Recently, there have been some papers generalizing gradient Ricci solitons, for instance [3] on quasi-Einstein manifolds, [2] on τ-quasi-Einstein manifolds and [11, 15, 18] on Ricci-harmonic solitons. A natural problem is whether there exist similar results for **τ-quasi Ricci-harmonic metrics**. Wang [16] introduced the following τ-quasi Ricci-harmonic metrics.

Definition 1.2 — [16]. Let (N^n, h) be a fixed Riemannian manifold and (M, g) be a complete Riemannian manifold. A metric g of M is a **τ-quasi Ricci-harmonic** (with respect to h), if for some map $\phi : (M, g) \to (N, h)$, some potential function $f : M \to \mathbb{R}$ and some constant λ, g satisfies the following coupled system

$$\begin{cases} Ric_{f,\tau} - \alpha\nabla\phi \otimes \nabla\phi = \lambda g, \\ \tau_g\phi = \langle\nabla\phi, \nabla f\rangle, \end{cases} \tag{1.4}$$

where α is a time-dependent coupling constant.

He derived some rigid properties for the compact τ-quasi Ricci-harmonic metrics. Moreover he derived the lower bound estimates of the generalized scalar curvature for τ-quasi Ricci-harmonic metric in the noncompact case. In this paper, continuing with this topic and motivated by [1-3, 5], we derive some interesting rigid properties for the compact τ-quasi Ricci-harmonic metric which are different from Wang's. Motivated by a recent result of [10], we will show that under a suitable assumption the possible values of the constant generalized scalar curvature are in fact quantified and they are expressed by the constant λ.

This paper is organized as follows: In Section 2 we establish several basic formulae. In Section 3, based on these formulae, we derive rigid properties on compact manifolds. In Section 4 we show that if a $\tau(\tau > 0)$-quasi Ricci-harmonic metric possesses constant generalized scalar curvature then we determine the generalized scalar curvature in explicit form.

2. Identities on τ-quasi Ricci-harmonic Metrices

In this section, we first give some lemmas which will be used later. Before proceeding, we make an observation: from the first equation of (1.4), we have

$$Ric_\phi(\nabla f, \nabla f) + \langle \nabla_{\nabla f}\nabla f, \nabla f\rangle = \frac{1}{\tau}|\nabla f|^4 + \lambda|\nabla f|^2, \tag{2.1}$$

where $\langle , \rangle$ and $|\ \ |$ stand for the metric g and its associated norm, respectively. Taking trace of both members of the first equation of (1.4) we deduce

$$R_\phi + \Delta f - \frac{1}{\tau}|\nabla f|^2 = \lambda m. \tag{2.2}$$

Thereby we derive

$$\langle \nabla f, \nabla R_\phi\rangle + \langle \nabla f, \nabla \Delta f\rangle = \frac{2}{\tau}\langle \nabla f, \nabla_{\nabla f}\nabla f\rangle. \tag{2.3}$$

One notices that combining equations the first equation of (1.4) and (2.2) we infer

$$\nabla^2 f - \frac{\Delta f}{m} g = \frac{1}{\tau}\big(df \otimes df - \frac{1}{m}|\nabla f|^2 g\big) - \big(Ric_\phi - \frac{R_\phi}{m} g\big). \tag{2.4}$$

The following Lemma was given in [16].

Lemma 2.1 — [16]. Let g be a τ-quasi Ricci-harmonic metric defined in Definition 1.2. Then one can get

$$\frac{1}{2}\nabla R_\phi = \frac{\tau - 1}{\tau} Ric_\phi(\nabla f) + \frac{1}{\tau}[R_\phi - (m-1)\lambda]\nabla f \tag{2.5}$$

and

$$\begin{aligned}\frac{1}{2}\Delta R_\phi =&\frac{\tau+2}{2\tau}\langle\nabla R_\phi,\nabla f\rangle-\frac{\tau-1}{\tau}\alpha(\tau_g\phi)^2-\frac{\tau-1}{\tau}|Ric_\phi-\frac{1}{m}R_\phi g|^2\\&-\frac{m+\tau-1}{m\tau}(R_\phi-m\lambda)\big(R_\phi-\frac{m(m-1)}{m+\tau-1}\lambda\big).\end{aligned}\tag{2.6}$$

Moreover, there exists a constant μ such that

$$R_\phi+\frac{\tau-1}{\tau}|\nabla f|^2-(m-\tau)\lambda=\mu e^{\frac{2}{\tau}f}\tag{2.7}$$

and

$$\Delta f+|\nabla f|^2-\tau\lambda+\mu e^{\frac{2}{\tau}f}=0.\tag{2.8}$$

From (1.4) we have the following lemma.

