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Cat-VALUED SHEAVES

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Let $\widetilde{\mathcal{O}}(\mathbf{B})$ be the category of (open) subcategories of a topological groupoid $\mathbf{B}$. In this paper we study **Cat**-valued sheaves over category $\widetilde{\mathcal{O}}(\mathbf{B})$. The paper introduces a notion of categorical union, such that the categorical union of subcategories is a subcategory. We use this definition of categorical unions to define a categorical cover of a topological category. Instead of assuming a Grothendieck topology, we define **Cat**-valued sheaves in terms of the categorical cover defined in this paper. The main result is the following. For a fixed category $\mathbf{C}$, the categories of local functorial sections from $\mathbf{B}$ to $\mathbf{C}$ define a **Cat**-valued sheaf on $\widetilde{\mathcal{O}}(\mathbf{B})$. Replacing $\mathbf{C}$ with a categorical group $\mathcal{G}$, we find a **CatGrp**-valued sheaf on $\widetilde{\mathcal{O}}(\mathbf{B})$. We also relate and distinguish our construction with the notion of stacks.

Key words : Presheaves; sheaves; union of subcategories; categorical groups.

1. Introduction

This paper is a sequel to [15], which deals with **Cat**-valued presheaves and **Cat**-valued sieves. For this paper we will follow-up the definition of **Cat**-valued presheaves introduced in [15] and develop the notion of **Cat**-valued sheaves. We have reviewed and recalled all the required results (without proofs) of [15] here as well, so this paper can be read as a more or less self contained piece. Before we get into the topic of this paper, let us recollect the thoughts and motivation, explained in Section 1 of [15], behind the introduction of **Cat**-valued presheaves.

Traditionally a presheaf is defined, for a given category $\mathbf{C}$, as a contravariant functor [9, 24]

$$R : \mathbf{C}^{\mathrm{op}} \longrightarrow \mathbf{Set}, \tag{1.1}$$

where **Set** is the (locally small) category of small sets. **C** is typically chosen to be the category $\widetilde{\mathcal{O}}(B)$ of open subsets of a topological space B, that is,

$$\begin{aligned} &\mathrm{Obj}\Big(\widetilde{\mathcal{O}}(B)\Big) := \{\text{Open } U \mid U \subset B\}, \\ &\mathrm{Hom}(U, V) := \{\text{Continuous } f : U \longrightarrow V \mid \text{open } U, V \subset B\}. \end{aligned} \tag{1.2}$$

In (1.1) instead of **Set**, one may possibly consider any other *concrete category* such as the category of (small) groups **Grp**, the category of (small) vector spaces **Vect** or the category of (small) rings **Ring**, and accordingly one gets "presheaf of groups", "presheaf of vector spaces" or "presheaf of rings" respectively. We will often adopt an alternate terminology for them, such as, presheaf of groups will be called **Grp**-valued presheaf, and likewise

- presheaf of vector spaces= **Vect**-valued presheaf
- presheaf of rings=**Ring**-valued presheaf

and so on.

Now suppose instead of a topological space B, we are interested in a "topological category" **B**. By a topological category we mean a category whose object and morphism sets are both topological spaces. Note that this taxonomy of topological category is not universal in literature. We adopt the definition of [11]. In this context a natural object of interest would be the category $\widetilde{\mathcal{O}}(\mathbf{B})$ of subcategories of **B**. A natural choice would be to consider the category **Cat** of small categories as the codomain of a presheaf in this context, rather than **Set** (or any other concrete category). However **Cat** is not a concrete category, and we cannot proceed with the existing definition of presheaf given in (1.1), and we need a new framework. In [15] we propose such a framework for **Cat**-*valued presheaves*, and develop a corresponding theory of **Cat**-*valued sieves.* In this paper we establish the corresponding notion of sheaves, namely **Cat**-*valued sheaves.* It should be pointed out here that a stack is "morally" a generalization of the notion of ordinary sheaves to categorical sheaves [12, 13, 36]. However our motivation and construction is slightly different than that of stacks. This issue has been discussed in Section 6.

"Categorical intersection" of two subcategories, defined simply as intersection of object sets and morphism sets, is a subcategory. However, unlike subsets, the union of two subcategories, defined naively as the union of object sets and morphism sets, is not a subcategory. But, in order to define a sheaf over a topological category, we need a "reasonable definition" of a cover of a topological category, and in turn that requires a "reasonable definition" of the union of (open) subcategories. So, before advancing to **Cat**-valued sheaves from **Cat**-valued presheaves, we work out a definition of "categorical union" of subcategories. The categorical unions and categorical intersections satisfy usual inter-relations enjoyed by their set theoretic counterparts. We also define "open subcategories", and "open categorical cover" of an open subcategory. We construct an example of "**Cat**-valued sheaf of functorial sections" for a fixed category $\mathbf{C}$, where an object $\mathbf{U} \in \mathrm{Obj}(\widetilde{\mathcal{O}}(\mathbf{B}))$ is sent to the category of functors from $\mathbf{U}$ to $\mathbf{C}$. Since the categorical union of subcategories is not merely union of morphism sets and object sets, it is really a "non-trivial" task to establish that categories of local functorial sections indeed define a sheaf (that is, they satisfy appropriate "locality" and "gluing" conditions). In fact at first sight it may seem that this construction is not going to work. However, it is remarkable that despite all the intricacies, it turns out that local functorial sections (and natural transformations between them) do define a **Cat**-valued sheaf. Theorem 4,2, which is the main result of this paper, establishes the preceding discussions.

As an application of Theorem 4.2 we also consider the "sheaves of categorical groups" or "**CatGrp**-valued sheaves", where **CatGrp** is the category of categorical groups, treated as a full subcategory (identified via an obvious full faithful forgetful functor $\mathbf{CatGrp} \longrightarrow \mathbf{Cat}$) of **Cat**. Let $\mathcal{C}$ be a category of a set of small categories; that is, the objects are a set of small categories and the morphisms are functors between them. We work with a **Cat**-*valued presheaf* over $\mathcal{C}$ given by a contravariant functor:

$$\mathcal{R} : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Cat}. \tag{1.3}$$

The background motivation for our construction stems from the study of categorical geometry [8, 16, 17, 20, 35] on a *path space groupoid* of a given smooth manifold M; that is, a category $\mathbb{P}M$, whose object space is the manifold M and morphisms are certain equivalence classes of smooth paths. The usual compact-open topology defines a topology on $\mathrm{Mor}(\mathbb{P}M)$. The path space groupoid over a smooth manifold naturally occurs providing the background geometry in higher gauge theories [7, 23, 29, 31, 34] and non abelian gerbe theories [1-6, 10, 32, 33]. Recently "locally defined subcategories" of $\mathbb{P}M$ also appeared in the context of local trivializations of categorical principal bundles [17, 21]. In the context of this paper, it would be of particular interest to consider the case when $\mathcal{C} = \widetilde{\mathcal{O}}(\mathbb{P}M)$. However considering the length of this paper, we do not pursue a detailed and rigorous treatment of $\mathbb{P}M$. For that reason, though the framework developed here is perhaps most suitable for $\mathcal{C} = \widetilde{\mathcal{O}}(\mathbb{P}M)$, we tread on a more abstract approach and try to minimize the reference to $\mathbb{P}M$. Occasionally when we have to recall $\mathbb{P}M$, we would give a heuristic description without going into the technicalities. For a detailed exposition on the topic, we refer to [5, 18, 30].

Notation and convention

We borrow our notation from [15]. Let $\mathbf{C}$ and $\mathbf{D}$ be a given pair of categories.

$$\mathrm{Fun}(\mathbf{C}, \mathbf{D}) \tag{1.4}$$

will denote the set of all functors from $\mathbf{C}$ to $\mathbf{D}$. The set of all natural transformations between functors from $\mathbf{C}$ to $\mathbf{D}$ will be denoted as

$$\mathcal{N}(\mathbf{C}, \mathbf{D}). \tag{1.5}$$

If $\theta_1, \theta_2 : \mathbf{C} \longrightarrow \mathbf{D}$ are a pair of functors, then

$$\mathrm{Nat}(\theta_1, \theta_2) \tag{1.6}$$

is the set of all natural transformations between θ_1 and θ_2. We will often denote a natural transformation Φ from a functor θ_1 to another functor θ_2 as

$$\Phi : \theta_1 \Longrightarrow \theta_2. \tag{1.7}$$

The category of functors will be denoted as

$$\mathcal{F}(\mathbf{C}, \mathbf{D}); \tag{1.8}$$

that is

$$\begin{aligned} \mathrm{Obj}\Big(\mathcal{F}(\mathbf{C}, \mathbf{D})\Big) &= \mathrm{Fun}(\mathbf{C}, \mathbf{D}) \\ \mathrm{Mor}\Big(\mathcal{F}(\mathbf{C}, \mathbf{D})\Big) &= \mathcal{N}(\mathbf{C}, \mathbf{D}). \end{aligned} \tag{1.9}$$

Given a morphism f in some category, $s(f), t(f)$ will respectively denote the source of f and target of f; that is,

$$s(f) \xrightarrow{f} t(f).$$

$\emptyset$ will denote the *empty category*; i.e. a category whose object and morphism sets are null sets.