Lemma 2.2 — Let g be a τ-quasi Ricci-harmonic metric defined in Definition 1.2. Then one can get

$$\frac{1}{2}\Delta|\nabla f|^2=|\nabla^2 f|^2-Ric_\phi(\nabla f,\nabla f)+\frac{2}{\tau}|\nabla f|^2\Delta f+\alpha(\tau_g\phi)^2,\tag{2.9}$$

$$\nabla(R_\phi+|\nabla f|^2-2\lambda f)=\frac{2}{\tau}[\nabla_{\nabla f}\nabla f+(|\nabla f|^2-\Delta f)\nabla f],\tag{2.10}$$

$$\frac{1}{2}\langle\nabla R_\phi,\nabla f\rangle=\frac{\tau-1}{\tau}Ric_\phi(\nabla f,\nabla f)+\frac{1}{\tau}[R_\phi-(m-1)\lambda]|\nabla f|^2\tag{2.11}$$

and

$$\begin{aligned}\frac{1}{2}\Delta R_\phi =&-|\nabla^2 f-\frac{\Delta f}{m}g|^2-\frac{(\Delta f)^2}{m}+\lambda\Delta f-\alpha(\tau_g\phi)^2+\langle\nabla R_\phi,\nabla f\rangle\\&+\frac{1}{2}\langle\nabla f,\nabla\Delta f\rangle+\frac{1}{\tau}div(\nabla_{\nabla f}\nabla f-\Delta f\nabla f).\end{aligned}\tag{2.12}$$

PROOF : By taking the divergence of the first equation of (1.4), we have

$$divRic+div\nabla^2 f-\frac{1}{\tau}div(df\otimes df)-\alpha div(\nabla\phi\otimes\nabla\phi)=\lambda g.\tag{2.13}$$

By the contracted second Bianchi identity

$$\nabla R=2divRic,\tag{2.14}$$

the fact that

$$div\nabla^2 f=Ric(\nabla f)+\nabla\Delta f,\tag{2.15}$$

$$\nabla|\nabla\phi|^2=2\nabla_{\nabla\phi}\nabla\phi,$$

$$div(\nabla\phi\otimes\nabla\phi)=\tau_g\phi\nabla\phi+\nabla_{\nabla\phi}\nabla\phi$$

and

$$div(\nabla f\otimes\nabla f)=\Delta f\nabla f+\nabla_{\nabla f}\nabla f,$$

we have that

$$\nabla R+2Ric(\nabla f)+2\nabla\Delta f-\frac{2}{\tau}\Delta f\nabla f-\frac{2}{\tau}\nabla_{\nabla f}\nabla f-2\alpha\tau_g\phi\nabla\phi-2\alpha\nabla_{\nabla\phi}\nabla\phi=0. \tag{2.16}$$

In particular one deduces

$$\begin{aligned}&\langle\nabla R,\nabla f\rangle+2Ric(\nabla f,\nabla f)+2\langle\nabla\Delta f,\nabla f\rangle-\frac{2}{\tau}\langle\Delta f\nabla f,\nabla f\rangle\\ &-\frac{2}{\tau}\langle\nabla_{\nabla f}\nabla f,\nabla f\rangle-2\alpha\tau_g\phi\langle\nabla\phi,\nabla f\rangle-2\alpha\langle\nabla_{\nabla\phi}\nabla\phi,\nabla f\rangle=0.\end{aligned} \tag{2.17}$$

From (2.3) we have

$$\langle\nabla R,\nabla f\rangle=-\langle\nabla f,\nabla\Delta f\rangle+\frac{1}{\tau}\langle\nabla f,\nabla|\nabla f|^2\rangle+\alpha\langle\nabla f,\nabla|\nabla\phi|^2\rangle. \tag{2.18}$$

Plugging (2.18) into (2.17) leads to

$$\langle\nabla\Delta f,\nabla f\rangle+2Ric(\nabla f,\nabla f)-2\alpha(\nabla\phi\otimes\nabla\phi)(\nabla f,\nabla f)-\frac{2}{\tau}\langle\Delta f\nabla f,\nabla f\rangle=0. \tag{2.19}$$

Next using Bochner's formula

$$\frac{1}{2}\Delta|\nabla f|^2=|\nabla^2 f|^2+\langle\nabla f,\nabla\Delta f\rangle+\mathrm{Ric}(\nabla f,\nabla f)$$

and the last identity we conclude

$$\frac{1}{2}\Delta|\nabla f|^2-|\nabla^2 f|^2+\mathrm{Ric}(\nabla f,\nabla f)-2\alpha(\nabla\phi\otimes\nabla\phi)(\nabla f,\nabla f)-\frac{2}{\tau}\langle\Delta f\nabla f,\nabla f\rangle=0,$$

which finishes the first statement of the lemma.

Noticing that $Ric_\phi(\nabla f)+\nabla_{\nabla f}\nabla f=\frac{1}{\tau}|\nabla f|^2\nabla f+\lambda\nabla f$ we use (2.5) to write

$$\begin{aligned}&\frac{1}{2}\nabla(R_\phi+|\nabla f|^2)\\ =&-\frac{1}{\tau}Ric_\phi(\nabla f)+\frac{1}{\tau}[R_\phi-(m-1)\lambda]\nabla f+\frac{1}{\tau}|\nabla f|^2\nabla f+\lambda\nabla f.\end{aligned}$$

Thus, using equation (2.2) once more, we achieve

$$\begin{aligned}
&\nabla(R_\phi+|\nabla f|^2-2\lambda f)\\
=&\frac{2}{\tau}\{|\nabla f|^2\nabla f-Ric_\phi(\nabla f)+[R_\phi-(m-1)\lambda]\nabla f\}\\
=&\frac{2}{\tau}\{[|\nabla f|^2+R_\phi-(m-1)\lambda]\nabla f-Ric_\phi(\nabla f)\}\\
=&\frac{2}{\tau}\{(|\nabla f|^2+\frac{1}{\tau}|\nabla f|^2-\Delta f+\lambda)\nabla f-Ric_\phi(\nabla f)\}\\
=&\frac{2}{\tau}\{(|\nabla f|^2-\Delta f)\nabla f+\frac{1}{\tau}|\nabla f|^2\nabla f+\lambda\nabla f-Ric_\phi(\nabla f)\}\\
=&\frac{2}{\tau}\{(|\nabla f|^2-\Delta f)\nabla f+\nabla_{\nabla f}\nabla f\},
\end{aligned} \tag{2.20}$$

which finishes the second statement of the lemma.