Overview of the sections

We now give a brief overview of each section of this paper.

- In Section 2 we review some of the constructions and results from [15]. We work with a category of a set of small categories, $\mathcal{C}$. In particular, in this section we recall the definition of a **Cat**-valued presheaf over $\mathcal{C}$ introduced in [15]. We state a version of Yoneda embedding in Proposition 2.3.

- Section 3 prepares the stage for the next section. We first define the categorical union of subcategories, and Proposition 3.1 ensures that the categorical union and intersection satisfy Boolean relations (the proof is carried out in the Appendix). We take $\mathbf{B}$ to be a groupoid. We call a subcategory open if both object and morphism sets are open, and the subcategory is a groupoid. This particular definition of open subcategories is essential to have a well defined sheaf of functorial sections. We introduce the notion of an open categorical cover and provide some examples for the same.

- Section 4 is the central theme of this paper. The ultimate goal in this section is to construct the (**Cat**-valued) sheaf of functorial sections. We define a **Cat**-valued sheaf over $\widetilde{\mathcal{O}}(\mathbf{B})$ to be a **Cat**-valued presheaf satisfying certain locality and gluing conditions (described in (4.6)-(4.13)).

  Next we fix a category $\mathbf{C}$. Our goal is to construct the (**Cat**-valued) sheaf of functorial sections to category $\mathbf{C}$. Finally in Theorem 4.2 we prove that functorial sections with respect to category $\mathbf{C}$ over $\mathbf{B}$ define a **Cat**-valued sheaf.

- In Section 5 we first give a brief review of categorical groups, and show (Proposition 5.1) that $\mathcal{G}^{\mathbf{U}} := \mathcal{F}(\mathbf{U}, \mathcal{G})$ form a categorical group, where $\mathcal{G}$ is a fixed categorical group and $\mathbf{U}$ any category. We define (category of categorical groups) **CatGrp**-valued presheaves. In Proposition 5.2 we show that if we replace $\mathbf{C}$ of Section 3 by a categorical group $\mathcal{G}$, then we get a **CatGrp**-valued sheaf on $\widetilde{\mathcal{O}}(\mathbf{B})$.

- In Section 6 we give a brief review of the notion of stacks and compare our construction with that of stacks.

- The Appendix contains the proof of Proposition 3.1.

- We end this paper with some concluding remarks to highlight potential uses of our construction and results in this paper.

Summary of the paper

Main objective of this paper is to propose a framework for **Cat**-valued sheaves and construct an example of "**Cat**-valued sheaf of (local) functorial sections" defined on a topological groupoid **B** with respect to a fixed category **C**. Our starting point is the **Cat**-valued presheaf on $\mathcal{C}$ introduced in [15] and recalled in Section 2.

A **Cat**-valued presheaf on $\mathcal{C}$ is a contravariant functor

$$\mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Cat}.$$

With the motivation to define **Cat**-valued sheaves on $\widetilde{\mathcal{O}}(\mathbf{B})$, we introduce a new notion of categorical union of subcategories, which ensures that union of subcategories is a subcategory. We show that categorical unions thus defined and intersections of subcategories are consistent with the standard set theoretic relation. We turn to defining open subcategories and categorical cover of an open subcategory of a topological groupoid **B** and illustrate the definitions with several examples. Then we define a **Cat**-valued sheaf on $\widetilde{\mathcal{O}}(\mathbf{B})$ to be a **Cat**-valued presheaf satisfying certain "gluing" and "locality" conditions.

We fix a category **C** and consider the functor categories $\mathbf{C}^{\mathbf{U}} := \mathcal{F}(\mathbf{U}, \mathbf{C})$ defined by open subcategories **U** of **B**. We show that the prescription

$$\mathbf{U} \mapsto \mathbf{C}^{\mathbf{U}}$$

then produces a **Cat**-valued sheaf on $\widetilde{\mathcal{O}}(\mathbf{B})$. We call them **Cat**-valued sheaf of *functorial sections* on $\widetilde{\mathcal{O}}(\mathbf{B})$.

As a special case of above construction, taking **C** to be a categorical group $\mathcal{G}$, we obtain a **CatGrp**-valued sheaf on $\widetilde{\mathcal{O}}(\mathbf{B})$.

2. **Cat**-valued Presheaves

For ease of reference, in this section we briefly review and collect some of the relevant material from [15]. Some parts would be verbatim from [15]. We keep the details to a bare minimum, and skip the proofs. We will provide references to corresponding Proposition/Theorem numbers in [15].

Let $\mathcal{C}$ be a category of a collection of (small) categories; that is objects are a set of (small) categories and morphisms are functors between them. Later we will mostly be dealing with the category $\widetilde{\mathcal{O}}(\mathbf{B})$, where $\mathbf{B}$ is a given topological category and,

$$\begin{aligned}
&\mathrm{Obj}\Big(\widetilde{\mathcal{O}}(\mathbf{B})\Big) := \{\text{ “open” } \mathbf{U} | \mathbf{U} \subset \mathbf{B}\} = \text{ set of all “open subcategories” of } \mathbf{B}, \\
&\text{[Open subcategories are defined in Section 3]} \\
&\mathrm{Hom}(\mathbf{U},\mathbf{V}) = \{\text{“continuous”functor } \mathbf{\Theta} : \mathbf{U} \longrightarrow \mathbf{V} | \text{open } \mathbf{U},\mathbf{V} \subset \mathbf{B}\}.
\end{aligned} \tag{2.1}$$

We will also work with the category $\mathcal{O}(\mathbf{B})$, whose objects are same as those of $\widetilde{\mathcal{O}}(\mathbf{B})$; but the only morphism between any two subcategories (objects) is the inclusion functor if one is subcategory of the other, otherwise no morphism exists. Thus:

$$\begin{aligned}
&\mathrm{Obj}\Big(\mathcal{O}(\mathbf{B})\Big) := \{\text{open } \mathbf{U} | \mathbf{U} \subset \mathbf{B}\} = \text{set of all open subcategories of } \mathbf{B}, \\
&\mathrm{Hom}(\mathbf{U},\mathbf{V}) = \{\mathbf{i} : \mathbf{U} \hookrightarrow \mathbf{V} | \mathbf{U} \subset \mathbf{V} \subset \mathbf{B}\}, \\
&\qquad\qquad\qquad \text{where } \mathbf{i} \text{ is the inclusion functor, and} \\
&\mathrm{Hom}(\mathbf{U},\mathbf{V}) = \emptyset, \text{if } \mathbf{U} \not\subset \mathbf{V}.
\end{aligned} \tag{2.2}$$

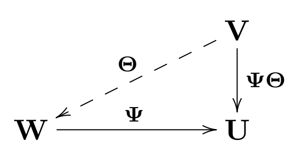
Let **Cat** be the category of all (small) categories. Analogous to the contravariant Hom-functor in a set theoretic framework, we have the contravariant functor $\mathcal{F}_{\mathbf{U}} : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Cat}$, corresponding to each $\mathbf{U} \in \mathrm{Obj}(\mathcal{C})$ as follows.

$$\begin{aligned}
\mathcal{F}_{\mathbf{U}} :\ & \mathrm{Obj}(\mathcal{C}) \longrightarrow \mathrm{Obj}(\mathbf{Cat}) \\
& \mathbf{V} \mapsto \mathcal{F}(\mathbf{V},\mathbf{U})
\end{aligned} \tag{2.3}$$

$$\begin{aligned}
\mathcal{F}_{\mathbf{U}} :\ & \mathrm{Mor}(\mathcal{C}) \longrightarrow \mathrm{Mor}(\mathbf{Cat}) \\
& \Big(\mathbf{V} \xrightarrow{\mathbf{\Theta}} \mathbf{W}\Big) \mapsto \Big(\mathcal{F}(\mathbf{W},\mathbf{U}) \xrightarrow{\mathcal{F}_{\mathbf{U}}(\mathbf{\Theta})} \mathcal{F}(\mathbf{V},\mathbf{U})\Big),
\end{aligned} \tag{2.4}$$

where $\mathbf{\Theta}$ is a functor from the category $\mathbf{V}$ to $\mathbf{W}$, and (2.4) is specified by following two equations.

$$\begin{aligned}
\mathcal{F}_{\mathbf{U}}(\mathbf{\Theta}) : \mathrm{Obj}\Big(\mathcal{F}(\mathbf{W},\mathbf{U})\Big) &\longrightarrow \mathrm{Obj}\Big(\mathcal{F}(\mathbf{V},\mathbf{U})\Big), \\
\mathbf{\Psi} &\mapsto \mathbf{\Psi\Theta},
\end{aligned} \tag{2.5}$$



and

$$\begin{aligned} &\mathcal{F}_{\mathbf{U}}(\boldsymbol{\Theta}) : \mathrm{Mor}\Big(\mathcal{F}(\mathbf{W},\mathbf{U})\Big) \longrightarrow \mathrm{Mor}\Big(\mathcal{F}(\mathbf{V},\mathbf{U})\Big),\\ &\mathcal{F}_{\mathbf{U}}(\boldsymbol{\Theta}) : \mathcal{N}(\mathbf{W},\mathbf{U}) \longrightarrow \mathcal{N}(\mathbf{V},\mathbf{U}),\\ &\big(\mathcal{F}_{\mathbf{U}}(\boldsymbol{\Theta})\big)(\mathcal{S}) := \mathcal{S}\boldsymbol{\Theta} \in \mathcal{N}(\mathbf{V},\mathbf{U}). \end{aligned} \tag{2.6}$$

Following proposition confirms that indeed $\mathcal{F}_{\mathbf{U}} : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Cat}$ is a functor.