The third statement of the lemma follows from (2.5).

Initially we compute the divergence of identity (2.10) of Lemma 2.2 to obtain

$$\begin{aligned}
\Delta R_\phi+\Delta|\nabla f|^2-2\lambda\Delta f=&\frac{2}{\tau}\{div(\nabla_{\nabla f}\nabla f)+(|\nabla f|^2-\Delta f)\Delta f\\
&+\langle\nabla(|\nabla f|^2-\Delta f),\nabla f\rangle\}.
\end{aligned}$$

Using Bochner's formula

$$\frac{1}{2}\Delta|\nabla f|^2=|\nabla^2 f|^2+\langle\nabla f,\nabla\Delta f\rangle+Ric(\nabla f,\nabla f)$$

and writing

$$|\nabla^2 f|^2=|\nabla^2 f-\frac{\Delta f}{m}g|^2+\frac{1}{m}(\Delta f)^2,$$

we have

$$\begin{aligned}
\frac{1}{2}\Delta R_\phi=&-Ric(\nabla f,\nabla f)-|\nabla^2 f-\frac{\Delta f}{m}g|^2-\frac{1}{m}(\Delta f)^2\\
&+\lambda\Delta f-\langle\nabla f,\nabla\Delta f\rangle+\frac{2}{\tau}\langle\nabla_{\nabla f}\nabla f,\nabla f\rangle\\
&+\frac{1}{\tau}\{div(\nabla_{\nabla f}\nabla f)+(|\nabla f|^2-\Delta f)\Delta f-\langle\nabla\Delta f,\nabla f\rangle\}.
\end{aligned}$$

Next, we invoke equation (2.2) to write

$$\begin{aligned}
\langle\nabla f,\nabla\Delta f\rangle=&\langle\nabla(m\lambda+\frac{1}{\tau}|\nabla f|^2-R_\phi),\nabla f\rangle\\
=&\frac{2}{\tau}\langle\nabla_{\nabla f}\nabla f,\nabla f\rangle-\langle\nabla R_\phi,\nabla f\rangle.
\end{aligned}$$

Noticing that $div(\nabla f\Delta f)=(\Delta f)^2+\langle\nabla f,\nabla\Delta f\rangle$. Then the last relation for $\frac{1}{2}\Delta R_\phi$ turns into

$$\begin{aligned}\frac{1}{2}\Delta R_\phi=&-Ric(\nabla f,\nabla f)-|\nabla^2 f-\frac{\Delta f}{m}g|^2-\frac{(\Delta f)^2}{m}+\lambda\Delta f+\langle\nabla R_\phi,\nabla f\rangle\\&+\frac{1}{\tau}|\nabla f|^2\Delta f+\frac{1}{\tau}div(\nabla_{\nabla f}\nabla f-\Delta f\nabla f).\end{aligned}\tag{2.21}$$

Substituting (2.11) in (2.21), we obtain

$$\begin{aligned}\frac{1}{2}\Delta R_\phi=&-Ric(\nabla f,\nabla f)-|\nabla^2 f-\frac{\Delta f}{m}g|^2-\frac{(\Delta f)^2}{m}+\lambda\Delta f+\frac{1}{2}\langle\nabla R_\phi,\nabla f\rangle\\&+\frac{1}{\tau}|\nabla f|^2\Delta f+\frac{1}{\tau}div(\nabla_{\nabla f}\nabla f-\Delta f\nabla f)+\frac{\tau-1}{\tau}Ric(\nabla f,\nabla f)\\&+\frac{1}{\tau}[R_\phi-(m-1)\lambda]|\nabla f|^2.\end{aligned}$$

From here we deduce

$$\begin{aligned}\frac{1}{2}\Delta R_\phi=&-\frac{1}{\tau}Ric(\nabla f,\nabla f)-|\nabla^2 f-\frac{\Delta f}{m}g|^2-\frac{(\Delta f)^2}{m}+\lambda\Delta f+\frac{1}{2}\langle\nabla R_\phi,\nabla f\rangle\\&+\frac{1}{\tau}div(\nabla_{\nabla f}\nabla f-\Delta f\nabla f)+\frac{1}{\tau}\lambda|\nabla f|^2+\frac{1}{\tau}(R_\phi+\Delta f-m\lambda)|\nabla f|^2.\end{aligned}$$

Next, using (2.2), we infer

$$\begin{aligned}\frac{1}{2}\Delta R_\phi=&-|\nabla^2 f-\frac{\Delta f}{m}g|^2-\frac{(\Delta f)^2}{m}+\lambda\Delta f+\frac{1}{2}\langle\nabla R_\phi,\nabla f\rangle\\&+\frac{1}{\tau}\{-Ric(\nabla f,\nabla f)+\frac{1}{\tau}|\nabla f|^4+\lambda|\nabla f|^2+div(\nabla_{\nabla f}\nabla f-\Delta f\nabla f)\}.\end{aligned}$$

On the other hand, using equation (2.1) and (2.3), we have

$$-Ric(\nabla f,\nabla f)+\frac{1}{\tau}|\nabla f|^4+\lambda|\nabla f|^2=\langle\nabla_{\nabla f}\nabla f,\nabla f\rangle=\frac{\tau}{2}(\langle\nabla f,\nabla R_\phi\rangle+\langle\nabla f,\nabla\Delta f\rangle).$$

Substituting this in the above formula for ΔR_ϕ, noticing that $Ric_\phi(\nabla f,\nabla f)+\alpha(\tau_g\phi)^2=Ric(\nabla f,\nabla f)$, we get the expression in the statement, which completes the proof of the lemma.

Remark 2.3 : Letting $\tau\to\infty$ leads to some basic formulas for the gradient Ricci-harmonic soliton metric (cf. also [15]).

Lemma 2.4 — [15]. Let g be a gradient Ricci-harmonic soliton metric defined in Definition 1.1. Then one can get

$$R_\phi+\Delta f=\lambda m.\tag{2.22}$$

$$R_\phi+|\nabla f|^2-2\lambda f=C,\tag{2.23}$$

where C is a constant.

Remark 2.5 : (1) On a compact gradient Ricci-harmonic soliton, equation (2.22) gives

$$\int\limits_{M^m} (m\lambda - R_\phi)\, d\upsilon_g = 0, \tag{2.24}$$

where $d\upsilon_g$ is the volume form of (M, g).

(2) For a gradient Ricci-harmonic soliton (M, g) it is always possible to choose the potential function f satisfying

$$R_\phi + |\nabla f|^2 - 2\lambda f = 0. \tag{2.25}$$

A gradient Ricci-harmonic soliton (M, g) with such a potential function is simply called **a gradient Ricci-harmonic soliton with normalized potential**.

3. Some Rigidity Results on τ-quasi Ricci-harmonic Metrics

In this section we shall show some integral formulae for a compact Ricci-harmonic metric. Those formulae enable us to derive some rigidity results for such a class of manifolds.

As a consequence of (2.12), we deduce the following integral formulae.

Proposition 3.1 — Let g be a τ-quasi Ricci-harmonic metric defined in Definition 1.2 on compact orientable manifold M. Then one can get

(1)

$$\int\limits_{M^m} |\nabla^2 f - \frac{\Delta f}{m} g|^2\, d\upsilon_g + \int\limits_{M^m} \alpha(\tau_g\phi)^2\, d\upsilon_g = \frac{m-2}{2m}\int\limits_{M^m} \langle \nabla R_\phi, \nabla f\rangle\, d\upsilon_g - \frac{m+2}{2\tau m}\int\limits_{M^m} |\nabla f|^2 \Delta f\, d\upsilon_g.$$

(2)

$$\int\limits_{M^m} |\nabla^2 f - \frac{\Delta f}{m} g|^2\, d\upsilon_g + \frac{m+2}{2m}\int\limits_{M^m} (\Delta f)^2\, d\upsilon_g + \int\limits_{M^m} \alpha(\tau_g\phi)^2\, d\upsilon_g = \int\limits_{M^m} \langle \nabla R_\phi, \nabla f\rangle\, d\upsilon_g.$$

Proof : Since M is compact, we can use (2.12) and Stokes' formula to infer

$$\begin{aligned}\int\limits_{M^m} |\nabla^2 f - \frac{\Delta f}{m} g|^2\, d\upsilon_g &= \int\limits_{M^m} (\lambda - \frac{\Delta f}{m})\Delta f\, d\upsilon_g - \int\limits_{M^m} \alpha(\tau_g\phi)^2\, d\upsilon_g \\ &\quad + \frac{1}{2}\int\limits_{M^m} \langle \nabla R_\phi, \nabla f\rangle\, d\upsilon_g + \frac{1}{2}\int\limits_{M^m} \langle \nabla f, \nabla(R_\phi + \Delta f)\rangle\, d\upsilon_g.\end{aligned} \tag{3.1}$$

Next, we use relation (2.2) and Stokes' formula to write

$$\begin{aligned}\int_{M^m} (\lambda - \frac{\Delta f}{m})\Delta f \, d\upsilon_g =& \frac{1}{m}\int_{M^m} (R_\phi - \frac{1}{\tau}|\nabla f|^2)\Delta f \, d\upsilon_g \\ =& -\frac{1}{m}\int_{M^m} \langle \nabla R_\phi, \nabla f\rangle \, d\upsilon_g + \frac{1}{m\tau}\int_{M^m} \langle \nabla f, \nabla |\nabla f|^2\rangle \, d\upsilon_g.\end{aligned} \tag{3.2}$$

On the other hand, we notice that equation (2.2) yields $\nabla(R_\phi + \Delta f) = \frac{1}{\tau}\nabla|\nabla f|^2$. By using this datum on the previous equation, we have

$$\begin{aligned}\int_{M^m} |\nabla^2 f - \frac{\Delta f}{m} g|^2 \, d\upsilon_g +& \int_{M^m} \alpha(\tau_g\phi)^2 \, d\upsilon_g \\ =& \frac{m-2}{2m}\int_{M^m} \langle \nabla R_\phi, \nabla f\rangle \, d\upsilon_g - \frac{m+2}{2\tau m}\int_{M^m} |\nabla f|^2 \Delta f \, d\upsilon_g,\end{aligned}$$

which ends the first assertion.