Proposition 2.1 — [Proposition 2.1 [15]]. Let $\mathcal{C}$ be a category of a given set of (small)categories and **Cat** be the category of all (small) categories. Then, for each $\mathbf{U} \in \mathrm{Obj}(\mathcal{C})$, we have a contravariant functor $\mathcal{F}_{\mathbf{U}} : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Cat}$ defined as in (2.3), (2.5) and (2.6).

Moreover the functors $\mathcal{F}_{\mathbf{U}}$ are consistent with the Yoneda lemma.

Proposition 2.2 — [Proposition 2.2 [15]]. There exists an isomorphism between $\mathrm{Fun}(\mathbf{U},\mathbf{V})$ and $\mathrm{Nat}(\mathcal{F}_{\mathbf{U}},\mathcal{F}_{\mathbf{V}})$:

$$\mathrm{Fun}(\mathbf{U},\mathbf{V}) \cong \mathrm{Nat}(\mathcal{F}_{\mathbf{U}},\mathcal{F}_{\mathbf{V}}). \tag{2.7}$$

2.1 *Presheaves of categories*

Two prominent directions of enquiry, which immediately emerge out of the definition of presheaves, are *sheaves* and *sieves*. In [15] we have introduced the notion of **Cat**-valued presheaf to study the **Cat**-valued sieves. Here we will recall the definition of **Cat**-valued presheaf given in [15]. In this paper our focus will be **Cat**-valued sheaves.

As before, let $\mathcal{C}$ be a category of a set of (small) categories, and **Cat** be the category of all small categories. We work with following definition of *presheaves of categories* over $\mathcal{C}$. A *presheaf of categories* (or, a **Cat**-*valued presheaf*), over the category $\mathcal{C}$, is a functor

$$\mathcal{R} : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Cat}. \tag{2.8}$$

An immediate consequence of the definition above and Proposition 2.1 is the following.

Lemma 2.1 [Corollary 3.1, [15]]. For each $\mathbf{U} \in \mathrm{Obj}(\mathcal{C})$, the functor

$$\mathcal{F}_{\mathbf{U}} : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Cat}$$

in Proposition 2.1 is a presheaf of categories, over the category $\mathcal{C}$.

Let $\mathbf{Prsh}(\mathcal{C}, \mathbf{Cat}) := \mathcal{F}(\mathcal{C}^{\mathrm{op}}, \mathbf{Cat})$ denote the category of **Cat**-valued presheaves, over the category $\mathcal{C}$; that is,

$$\begin{aligned} \mathrm{Obj}\Big(\mathbf{Prsh}(\mathcal{C}, \mathbf{Cat})\Big) &= \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Cat}), \\ \mathrm{Mor}\Big(\mathbf{Prsh}(\mathcal{C}, \mathbf{Cat})\Big) &= \mathcal{N}(\mathcal{C}^{\mathrm{op}}, \mathbf{Cat}). \end{aligned} \tag{2.9}$$

Using Lemma 2.1 and Proposition 2.2 we prove a version of Yoneda embedding of $\mathcal{C}$ in $\mathbf{Prsh}(\mathcal{C}, \mathbf{Cat})$.

Proposition 2.3 — [Theorem 3.2, [15]]. Let $\mathcal{C}$ be a category of a collection of (small) categories, and **Cat** be the category of all small categories. Let $\mathbf{Prsh}(\mathcal{C}, \mathbf{Cat}) := \mathcal{F}(\mathcal{C}^{\mathrm{op}}, \mathbf{Cat})$ be the category of **Cat**-valued presheaves over the category $\mathcal{C}$. Then there exists a full and faithful functor

$$\mathcal{C} \longrightarrow \mathbf{Prsh}(\mathcal{C}, \mathbf{Cat}).$$

In other words, $\mathcal{C}$ can be identified as a full subcategory of $\mathbf{Prsh}(\mathcal{C}, \mathbf{Cat})$.

Instead of working with the entire category **Cat**, one can consider a presheaf of subcategories of **Cat**. For example, one may define a presheaf of ”categorical groups”, over $\mathcal{C}$, to be a contravariant functor from $\mathcal{C}$ to **CatGrp** :

$$\mathcal{R} : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{CatGrp},$$

where **CatGrp** is the category of categorical groups. We will denote “category of presheaves of categorical groups” by

$$\mathbf{Prsh}(\mathcal{C}, \mathbf{CatGrp}).$$

In section 5 we will construct such an example of presheaf of categorical groups.

3. Categorical Cover

In the next section we are going to introduce the notion of “**Cat**-valued sheaves”. As a prerequisite we need a “reasonable definition” for a cover of a sub category. This section will provide the necessary frame-work for union of subcategories (namely “*categorical union*”) of a given category, and *categorical cover* of a “topological category”.

3.1 *Categorical cover*

Suppose **B** is a category, such that both object and morphism spaces are topological spaces. We will call such categories *topological categories* [11]. If the category is also a groupoid, then we call it a *topological groupoid*. For example, a Lie groupoid is a topological groupoid. We will soon see another example of a topological groupoid, the "path space groupoid" of a topological space, which would be more relevant for this paper. It is easy to see that the *intersection of two subcategories* $\mathbf{U}$ and $\mathbf{V}$, defined as:

$$\begin{aligned} \mathrm{Obj}(\mathbf{U} \cap \mathbf{V}) &:= \mathrm{Obj}(\mathbf{U}) \cap \mathrm{Obj}(\mathbf{V}), \\ \mathrm{Mor}(\mathbf{U} \cap \mathbf{V}) &:= \mathrm{Mor}(\mathbf{U}) \cap \mathrm{Mor}(\mathbf{V}), \end{aligned} \tag{3.1}$$

is also a subcategory. However, if we define the union of two subcategories in similar fashion,

$$\begin{aligned} \mathrm{Obj}(\mathbf{U} \cup \mathbf{V}) &:= \mathrm{Obj}(\mathbf{U}) \cup \mathrm{Obj}(\mathbf{V}), \\ \mathrm{Mor}(\mathbf{U} \cup \mathbf{V}) &:= \mathrm{Mor}(\mathbf{U}) \cup \mathrm{Mor}(\mathbf{V}), \end{aligned} \tag{3.2}$$

then this union fails to be a subcategory. One can see that the obstruction, for the above union of subcategories to be a subcategory, is precisely the $\mathrm{Obj}(\mathbf{U}) \cap \mathrm{Obj}(\mathbf{V})$. Because of the following reason. Suppose $a \xrightarrow{f_1} b \in \mathrm{Mor}(\mathbf{U})$ and $b \xrightarrow{f_2} c \in \mathrm{Mor}(\mathbf{V})$, where $b \in \mathrm{Obj}(\mathbf{U}) \cap \mathrm{Obj}(\mathbf{V})$. Then $f_2, f_1 \in \mathrm{Mor}(\mathbf{U}) \cup \mathrm{Mor}(\mathbf{V})$, but $f_2 \circ f_1 \notin \mathrm{Mor}(\mathbf{U}) \cup \mathrm{Mor}(\mathbf{V})$. Note that if $\mathbf{U} \cap \mathbf{V}$ is empty, then we have a well define union of subcategories defined as in (3.2).

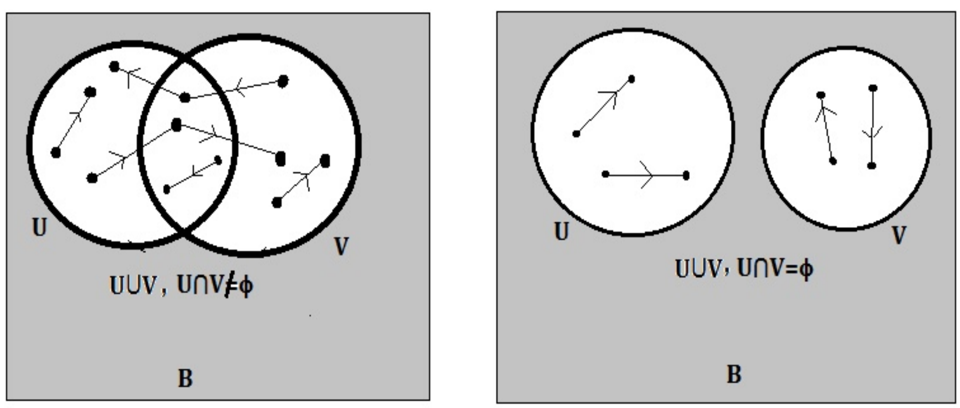


Figure 1: A schematic diagram for categorical union of a pair of subcategories

As a remedy to this problem we define the *categorical union* of two subcategories $\mathbf{U}$ and $\mathbf{V}$ to be the smallest subcategory containing $\mathbf{U}$ and $\mathbf{V}$. Explicitly $\mathbf{U} \cup \mathbf{V}$ has following description.

$$\begin{aligned} \mathrm{Obj}(\mathbf{U} \cup \mathbf{V}) &:= \mathrm{Obj}(\mathbf{U}) \cup \mathrm{Obj}(\mathbf{V}), \\ \mathrm{Mor}(\mathbf{U} \cup \mathbf{V}) &:= \mathbf{Gen}\Big(\mathrm{Mor}(\mathbf{U}) \cup \mathrm{Mor}(\mathbf{V})\Big), \end{aligned} \tag{3.3}$$

where

$$\mathbf{Gen}\Big(\mathrm{Mor}(\mathbf{U})\cup\mathrm{Mor}(\mathbf{V})\Big)=\Big\{f_2\circ f_1|f_2,f_1\in\mathrm{Mor}(\mathbf{U})\cup\mathrm{Mor}(\mathbf{V}),s(f_2)=t(f_1)\Big\};\tag{3.4}$$

that is, $\mathrm{Mor}(\mathbf{U}\cup\mathbf{V})$ is the subset of $\mathrm{Mor}(\mathbf{B})$, generated by $\mathrm{Mor}(\mathbf{U})$ and $\mathrm{Mor}(\mathbf{V})$. Note that $\mathrm{Mor}(\mathbf{U})\cup\mathrm{Mor}(\mathbf{V})\subset\mathbf{Gen}\Big(\mathrm{Mor}(\mathbf{U})\cup\mathrm{Mor}(\mathbf{V})\Big)$, and (3.4) can be rewritten as

$$\begin{aligned}&\mathbf{Gen}\Big(\mathrm{Mor}(\mathbf{U})\cup\mathrm{Mor}(\mathbf{V})\Big)=\mathrm{Mor}(\mathbf{U})\bigcup\mathrm{Mor}(\mathbf{V})\bigcup\\&\Big\{f_2\circ f_1|f_2\in\mathrm{Mor}(\mathbf{U}),f_1\in\mathrm{Mor}(\mathbf{V}),s(f_2)=t(f_1)\in\mathrm{Obj}(\mathbf{U})\cap\mathrm{Obj}(\mathbf{V})\Big\}\bigcup\\&\Big\{f_2\circ f_1|f_2\in\mathrm{Mor}(\mathbf{V}),f_1\in\mathrm{Mor}(\mathbf{U}),s(f_2)=t(f_1)\in\mathrm{Obj}(\mathbf{U})\cap\mathrm{Obj}(\mathbf{V})\Big\}.\end{aligned}\tag{3.5}$$

For non-intersecting $\mathbf{U}$ and $\mathbf{V}$, we have

$$\mathbf{Gen}\Big(\mathrm{Mor}(\mathbf{U})\cup\mathrm{Mor}(\mathbf{V})\Big)=\mathrm{Mor}(\mathbf{U})\cup\mathrm{Mor}(\mathbf{V}),\qquad\text{if }\ \mathbf{U}\cap\mathbf{V}=\emptyset.$$

It is obvious that the categorical union of $\mathbf{U},\mathbf{V}$, defined in (3.3), is a subcategory of $\mathbf{B}$. Henceforth by union of subcategories we will always understand the categorical union of subcategories.

It would be often convenient to visualize $\mathbf{B}$ as the *path space category (groupoid)* on a (smooth) topological space B. We will not discuss the path space category here in detail. [5, 4, 18] can be consulted for the same. Intuitively, the path space category of a topological space B is the category whose objects are elements of B and morphisms are paths on B (modulo "certain equivalence relations"), and composition is given by usual concatenation of paths. The equivalence relation is typically chosen to be "thin homotopy equivalence" [29] or "back-track equivalence" [18]. The later is a slightly weaker condition compare to the former (see Section 3 of [18] for a comparison between the two). Since every "path" can be reversed, this category in fact is a groupoid. The morphism space of a path space groupoid is equipped with usual compact-open topology. We will denote the path space category(groupoid) on B as $\mathbb{P}B$,

$$\begin{aligned}&\mathrm{Obj}(\mathbb{P}B)=B,\\&\mathrm{Mor}(\mathbb{P}B)=\{\text{paths on }B\}/\text{"some equivalence relations"}.\end{aligned}\tag{3.6}$$

Similarly given any open subset $U\subset B$, we can define the subcategory $\mathbb{P}U$ to be the path space groupoid corresponding to U. To relate this to the categorical union of subcategories, let us consider $\mathbf{U}:=\mathbb{P}U$ and $\mathbf{V}:=\mathbb{P}V$ to be the path space categories on open subsets

$U \subset B$ and $V \subset B$, respectively. Obviously $\mathbf{U}, \mathbf{V}$ are subcategories of $\mathbb{P}B := \mathbf{B}$. So, in this case $\mathbf{Gen}\Big(\mathrm{Mor}(\mathbf{U}) \cup \mathrm{Mor}(\mathbf{V})\Big)$ is basically the set of all paths (modulo "certain equivalence relations") lying on $U \cup V \subset B$. In other words, $\mathbf{U} \cup \mathbf{V}$ is the path space category $\mathbb{P}(U \cup V)$ of $U \cup V \subset B$. On a related note we observe that even without the smoothness condition on B we can construct the path space groupoid $\mathbb{P}B$ by taking the morphism space to be the space of homotopy equivalent paths on B, which is of course stronger equivalence condition than the previous two.

The intersection and (categorical) union of subcategories respectively defined in (3.1), (3.3) satisfy usual set theoretic Boolean relations of unions and intersections. Here we list some of them, and have provided a proof for the following proposition in the Appendix.

Proposition 3.1 — Let $\mathbf{U}, \mathbf{V}, \mathbf{W}$ be subcategories of a category $\mathbf{B}$. Suppose intersection and (categorical) union of subcategories are respectively defined as in (3.1), (3.3). Then following relations hold:

(i) $\mathbf{U} \cup (\mathbf{V} \cup \mathbf{W}) = (\mathbf{U} \cup \mathbf{V}) \cup \mathbf{W}$ and $\mathbf{U} \cap (\mathbf{V} \cap \mathbf{W}) = (\mathbf{U} \cap \mathbf{V}) \cap \mathbf{W}$,

(ii) $\mathbf{U} \cup (\mathbf{V} \cap \mathbf{W}) = (\mathbf{U} \cup \mathbf{V}) \cap (\mathbf{U} \cup \mathbf{W})$.

(iii) $\mathbf{U} \cap (\mathbf{V} \cup \mathbf{W}) = (\mathbf{U} \cap \mathbf{V}) \cup (\mathbf{U} \cap \mathbf{W})$,

Proof : See the Appendix. □

Let $\mathbf{U}$ be a subcategory of a topological groupoid $\mathbf{B}$, such that $\mathrm{Obj}(\mathbf{U})$ and $\mathrm{Mor}(\mathbf{U})$ are open (in $\mathrm{Obj}(\mathbf{B})$ and $\mathrm{Mor}(\mathbf{B})$, respectively). Then we call $\mathbf{U}$ to be an *open subcategory* of $\mathbf{B}$, if $\mathbf{U}$ is also a groupoid. The following obvious result would be useful later.