Proceeding, we can use (2.2) to infer $\nabla|\nabla f|^2 = \tau\nabla R_\phi + \tau\nabla\Delta f$. So we obtain from equation (3.2) that

$$\int_{M^m} (\lambda - \frac{\Delta f}{m})\Delta f \, d\upsilon_g = -\frac{1}{\tau}\int_{M^m} (\Delta f)^2 \, d\upsilon_g. \tag{3.3}$$

We use Stokes' formula to write

$$\int_{M^m} \langle \nabla f, \nabla(R_\phi + \Delta f)\rangle \, d\upsilon_g = \int_{M^m} \langle \nabla R_\phi, \nabla f\rangle \, d\upsilon_g - \int_{M^m} (\Delta f)^2 \, d\upsilon_g. \tag{3.4}$$

Plugging (3.3) and (3.4) into (3.1) leads to (2). So we finish the proof of Proposition 3.1.

Before to announce the next results we point out that they are generalizations of ones found in [1] and [2] for quasi-Einstein metrics. First, we have the following theorem.

***Theorem* 3.2** — *Let g be a τ-quasi Ricci-harmonic metric defined in Definition 1.2 on compact manifold M. Then ϕ is harmonic and M is trivial provided one of the following conditions holds*:

(1) $\int_{M^m} Ric_\phi(\nabla f, \nabla f) \, d\upsilon_g \le \frac{2}{\tau}\int_{M^m} |\nabla f|^2 \Delta f \, d\upsilon_g$.

(2) $R_\phi \ge \lambda m$ or $R_\phi \le \lambda m$.

PROOF : First we integrate the identity (2.9) derived in Lemma 2.2 and we use Stokes' formula to infer

$$\int_{M^m} |\nabla^2 f| \, d\upsilon_g + \int_{M^m} \alpha(\tau_g\phi)^2 \, d\upsilon_g = \int_{M^m} Ric_\phi(\nabla f, \nabla f) \, d\upsilon_g - \frac{2}{\tau}\int_{M^m} |\nabla f|^2 \Delta f \, d\upsilon_g. \tag{3.5}$$

On the other hand, since we are assuming that the right hand of above identity is less than or equal to zero, we obtain $\nabla^2 f = 0$ and $\tau_g \phi = 0$. Therefore, $\Delta f = 0$, which implies by Hopf's theorem that f is constant and it implies ϕ is harmonic map, so we finish the establishment of the first assertion.

Proceeding one notices that for $\tau = \infty$, using equation (2.2) the result follows. When $0 < \tau < \infty$, consider $u = e^{-\frac{f}{\tau}}$. Then we have

$$\nabla u = -\frac{1}{\tau} e^{-\frac{f}{\tau}} \nabla f,$$

$$\frac{\tau}{u} \nabla^2 u = -\nabla^2 f + \frac{1}{\tau} df \otimes df.$$

Therefore the first equation of (1.4) can be rewritten as

$$Ric_\phi - \frac{\tau}{u} \nabla^2 u = \lambda g. \tag{3.6}$$

Taking trace of (3.6) we have

$$\Delta u = \frac{u}{\tau}(R_\phi - m\lambda). \tag{3.7}$$

Integrating this equation with respect to dv_g we get

$$\int_{M^n} \frac{u}{\tau}(R_\phi - m\lambda)\, dv_g = 0.$$

Since $u > 0$ this immediately gives the above result. From which we complete the proof of the theorem.

Now, if M is a τ-quasi Ricci-harmonic metric and τ is finite, we shall present conditions in order to obtain $\nabla f \equiv 0$. We first give two lemmas which will be used later.

Lemma 3.3 — [4]. Let X be a smooth vector field on the m dimensional complete, noncompact, oriented Riemannian manifold M, such that $divX$ does not change sign on M. If $|X| \in L^1(M)$, then $divX = 0$ on M.

Lemma 3.4 — [1]. Let (M^m, g) be a Riemannian manifold and $X \in \mathfrak{X}(M)$. Then if $(X^b \otimes X^b) = \rho g$ for some smooth function $\rho : M \to \mathbb{R}$, then $\rho = |X|^2 = 0$. In particular, the unique solution of the equation $df \otimes df = \rho g$ is f constant.

***Theorem* 3.5** — *Let g be a τ-quasi Ricci-harmonic metric defined in Definition 1.2 on complete manifold M and τ is finite. Then $\nabla f \equiv 0$, if one of the following conditions holds:*

(1) M is non compact, $m\lambda \geq R_\phi$ and $|\nabla f| \in L^1(M)$. In particular, M is a harmonic Einstein manifold.

(2) (M, g) is a harmonic Einstein manifold and ∇f is a conformal vector filed.

PROOF : Taking into account identity (2.2) we obtain

$$\tau\Delta f = |\nabla f|^2 + \tau(\lambda m - R_\phi). \tag{3.8}$$

By one hand $\tau\Delta f \geq 0$, since $(m\lambda - R_\phi) \geq 0$. On the other hand, if $|\nabla f| \in L^1(M)$, by Lemma 3.3, we have that $\Delta f = 0$. Next, we may use equation (3.8) to conclude that $\nabla f \equiv 0$, as well as $m\lambda = R_\phi$. Therefore, f is constant and M is a harmonic Einstein manifold, which gives the first assertion.

Now let us suppose that (M, g) is a harmonic Einstein manifold, in particular a surface has this propriety. If ∇f is a conformal vector field with conformal factor ρ, then $\nabla^2 f = \rho g$, where $\rho = \frac{1}{m}\Delta f$. Since $Ric_\phi = \frac{R_\phi}{m} g$ we deduce from equation (2.11) that

$$\frac{1}{\tau}\big(df \otimes df - \frac{1}{m}|\nabla f|^2 g\big) = 0. \tag{3.9}$$

But, using that τ is finite, we can apply Lemma 3.4 to conclude that $\nabla f \equiv 0$, which completes the proof of the theorem.