Lemma 3.1 — Suppose $\mathbf{U}$ is an open subcategory of a topological groupoid $\mathbf{B}$, and for $a \xrightarrow{f_1} b, b \xrightarrow{f_2} c, a \xrightarrow{f_1'} b', b' \xrightarrow{f_2'} c \in \mathrm{Mor}(\mathbf{U})$, we have

$$f_2 \circ f_1 = f_2' \circ f_1'. \tag{3.7}$$

Then there exists an isomorphism $b \xrightarrow{g_0} b' \in \mathrm{Mor}(\mathbf{U})$, such that

$$\begin{aligned} g_0 \circ f_1 &= f_1' \\ f_2 \circ g_0^{-1} &= f_2'. \end{aligned} \tag{3.8}$$

Proof : Directly follows from the fact that $\mathbf{U}$ is a groupoid. □

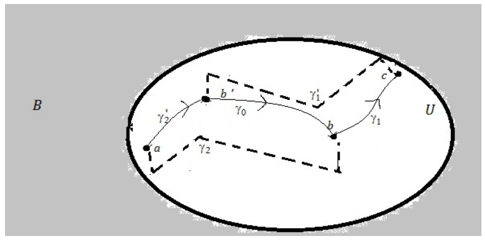


Figure 2:

In the context of the path space groupoid $\mathbb{P}B$ over a smooth topological space B, there is an elegant approach, namely "*back-track erasing*" [18, 26], to deal with such a situation and the condition in (3.7)-(3.8) is ensured by imposing certain equivalence relation on the space of paths. Roughly the idea is as follows. If U is an open subset of B, and $\mathbf{U} := \mathbb{P}U$ is the path space category over U , that is

$$\begin{aligned} &\mathrm{Obj}(\mathbb{P}U) = U, \\ &\mathrm{Mor}(\mathbb{P}U) = \{\text{paths on } U\}/\text{“some equivalence relations”}. \end{aligned} \tag{3.9}$$

Suppose two pairs of paths (modulo equivalence), $a \xrightarrow{\gamma_1} b, b \xrightarrow{\gamma_2} c \in \mathrm{Mor}(\mathbb{P}U)$, and $a \xrightarrow{\gamma_1'} b', b' \xrightarrow{\gamma_2'} c \in \mathrm{Mor}(\mathbb{P}U)$ such that

$$\gamma_2 \circ \gamma_1 = \gamma_2' \circ \gamma_1',$$

then we have the path segment $b \xrightarrow{\gamma_0} b'$ between b and b', satisfying

$$\begin{aligned} \gamma_0 \circ \gamma_1 &= \gamma_1' \\ \gamma_2 \circ \gamma_0^{-1} &= \gamma_2'. \end{aligned} \tag{3.10}$$

Figure 2 illustrates (3.10).

For future reference we make following observation.

Lemma 3.2 — Let $\mathbf{U}, \mathbf{V}$ be open subcategories of a topological groupoid $\mathbf{B}$. Then $\mathbf{U} \cup \mathbf{V}$ and $\mathbf{U} \cap \mathbf{V}$ are also open subcategories of $\mathbf{B}$.

PROOF : Straightforward verification. □

Let $\mathbf{U}$ be an open subcategory of a topological groupoid $\mathbf{B}$. An indexed family of subcategories $\{\mathbf{U}_\alpha\}_{\alpha \in I}$ is an *open categorical cover* of $\mathbf{U}$, if each $\mathbf{U}_\alpha$ is an open subcategory of $\mathbf{U}$, and

$$\mathbf{U} = \bigcup_{\alpha} \mathbf{U}_\alpha, \tag{3.11}$$

where the union of subcategories is taken according to the definition in (3.3).

Following examples illustrate the idea behind an open categorical cover of a topological category.

Example 3.1 : We call a category $\mathbf{A}$ *trivially discrete*, if the objects form a set, and the only morphisms in $\mathrm{Mor}(\mathbf{A})$ are identity morphisms:

$$\mathrm{Mor}(\mathbf{A}) = \{1_a | a \in \mathrm{Obj}(\mathbf{A})\} \simeq \mathrm{Obj}(\mathbf{A}).$$

Let B be a topological space with an open cover $\{U_i\}_{i \in I}$, and B^{dis} be the corresponding trivially discrete category. Then, it is obvious that the set of trivially discrete categories, corresponding to $\{U_i\}_{i \in I}$, defines a categorical cover of B^{dis}.

Example 3.2 : Suppose $B := M$ is a smooth manifold with an open cover $\{U_i\}_{i \in I}$. Let U_{ik} denotes the intersection of U_i and U_k,

$$U_{ik} := U_i \cap U_k.$$

Let $\mathbf{B} := \mathbb{P}M$ be the path space category over M; that is, (skipping the technicalities)

$$\begin{aligned} &\mathrm{Obj}(\mathbb{P}M) = M \\ &\mathrm{Mor}(\mathbb{P}M) = \{C^1\text{-paths on } M\}/\text{“some equivalence relations”}. \end{aligned}$$

Let, for each $i \in I$,

$$\mathbf{U}_i := \mathbb{P}U_i$$

be the path space category on the open set U_i :

$$\begin{aligned} &\mathrm{Obj}(\mathbf{U}_i) = U_i, \\ &\mathrm{Mor}(\mathbf{U}_i) = \{C^1\text{-paths on } U_i\}/\text{“some equivalence relations”}. \end{aligned}$$

Clearly each $\mathbf{U}_i$ is a subcategory of $\mathbb{P}M$. Observe that every path on M has to locally lie on some U_i. That means, for each path γ on M, we have $J \subset I$, such that γ entirely lies on $\bigcup_{j \in J} U_j$. Suppose $J = \{j_1, \cdots, j_k\}$, and

$$\gamma_{j_r} := \gamma|_{U_{j_r}} = \text{restriction of } \gamma \text{ to } U_{j_r}.$$

So, γ can be written as composition of a sequence of paths $(\gamma_{j_k}, \cdots, \gamma_{j_1})$:

$$\gamma = \gamma_{j_k} \circ \cdots \circ \gamma_{j_1}.$$

Note that $\gamma_{j_r} \in \mathrm{Mor}(\mathbf{U}_{j_r})$. In other words, according to the definition in (3.3),

$$\gamma \in \mathrm{Mor}\left(\bigcup_{j \in J} \mathbf{U}_j\right).$$

Also as explained before each $\mathbf{U}_i$ is a groupoid. Hence, as a consequence, $\{\mathbf{U}_i\}_{i \in I}$ define a categorical cover of $\mathbb{P}M$,

$$\bigcup_{i \in I} \mathbf{U}_i = \mathbb{P}M. \qquad \square$$

Example 3.3 : This is rather an example of what is not an open categorical cover. Let B be a smooth topological space with an open cover $\{U_i\}_{i \in I}$. Let U_{ik} denotes the intersection of U_i and U_k,

$$U_{ik} := U_i \cap U_k.$$

Let $\mathbf{B}$ be the path space grouopid $\mathbb{P}B$. Let $U_i^j, \ \ i, j \in I$, be the set of all morphisms whose sources are in U_i and targets are in U_j :

$$U_i^j := \{f \in \mathrm{Mor}(\mathbf{B}) | \mathrm{source}(f) \in U_i, \mathrm{target}(f) \in U_j\} \subset \mathrm{Mor}(\mathbf{B}). \tag{3.12}$$

Note that also,

$$U_i^j \cap U_k^l = U_{ik}^{jl},$$

where U_{ik}^{jl} is the set of morphisms starting in U_{ik}, and terminating U_{jl}. Since $\{U_i\}$ is an open cover of B, every morphism in $\mathbf{B}$ has to start in some U_i and terminate in some U_j. In other words,

$$\bigcup_{i,j \in I} U_i^j = \mathrm{Mor}(\mathbf{B}).$$

Now let us define a category $\mathbf{U}_i^k$, for each $i, k \in I$, as follows:

$$\begin{aligned} &\mathrm{Obj}(\mathbf{U}_i^j) = U_i \cup U_j, \\ &\mathrm{Mor}(\mathbf{U}_i^j) = U_i^j \cup \{1_p | p \in U_i \cup U_j\}. \end{aligned} \tag{3.13}$$

Thus each $\mathbf{U}_i^j$ is a subcategory of $\mathbf{U}$, and

$$\bigcup_{i,j \in I} \mathbf{U}_i^j = \mathbf{B}, \tag{3.14}$$

But $\mathbf{U}_i^j$ is not a groupoid. So according to our definition $\mathbf{U}_i^j$s are not open subcategories. Hence $\{\mathbf{U}_i^j\}_{i,j \in I}$ is not an (open)categorical cover of $\mathbf{B}$. $\square$

4. **Cat**-valued Sheaves and Functorial Sections

In this section first we will introduce the notion of "**Cat**-valued sheaves", and finally, our intention is to construct an example of **Cat** valued sheaf of "functorial sections".

4.1 **Cat**-*valued sheaves*

Suppose $\mathbf{B}$ is a topological groupoid. Let us consider the category $\widetilde{\mathcal{O}}(\mathbf{B})$ of open subcategories:

$$\begin{aligned}\mathrm{Obj}\Big(\widetilde{\mathcal{O}}(\mathbf{B})\Big) &:= \{\text{open } \mathbf{U} | \mathbf{U} \subset \mathbf{B}\} \\ &= \text{set of all open subcategories of } \mathbf{B},\end{aligned} \tag{4.1}$$
$$\mathrm{Hom}(\mathbf{U}, \mathbf{V}) = \{\mathbf{\Theta} : \mathbf{U} \longrightarrow \mathbf{V} | \text{ open } \mathbf{U}, \mathbf{V} \subset \mathbf{B}, \mathbf{\Theta} \text{ continuous}\}.$$

By a continuous functor we mean a functor which is continuous both on objects and morphisms. Henceforth $\widetilde{\mathcal{O}}(\mathbf{B})$ will always denote the category of open subcategories of a topological groupoid $\mathbf{B}$ as described in (4.1).