As a consequence of the previous results we obtain the following corollary.

Corollary 3.6 — Let g be a τ-quasi Ricci-harmonic metric defined in Definition 1.2 on compact orientable manifold M and τ is finite. Then

(1) If $\int_{M^m} \langle \nabla R_\phi, \nabla f\rangle\, dv_g \leq 0$, then M is a harmonic Einstein manifold and ϕ is harmonic.

(2) If $n = 2$ and $\int_{M^n} |\nabla f|^2 \Delta f\, dv_g = 0$, then f is constant and ϕ is harmonic.

PROOF : If $\int_{M^m} \langle \nabla R_\phi, \nabla f\rangle\, dv_g \leq 0$, we deduce, from the second item of Proposition 3.1, that

$$\int_{M^m} |\nabla^2 f - \frac{\Delta f}{m} g|^2\, dv_g + \frac{m+2}{2m}\int_{M^m} (\Delta f)^2\, dv_g + \int_{M^m} \alpha(\tau_g\phi)^2\, dv_g = 0$$

This implies that $\nabla^2 f = \frac{\Delta f}{m} g$, $\Delta f = 0$ and $\tau_g\phi = 0$. Hence, we can know ϕ is harmonic map and apply Hopf's theorem to deduce that f is constant, which implies that M is a harmonic Einstein manifold, so we finish the establishment of the first assertion.

We notice that for $m = 2$, it is enough to suppose $\int_{M^m} |\nabla f|^2 \Delta f\, dv_g = 0$ to conclude that ∇f is conformal. But, using Theorem 3.5 we conclude that f is constant, which completes the proof of the theorem.

At last, we prove a characterization of compact shrinking trivial gradient Ricci-harmonic soliton metric. This is a generalization of one found in [5] for shrinking gradient Ricci soliton.

***Theorem* 3.7** — *An m-dimensional compact shrinking gradient Ricci-harmonic soliton (M, g) with normalized potential is trivial if and only if*

$$\int_{M^m} fR_\phi \, dv_g \leq \frac{1}{2}m^2\lambda Vol(M).$$

PROOF : It follows from equations (2.22) and (2.25) that

$$\begin{aligned} \frac{1}{2}\Delta f^2 =& f\Delta f + |\nabla f|^2 \\ =& f(\lambda m - R_\phi) + 2\lambda f - R_\phi \\ =& (m+2)\lambda f - fR_\phi - R_\phi. \end{aligned} \tag{3.10}$$

From (3.10), (2.24) and Stokes' formula, we have

$$\begin{aligned} \int_{M^m} fR_\phi \, dv_g =& \int_{M^m} [(m+2)\lambda f - R_\phi] \, dv_g \\ =& \int_{M^m} [(m+2)\lambda f - m\lambda] \, dv_g. \end{aligned} \tag{3.11}$$

From (3.11), we have

$$\int_{M^m} \lambda f \, dv_g = \frac{1}{m+2} \int_{M^m} (m\lambda + fR_\phi) \, dv_g. \tag{3.12}$$

Note that equations (2.24) and (2.25) imply

$$\int_{M^m} (2\lambda f - \lambda m) \, dv_g = \int_{M^m} |\nabla f|^2 \, dv_g. \tag{3.13}$$

Plugging (3.12) into (3.13) leads to

$$\int_{M^m} fR_\phi \, dv_g = \frac{1}{2}m^2\lambda Vol(M) + \frac{m+2}{2} \int_{M^m} |\nabla f|^2 \, dv_g. \tag{3.14}$$

If the condition $\int_{M^m} fR_\phi \, dv_g \leq \frac{1}{2}m^2\lambda Vol(M)$ holds, then we shall have

$$\int_{M^m} |\nabla f|^2 \, dv_g \leq 0, \tag{3.15}$$

which implies that the potential function f is a constant. Consequently, M is trival.

Conversely, if an m-dimensional compact and connected shrinking gradient Ricci-harmonic soliton is trivial, then $R_\phi = m\lambda$ and f is a constant. Therefore, by equation (2.25) we obtain $f = \frac{R_\phi}{2\lambda}$. Consequently, we have $\int\limits_{M^m} fR_\phi\, dv_g = \frac{1}{2}m^2\lambda Vol(M)$. This completes the proof of Theorem 3.7.

4. On τ-quasi Ricci-harmonic Metrics with Constant Generalized Scalar Curvature

In this section, focusing on the τ-quasi Ricci-harmonic metric, we consider them under the additional conditions that the manifold is of constant R_ϕ. Before to announce the next results we point out that they are generalizations of ones found in [10] and [3] for quasi-Einstein metrics. First, we have the following results.

By (2.5) we have

Proposition 4.1 — Let g be a τ-quasi Ricci-harmonic metric defined in Definition 1.2.

(1) If $\tau = 1$, then $R_\phi = \lambda(m-1)$ or f is a constant.

(2) If $\tau \neq 1$, a τ-quasi Ricci-harmonic metric has constant R_ϕ if and only if

$$Ric_\phi(\nabla f) = -\frac{1}{\tau - 1}[R_\phi - (m-1)\lambda]\nabla f.$$

By Theorem 3.2 we have

Corollary 4.2 — Let g be a τ-quasi Ricci-harmonic metric defined in Definition 1.2 on compact manifold M. If R_ϕ is constant, then (M, g) is trivial.

Remark 4.3 : It was proved in [3] that a compact quasi-Einstein metric with constant generalized scalar curvature is trivial.