Recall the definition of a **Cat**-valued presheaf defined in (2.8). Let $\mathcal{R}$ be a **Cat**-valued presheaf over $\widetilde{\mathcal{O}}(\mathbf{B})$,

$$\mathcal{R} : \widetilde{\mathcal{O}}(\mathbf{B})^{\mathrm{op}} \longrightarrow \mathbf{Cat}. \tag{4.2}$$

Let $\mathbf{U}$ be an open subcategory of $\mathbf{B}$, and $\{\mathbf{U}_\alpha\}_{\alpha \in I}$ be an open categorical cover of $\mathbf{U}$ (see (3.11)). Let

$$\mathbf{i}_\alpha : \mathbf{U}_\alpha \hookrightarrow \mathbf{U}, \qquad \alpha \in I$$

be the inclusion functor. Thus, $\mathcal{R}(\mathbf{i}_\alpha)$ defines a functor from the category $\mathcal{R}(\mathbf{U}) \in \mathrm{Obj}(\mathbf{Cat})$ to the category $\mathcal{R}(\mathbf{U}_\alpha) \in \mathrm{Obj}(\mathbf{Cat})$:

$$\mathcal{R}(\mathbf{i}_\alpha) : \mathcal{R}(\mathbf{U}) \longrightarrow \mathcal{R}(\mathbf{U}_\alpha), \tag{4.3}$$

for each $\alpha \in I$. Similarly, if $\mathbf{U}_\alpha \cap \mathbf{U}_\beta := \mathbf{U}_{\alpha\beta}$ is non empty, then we have a pair of inclusion functors (we use same notation for both of them)

$$\begin{aligned}\mathbf{i}_{\alpha\beta} : \mathbf{U}_{\alpha\beta} \hookrightarrow \mathbf{U}_\alpha,\\ \mathbf{i}_{\alpha\beta} : \mathbf{U}_{\alpha\beta} \hookrightarrow \mathbf{U}_\beta,\end{aligned} \tag{4.4}$$

and corresponding to them we have functors

$$\begin{aligned}\mathcal{R}(\mathbf{i}_{\alpha\beta}) : \mathcal{R}(\mathbf{U}_\alpha) \longrightarrow \mathcal{R}(\mathbf{U}_{\alpha\beta}),\\ \mathcal{R}(\mathbf{i}_{\alpha\beta}) : \mathcal{R}(\mathbf{U}_\beta) \longrightarrow \mathcal{R}(\mathbf{U}_{\alpha\beta}),\end{aligned} \tag{4.5}$$

respectively. We call $\mathcal{R}$ to be a **Cat**-*valued sheaf* over $\widetilde{\mathcal{O}}(\mathbf{B})$ provided following conditions are satisfied:

1. **[Locality]**

(i) If $\mathbf{\Psi}_1, \mathbf{\Psi}_2 \in \mathrm{Obj}\Big(\mathcal{R}(\mathbf{U})\Big)$, such that,

$$\mathcal{R}(\mathbf{i}_\alpha)(\mathbf{\Psi}_1) = \mathcal{R}(\mathbf{i}_\alpha)(\mathbf{\Psi}_2), \tag{4.6}$$

for all $\alpha \in I$, then,

$$\mathbf{\Psi}_1 = \mathbf{\Psi}_2. \tag{4.7}$$

(ii) If $\mathbf{\Psi}_1, \mathbf{\Psi}_2, \widetilde{\mathbf{\Psi}}_1, \widetilde{\mathbf{\Psi}}_2 \in \mathrm{Obj}(\mathcal{R}(\mathbf{U}))$ satisfy

$$\begin{aligned} \mathcal{R}(\mathbf{i}_\alpha)(\mathbf{\Psi}_1) &= \mathcal{R}(\mathbf{i}_\alpha)(\mathbf{\Psi}_2), \\ \mathcal{R}(\mathbf{i}_\alpha)(\widetilde{\mathbf{\Psi}}_1) &= \mathcal{R}(\mathbf{i}_\alpha)(\widetilde{\mathbf{\Psi}}_2), \end{aligned} \tag{4.8}$$

for all $\alpha \in I$, and $\Big(\mathbf{\Psi}_1 \xrightarrow{\mathcal{S}_1} \widetilde{\mathbf{\Psi}}_1\Big), \Big(\mathbf{\Psi}_2 \xrightarrow{\mathcal{S}_2} \widetilde{\mathbf{\Psi}}_2\Big) \in \mathrm{Mor}\Big(\mathcal{R}(\mathbf{U})\Big)$ such that,

$$\mathcal{R}(\mathbf{i}_\alpha)(\mathcal{S}_1) = \mathcal{R}(\mathbf{i}_\alpha)(\mathcal{S}_2), \tag{4.9}$$

for all $\alpha \in I$, then,

$$\mathcal{S}_1 = \mathcal{S}_2. \tag{4.10}$$

2. **[Gluing]**

(i) If for each $\alpha \in I$, a $\mathbf{\Psi}_\alpha \in \mathrm{Obj}\Big(\mathcal{R}(\mathbf{U}_\alpha)\Big)$ is given, such that, for any non empty $\mathbf{U}_{\alpha\beta}$

$$\mathcal{R}(\mathbf{i}_{\alpha\beta})(\mathbf{\Psi}_\alpha) = \mathcal{R}(\mathbf{i}_{\alpha\beta})(\mathbf{\Psi}_\beta), \tag{4.11}$$

then there exists a $\mathbf{\Psi} \in \mathrm{Obj}(\mathbf{U})$, such that

$$\mathcal{R}(\mathbf{i}_\alpha)(\mathbf{\Psi}) = \mathbf{\Psi}_\alpha, \qquad \forall \alpha \in I.$$

(ii) If for each $\alpha \in I$, a $\Big(\mathbf{\Psi} \xrightarrow{\mathcal{S}_\alpha} \widetilde{\mathbf{\Psi}}\Big) \in \mathrm{Mor}\Big(\mathcal{R}(\mathbf{U}_\alpha)\Big)$ is given, such that, for any non empty $\mathbf{U}_{\alpha\beta}$

$$\begin{aligned} \mathcal{R}(\mathbf{i}_{\alpha\beta})(\mathbf{\Psi}_\alpha) &= \mathcal{R}(\mathbf{i}_{\alpha\beta})(\mathbf{\Psi}_\beta), \\ \mathcal{R}(\mathbf{i}_{\alpha\beta})(\widetilde{\mathbf{\Psi}}_\alpha) &= \mathcal{R}(\mathbf{i}_{\alpha\beta})(\widetilde{\mathbf{\Psi}}_\beta), \end{aligned} \tag{4.12}$$

and

$$\mathcal{R}(\mathbf{i}_{\alpha\beta})(\mathcal{S}_\alpha) = \mathcal{R}(\mathbf{i}_{\alpha\beta})(\mathcal{S}_\beta). \tag{4.13}$$

then there exists a $\left(\mathbf{\Psi} \xrightarrow{\mathcal{S}} \widetilde{\mathbf{\Psi}}\right) \in \mathrm{Mor}(\mathbf{U})$, such that,

$$\mathcal{R}(\mathbf{i}_\alpha)(\mathcal{S}) = \mathcal{S}_\alpha.$$

Example 4.1 : The above definition of a **Cat**-valued sheaf is a generalization of the standard definition of a sheaf.

If we consider the trivially discrete (see Example 3.1 for the definition) topological category B^{dis} corresponding to a topological space B, then a **Cat**-valued sheaf over $\mathcal{O}(B^{\mathrm{dis}})$ is same as a sheaf in the traditional sense. □

We end this section with a more interesting example of a **Cat**-valued sheaf over $\widetilde{\mathcal{O}}(\mathbf{B})$.