Proposition 4.4 — Let g be a τ-quasi Ricci-harmonic metric defined in Definition 1.2 on manifold M and $\tau \geq 1$ and

(1) $\lambda = 0$, the generalized scalar curvature is constant and $\tau > 1$, then $Ric_\phi = 0$.

(2) $\lambda < 0$ and the generalized scalar curvature is constant, then

$$m\lambda \leq R_\phi \leq \frac{m(m-1)}{\tau + m - 1}\lambda$$

and when $\tau > 1$, R_ϕ equals either of the extreme values iff M is a harmonic Einstein manifold.

PROOF : Since R_ϕ is constant, from (2.6)

$$-\frac{m+\tau-1}{m\tau}(R_\phi - m\lambda)\left(R_\phi - \frac{m(m-1)}{m+\tau-1}\lambda\right) = \frac{\tau-1}{\tau}\alpha(\tau_g\phi)^2 + \frac{\tau-1}{\tau}\left|Ric_\phi - \frac{1}{m}R_\phi g\right|^2 \geq 0$$

So if $\lambda = 0$, $\tau > 1$, then $\tau_g\phi = 0$, $Ric_\phi = \dfrac{1}{m}R_\phi g$ and $R_\phi = 0$, thus $Ric_\phi = 0$ and ϕ is harmonic. If $\lambda < 0$, by $m\lambda \leq \dfrac{m(m-1)}{m+\tau-1}$, $R_\phi \in \left[m\lambda, \dfrac{m(m-1)}{m+\tau-1}\lambda\right]$. We have completed the proof of Proposition 4.4.

Remark 4.5 : It was proved in [3] that for an m-dimensional τ-quasi-Einstein manifold (M, g), if $\lambda < 0$ and the generalized scalar curvature R is constant, then R is in fact bounded and satisfies $m\lambda \leq R \leq \dfrac{m(m-1)}{m+\tau-1}\lambda$.

Continuing with this topic and motivated by a recent result of [10], we will show that under a suitable assumption the possible values of the constant generalized scalar curvature are in fact quantified.

***Theorem* 4.6** — *Let g be a τ-quasi Ricci-harmonic metric defined in Definition 1.2 on complete manifold M, and its generalized scalar curvature R_ϕ is constant. If $\tau > 1$ and $e^{-\frac{f}{\tau}}$ attains its maximum or minimum at some point, then there is $l \in \{0, 1, 2, \cdots, m\}$ such that*

$$R_\phi = \frac{\tau m - (\tau - m)l - m}{\tau + l - 1}\lambda.$$

PROOF : As before we assume $u = e^{-\frac{f}{\tau}}$. The fact that

$$\begin{aligned}\nabla R_\phi =& \nabla R - \alpha\nabla|\nabla\phi|^2 = 2divRic - 2\alpha\nabla_{\nabla\phi}\nabla\phi \\ =& 2divRic - 2\alpha[div(\nabla\phi\otimes\nabla\phi) - \tau_g\phi\nabla\phi] \\ =& 2divRic_\phi + 2\alpha\tau_g\phi\nabla\phi.\end{aligned} \tag{4.1}$$

Noting that both R_ϕ and λ are constants, by (2.15), (3.6) and (4.1), we have the computation

$$\begin{aligned}0 =& \frac{1}{2}\nabla R_\phi = divRic_\phi + \alpha\tau_g\phi\nabla\phi \\ =& div(\frac{\tau}{u}\nabla^2 u) + \alpha\tau_g\phi\nabla\phi \\ =& \frac{\tau}{u}div(\nabla^2 u) - \frac{\tau}{u^2}\nabla_{\nabla u}\nabla u + \alpha\tau_g\phi\nabla\phi \\ =& \frac{\tau}{u}[Ric(\nabla u) + \nabla\Delta u] - \frac{\tau}{u^2}\nabla_{\nabla u}\nabla u + \alpha\tau_g\phi\nabla\phi \\ =& \frac{\tau}{u}Ric_\phi(\nabla u) + \frac{\tau}{u}\nabla\Delta u - \frac{\tau}{u^2}\nabla_{\nabla u}\nabla u,\end{aligned} \tag{4.2}$$

where we use $\alpha\tau_g\phi\nabla\phi = -\frac{\tau}{u}\alpha(\nabla\phi\otimes\nabla\phi)(\nabla u)$. Substituting (3.7) into (4.2), we deduce

$$\frac{\tau}{u}Ric_\phi(\nabla u) + (R_\phi - m\lambda)\frac{\nabla u}{u} - \frac{\tau}{u^2}\nabla_{\nabla u}\nabla u = 0. \tag{4.3}$$

Using equation (3.6) again we get

$$Ric_\phi\nabla u - \frac{\tau}{u}\nabla_{\nabla u}\nabla u = \lambda\nabla u.$$

Combining this with (4.3) we have

$$(\tau\lambda - m\lambda + R_\phi)\frac{\nabla u}{u} + \frac{\tau(\tau-1)}{u^2}\nabla_{\nabla u}\nabla u = 0.$$

This yields

$$(m\lambda - \tau\lambda - R_\phi)\nabla(u^2) = \tau(\tau-1)\nabla|\nabla u|^2. \tag{4.4}$$

Then we find that if $\tau > 1$, the function u satisfies

$$|\nabla u|^2 = \frac{(m-\tau)\lambda - R_\phi}{\tau(\tau-1)}u^2 + C =: p(u), \tag{4.5}$$

where C is a constant and

$$\Delta u = \frac{u}{\tau}(R_\phi - m\lambda) =: q(u). \tag{4.6}$$

Both, (4.5) and (4.6), show that, if $\tau > 1$, u is an **isoparametric function** (see [17] for the definition of isoparametric function) on M.