4.2 **Cat**-*valued sheaf of functorial sections*

Fix a topological category $\mathbf{C}$. For any $\mathbf{U} \in \mathrm{Obj}\left(\widetilde{\mathcal{O}}(\mathbf{B})\right)$, let

$$\mathcal{F}(\mathbf{U}, \mathbf{C}) := \mathbf{C}^{\mathbf{U}}$$

be the category of functors from $\mathbf{U} \longrightarrow \mathbf{C}$; that is:

$$\begin{aligned} \mathrm{Obj}(\mathbf{C}^{\mathbf{U}}) &= \mathrm{Fun}(\mathbf{U}, \mathbf{C}) \\ \mathrm{Mor}(\mathbf{C}^{\mathbf{U}}) &= \mathcal{N}(\mathbf{U}, \mathbf{C}). \end{aligned} \tag{4.14}$$

Lemma 4.1 — For any $\left(\mathbf{U} \xrightarrow{\mathbf{\Theta}} \mathbf{V}\right) \in \mathrm{Mor}\left(\widetilde{\mathcal{O}}(\mathbf{B})\right)$, we have a functor

$$\mathcal{R}(\mathbf{\Theta}) : \mathbf{C}^{\mathbf{V}} \longrightarrow \mathbf{C}^{\mathbf{U}},$$

given as follows:

$$\begin{aligned} \mathcal{R}(\mathbf{\Theta}) :&\mathrm{Obj}(\mathbf{C}^{\mathbf{V}}) \longrightarrow \mathrm{Obj}(\mathbf{C}^{\mathbf{U}}), \\ &\mathbf{\Psi} \mapsto \mathbf{\Psi}\mathbf{\Theta}, \\ \mathcal{R}(\mathbf{\Theta}) :&\mathrm{Mor}(\mathbf{C}^{\mathbf{V}}) \longrightarrow \mathrm{Mor}(\mathbf{C}^{\mathbf{U}}), \\ &\mathcal{S} \mapsto \mathcal{S}\mathbf{\Theta}. \end{aligned} \tag{4.15}$$

Proof : Define a new category of categories $\widehat{\mathcal{C}}$ by attaching $\mathbf{C}$ with the original category $\widetilde{\mathcal{O}}(\mathbf{B})$:

$$\begin{aligned}\mathrm{Obj}(\widehat{\mathcal{C}}) &:= \mathrm{Obj}\Big(\widetilde{\mathcal{O}}(\mathbf{B})\Big) \cup \{\mathbf{C}\},\\ \mathrm{Mor}(\widehat{\mathcal{C}}) &:= \mathrm{Mor}\Big(\widetilde{\mathcal{O}}(\mathbf{B})\Big) \cup \Big(\bigcup_{\mathbf{U}\subset\mathbf{B}} \mathrm{Fun}(\mathbf{U},\mathbf{C})\Big) \cup \Big(\bigcup_{\mathbf{U}\subset\mathbf{B}} \mathrm{Fun}(\mathbf{C},\mathbf{U})\Big) \cup \Big(\mathrm{Fun}(\mathbf{C},\mathbf{C})\Big).\end{aligned}$$

Then according to Proposition 2.1, for $\mathbf{C} \in \mathrm{Obj}(\widehat{\mathcal{C}})$ we have the functor

$$\mathcal{F}_{\mathbf{C}} : \widehat{\mathcal{C}}^{\mathrm{op}} \longrightarrow \mathbf{Cat}. \tag{4.16}$$

In particular, if $\mathbf{U},\mathbf{V} \in \mathrm{Obj}\big(\widetilde{\mathcal{O}}(\mathbf{B})\big) \subset \mathrm{Obj}(\widehat{\mathcal{C}})$ and $\Big(\mathbf{U} \xrightarrow{\boldsymbol{\Theta}} \mathbf{V}\Big)$ then

$$\Big(\mathbf{U} \xrightarrow{\boldsymbol{\Theta}} \mathbf{V}\Big) \mapsto \Big(\mathbf{C}^{\mathbf{V}} \xrightarrow{\mathcal{F}_{\mathbf{C}}(\boldsymbol{\Theta})} \mathbf{C}^{\mathbf{U}}\Big),$$

since $\mathcal{F}_{\mathbf{C}}(\mathbf{U}) = \mathcal{F}(\mathbf{U},\mathbf{C}) = \mathbf{C}^{\mathbf{U}}$. It is immediate that $\mathcal{F}_{\mathbf{C}}$ restricted to $\widetilde{\mathcal{O}}(\mathbf{B})$ is in fact the $\mathcal{R}$ in the statement of the lemma;

$$\mathcal{F}_{\mathbf{C}}|_{\widetilde{\mathcal{O}}(\mathbf{B})} = \mathcal{R}.\Box$$

Therefore we have a **Cat**-valued presheaf $\mathcal{R}$:

$$\begin{aligned}\mathcal{R} :&\widetilde{\mathcal{O}}(\mathbf{B})^{\mathrm{op}} \longrightarrow \mathbf{Cat},\\ &\mathrm{Obj}\Big(\widetilde{\mathcal{O}}(\mathbf{B})\Big) \longrightarrow \mathrm{Obj}(\mathbf{Cat}),\\ &\mathbf{U} \mapsto \mathbf{C}^{\mathbf{U}},\\ &\mathrm{Mor}\Big(\widetilde{\mathcal{O}}(\mathbf{B})\Big) \longrightarrow \mathrm{Mor}(\mathbf{Cat}),\\ &\Big(\mathbf{U} \xrightarrow{\boldsymbol{\Theta}} \mathbf{V}\Big) \mapsto \Big(\mathbf{C}^{\mathbf{V}} \xrightarrow{\mathcal{R}(\boldsymbol{\Theta})} \mathbf{C}^{\mathbf{U}}\Big).\end{aligned} \tag{4.17}$$

Rest of this section will be devoted to prove (Theorem 4.2) that the **Cat**-valued presheaf, defined above, is in fact a **Cat**-valued sheaf.

Theorem* 4.2** — *Let* $\mathbf{B}$ *be a topological groupoid. Let* $\widetilde{\mathcal{O}}(\mathbf{B})$ *be the category of open subcategories. Let* $\mathcal{R}$ *be as defined in Lemma 4.1. Then* $\mathcal{R}$ *is a* **Cat-valued sheaf over* $\widetilde{\mathcal{O}}(\mathbf{B})$.

Note that if $\{\mathbf{U}_\alpha\}_\alpha$ is an open categorical cover of $\mathbf{B}$, then Lemma 3.2 ensures,

$$\begin{aligned}&\mathbf{U}_{\alpha\beta} := \mathbf{U}_\alpha \cap \mathbf{U}_\beta \in \mathrm{Obj}\big(\widetilde{\mathcal{O}}(\mathbf{B})\big),\\ &\mathbf{U}_\alpha \cup \mathbf{U}_\beta \in \mathrm{Obj}\big(\widetilde{\mathcal{O}}(\mathbf{B})\big).\end{aligned} \tag{4.18}$$

Before we carry out the proof of this theorem, we will work in a simpler situation to get an insight into the overall methodology of the proof.

We have to verify that $\mathcal{R}$ satisfies [**Locality**] and [**Gluing**] conditions. The crucial issue here is that, if $\{\mathbf{U}_\alpha\}$ is a categorical cover of $\mathbf{U}$, then in general $\bigcup_\alpha \mathrm{Mor}(\mathbf{U}_\alpha)$ is a subset of $\mathrm{Mor}\Big(\bigcup_\alpha \mathbf{U}_\alpha\Big) = \mathrm{Mor}(\mathbf{U})$. In other words, there may exist a morphism in $\mathrm{Mor}(\mathbf{U})$, which does not belong to any $\mathrm{Mor}(\mathbf{U}_\alpha)$. Now, if $\mathbf{\Psi}_1, \mathbf{\Psi}_2 \in \mathrm{Obj}(\mathcal{R}(\mathbf{U})) = \mathrm{Obj}(\mathbf{C}^{\mathbf{U}}) = \mathrm{Fun}(\mathbf{U}, \mathbf{C})$, such that $\mathbf{\Psi}_1, \mathbf{\Psi}_2$ coincide in each $\mathbf{U}_\alpha$, then it is not obvious that $\mathbf{\Psi}_1 = \mathbf{\Psi}_2$. Because, there is a possibility that $\mathbf{\Psi}_1, \mathbf{\Psi}_2$ are different on some

$$\begin{aligned} &f \in \mathrm{Mor}\Big(\bigcup_\alpha \mathbf{U}_\alpha\Big) = \mathrm{Mor}(\mathbf{U}), \\ &f \notin \bigcup_\alpha \mathrm{Mor}(\mathbf{U}_\alpha). \end{aligned} \tag{4.19}$$