The following discussion is similar to that in [10]. We denote

$$u_{max} = max\{u(x)|x\in M\} \quad \text{and} \quad u_{min} = min\{u(x)|x\in M\},$$

if they exist. Recall that for the isoparametric function u, the level sets

$$M_+(u) = \{x\in M|u(x) = u_{max}\} \quad \text{and} \quad M_-(u) = \{x\in M|u(x) = u_{min}\},$$

if they exist, are called the **focal varieties** of u. Now, without loss of generality we assume that u attains its maximum, thus $M_+(u)$ is nonempty and, according to [17], it is a smooth submanifold of M. As was proved by Wang [17] that the restriction of $\nabla^2 u$ to $M_+(u)$ has only two eigenvalues, 0 and $\frac{1}{2}p'(u)$, i.e., $\nabla^2 u(X) = 0$ for all $X\in TM_+(u)$, and $\nabla^2 u(Y) = \frac{1}{2}p'(u)Y$ for all $Y\in T^\perp M_+(u)$, where $TM_+(u)$ and $T^\perp M_+(u)$ denote the tangent bundle and normal bundle of $M_+(u)$, respectively.

The expression of $p(u)$ in (4.5) gives $\frac{1}{2}p'(u) = \frac{(m-\tau)\lambda - R_\phi}{\tau(\tau-1)}u$. Thus in our case

$$\frac{\tau}{u}\nabla^2 u(Y) = \begin{cases} 0, & \text{for} \quad Y \in TM_+(u), \\ \frac{(m-\tau)\lambda - R_\phi}{\tau - 1}Y, & \text{for} \quad Y \in T^\perp M_+(u). \end{cases} \tag{4.7}$$

Assume that $dimM_+ = m - l$, where $0 \le l \le m$. It follows from (3.6) and (4.7) that the restriction of the Ric_ϕ to $M_+(u)$ is of the form

$$Ric_\phi|_{M_+(u)} = \begin{pmatrix} \lambda I_{m-l} & 0 \\ 0 & \frac{(m-1)\lambda - R_\phi}{\tau-1} I_l \end{pmatrix}. \tag{4.8}$$

From (4.8) we deduce that R_ϕ satisfies $R_\phi = (m-l)\lambda + \frac{(m-1)\lambda - R_\phi}{\tau - 1}l$, so we have

$$R_\phi = \frac{\tau m - (\tau - m)l - m}{\tau + l - 1}\lambda.$$

We have completed the proof of Theorem 4.6.

The significance of this theorem is the following immediate consequence:

Corollary 4.7 — Let g be a nontrivial τ-quasi Ricci-harmonic metric on complete manifold M, and its generalized scalar curvature R_ϕ is constant, then $\lambda < 0$, or $\lambda = 0$ and $\mu > 0$. Moreover,

(1) if $\tau = 1$, then $R_\phi = (m-1)\lambda$.

(2) If $\tau > 1$, then $\lambda < 0$; and if moreover $e^{-\frac{f}{\tau}}$ attains its maximum or minimum at some point on M, then

$$R_\phi \in \left\{ \frac{\tau m - (\tau - m)l - m}{\tau + l - 1}\lambda \;\middle|\; l = 1, 2, \cdots, m \right\}.$$

PROOF : From Remark 3.6 and Corollary 4.6 of [16] one knows that constants λ and μ of a nontrivial τ-quasi-Ricci-harmonic metric on a noncompact manifold should satisfy $\lambda < 0$, or $\lambda = 0$ and $\mu > 0$. Moreover, by Corollary 4.2, a compact τ-quasi-Ricci-harmonic metric with constant generalized scalar curvature is trivial. Thus in our situation it must be the case $\lambda < 0$, or $\lambda = 0$ and $\mu > 0$.

If $\tau = 1$, from (1) of Proposition 4.1 and the nontriviality, we obtain $R_\phi = \lambda(m-1)$.

If $\tau > 1$, $\lambda = 0$ and $\mu > 0$, then from (1) of Proposition 4.4 we see that $Ric_\phi = 0$, by (3.7) we have and thus u is constant, a contradiction to the nontriviality.

Having proved that if $\tau > 1$ then $\lambda < 0$, we next assume that $e^{-\frac{f}{\tau}}$ attains its maximum or minimum at some point. It follows from Theorem 4.6 that

$$R_\phi \in \left\{ \frac{\tau m - (\tau - m)l - m}{\tau + l - 1}\lambda \mid l = 0, 1, 2, \cdots, m \right\}.$$

If $R_\phi = m\lambda$ (corresponding to $l = 0$), then by (2) of Proposition 4.4, M is a harmonic Einstein manifold and thus $Ric_\phi = \frac{1}{m}R_\phi g = \lambda g$. Hence, by (3.6) we have $\nabla^2 u = 0$, this equation has no complete nonconstant positive solution due to that the manifold is complete. Therefore, under the condition of nontriviality, $R_\phi = m\lambda$ is impossible. Hence we have

$$R_\phi \in \left\{ \frac{\tau m - (\tau - m)l - m}{\tau + l - 1}\lambda \mid l = 1, 2, \cdots, m \right\}.$$

We have completed the proof of Corollary 4.7.

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