But we will see, remarkably, that does not actually happen! And, even for an f as in (4.19), $\mathbf{\Psi}_1(f), \mathbf{\Psi}_2(f)$ are completely determined by the local data; i.e. if (4.6) holds for $\mathbf{\Psi}_1, \mathbf{\Psi}_2$, then $\mathbf{\Psi}_1(f) = \mathbf{\Psi}_2(f)$, for any $f \in \mathrm{Mor}(\mathbf{U})$ including one like in (4.19). Similarly, other conditions of [**Locality**] and [**Gluing**] also hold. To see this, at first we deal with a "test case", where an open subcategory $\mathbf{U}$ of $\mathbf{B}$ is covered by only two (open) subcategories $\{\mathbf{U}_1, \mathbf{U}_2\}$:

$$\mathbf{U}_1 \cup \mathbf{U}_2 = \mathbf{U}. \tag{4.20}$$

Before we proceed further, let us simplify our notations. If $\mathbf{U}$ is a subcategory of $\mathbf{V}$, and $\mathbf{i} : \mathbf{U} \hookrightarrow \mathbf{V}$ is the inclusion functor, then for $\mathbf{\Psi} \in \mathrm{Obj}(\mathbf{C}^{\mathbf{V}}), \mathcal{S} \in \mathrm{Mor}(\mathbf{C}^{\mathbf{V}})$, we will respectively denote,

$$\begin{aligned} \Big(\mathcal{R}(\mathbf{i})\Big)(\mathbf{\Psi}) &:= \mathbf{\Psi}|_{\mathbf{U}} \in \mathrm{Obj}(\mathbf{C}^{\mathbf{U}}), \\ \Big(\mathcal{R}(\mathbf{i})\Big)(\mathcal{S}) &:= \mathcal{S}|_{\mathbf{U}} \in \mathrm{Mor}(\mathbf{C}^{\mathbf{U}}). \end{aligned} \tag{4.21}$$

Proposition 4.1 — Let $\widetilde{\mathcal{O}}(\mathbf{B})$ be the category of open subcategories. Let $\mathcal{R}$ be as defined in Lemma 4.1. Let $\mathbf{U}$ be an open subcategory of $\widetilde{\mathcal{O}}(\mathbf{B})$ with categorical cover $\{\mathbf{U}_\alpha\}_{\alpha\in\{1,2\}}$. Then on $\mathbf{U}$ [**Locality**] and [**Gluing**] conditions, listed between (4.6)-(4.13), hold.

PROOF : First we verify [**Locality**] conditions.

Let $\mathbf{\Psi}_1, \mathbf{\Psi}_2 \in \mathrm{Obj}(\mathbf{C}^{\mathbf{U}}) = \mathrm{Fun}(\mathbf{U}, \mathbf{C})$, such that,

$$\mathbf{\Psi}_1|_{\mathbf{U}_\alpha} = \mathbf{\Psi}_2|_{\mathbf{U}_\alpha}, \qquad \alpha \in \{1, 2\}.$$

Recalling the definition of categorical union given in (3.3) we see: $\mathbf{\Psi}_1 = \mathbf{\Psi}_2$ on objects. Also, $\mathbf{\Psi}_1 = \mathbf{\Psi}_2$ on $\mathrm{Mor}(\mathbf{U}_1) \cup \mathrm{Mor}(\mathbf{U}_2) \subset \mathrm{Mor}(\mathbf{U})$. Now, consider an arbitrary $g \in \mathrm{Mor}(\mathbf{U})$, that means there exist $g_2 \in \mathrm{Mor}(\mathbf{U}_2)$ and $g_1 \in \mathrm{Mor}(\mathbf{U}_1)$, such that

$$g = g_2 \circ g_1.$$

[To be precise, $\tilde{g}_1 \in \mathrm{Mor}(\mathbf{U}_1), \tilde{g}_2 \in \mathrm{Mor}(\mathbf{U}_2)$, and $\tilde{g}_1 \circ \tilde{g}_2 = g$ is also an alternate possibility. But, this case can be dealt in exactly same fashion as the other. And, we will not explicitly consider it]. Thus,

$$\begin{aligned} \mathbf{\Psi}_1(g) &= \mathbf{\Psi}_1(g_2) \circ \mathbf{\Psi}_1(g_1) \quad [\because \mathbf{\Psi}_1 \text{is a functor}] \\ &= \mathbf{\Psi}_2(g_2) \circ \mathbf{\Psi}_2(g_1) \;\; [\because g_2 \in \mathrm{Mor}(\mathbf{U}_2), g_1 \in \mathrm{Mor}(\mathbf{U}_1), \\ &\qquad\qquad\qquad\qquad \text{and } \mathbf{\Psi}_1|_{\mathbf{U}_\alpha} = \mathbf{\Psi}_2|_{\mathbf{U}_\alpha}, \alpha \in \{1,2\}] \\ &= \mathbf{\Psi}_2(g_2 \circ g_1) \quad [\because \mathbf{\Psi}_2 \text{ is a functor}] \\ &= \mathbf{\Psi}_2(g). \end{aligned} \tag{4.22}$$

Thus $\mathbf{\Psi}_1 = \mathbf{\Psi}_2$ on $\mathbf{U}$.

Similarly, if $\big(\mathcal{S}_1 : \mathbf{\Psi}_1 \Longrightarrow \mathbf{\Psi}_2\big), \big(\mathcal{S}_2 : \mathbf{\Psi}_1 \Longrightarrow \mathbf{\Psi}_2\big) \in \mathrm{Mor}(\mathbf{C}^{\mathbf{U}}) = \mathcal{N}(\mathbf{U}, \mathbf{C})$, such that,

$$\mathcal{S}_1|_{\mathbf{U}_\alpha} = \mathcal{S}_2|_{\mathbf{U}_\alpha}, \qquad \alpha \in \{1,2\},$$

then for any $(a \xrightarrow{f} b) \in \mathrm{Mor}(\mathbf{U}_1) \cup \mathrm{Mor}(\mathbf{U}_2)$, we have the commuting diagram,

$$\begin{array}{ccc} \mathbf{\Psi}_1(a) & \xrightarrow{\mathbf{\Psi}_1(f)} & \mathbf{\Psi}_1(b) \\ \Big\downarrow{\scriptstyle \mathcal{S}(a)} & & {\scriptstyle \mathcal{S}(b)}\Big\downarrow \\ \mathbf{\Psi}_2(a) & \xrightarrow[\mathbf{\Psi}_2(f)]{} & \mathbf{\Psi}_2(b) \end{array}, \tag{4.23}$$

where we denote $\mathcal{S} := \mathcal{S}_1|_{\mathbf{U}_\alpha} = \mathcal{S}_2|_{\mathbf{U}_\alpha}, \alpha \in \{1,2\}$. Now suppose $(a \xrightarrow{g} c) \in \mathrm{Mor}(\mathbf{U})$ is an arbitrary morphism in $\mathrm{Mor}(\mathbf{U})$. That means we have $(a \xrightarrow{g_1} b) \in \mathrm{Mor}(\mathbf{U}_1), (b \xrightarrow{g_2} c) \in \mathrm{Mor}(\mathbf{U}_2)$, such that

$$g = g_2 \circ g_1.$$

$\mathcal{S}$ extends to $(a \xrightarrow{g} c) \in \mathrm{Mor}(\mathbf{U})$ as follows. We have commuting diagrams,

$$\begin{array}{ccc} \mathbf{\Psi}_1(a) & \xrightarrow{\mathbf{\Psi}_1(g_1)} & \mathbf{\Psi}_1(b) \\ \Big\downarrow{\scriptstyle \mathcal{S}(a)} & & {\scriptstyle \mathcal{S}(b)}\Big\downarrow \\ \mathbf{\Psi}_2(a) & \xrightarrow[\mathbf{\Psi}_2(g_1)]{} & \mathbf{\Psi}_2(b) \end{array}, \tag{4.24}$$

and,

$$\begin{array}{ccc} \mathbf{\Psi}_1(b) & \xrightarrow{\mathbf{\Psi}_1(g_2)} & \mathbf{\Psi}_1(c) \\ \Big\downarrow{\scriptstyle \mathcal{S}(b)} & & {\scriptstyle \mathcal{S}(c)}\Big\downarrow \\ \mathbf{\Psi}_2(b) & \xrightarrow[\mathbf{\Psi}_2(g_2)]{} & \mathbf{\Psi}_2(c) \end{array} . \tag{4.25}$$

Since $\mathbf{\Psi}_2, \mathbf{\Psi}_1$ are functors, composing above commuting diagrams we get the commuting diagram,

$$\begin{array}{ccc} \mathbf{\Psi}_1(a) & \xrightarrow{\mathbf{\Psi}_1(g)} & \mathbf{\Psi}_1(c) \\ \Big\downarrow{\scriptstyle \mathcal{S}(a)} & & {\scriptstyle \mathcal{S}(c)}\Big\downarrow \\ \mathbf{\Psi}_2(a) & \xrightarrow[\mathbf{\Psi}_2(g)]{} & \mathbf{\Psi}_2(c) \end{array} \qquad [\text{putting } g_2 \circ g_1 = g]. \tag{4.26}$$

Thus $\mathcal{S} = \mathcal{S}_1 = \mathcal{S}_2$ for the category $\mathbf{U}$.

Our next task is to verify **[Gluing]** conditions.

For that, suppose $\mathbf{\Psi}_\alpha \in \mathrm{Obj}(\mathbf{C}^{\mathbf{U}_\alpha}) = \mathrm{Fun}(\mathbf{U}_\alpha, \mathbf{C})$ given for $\alpha \in \{1, 2\}$, such that: $\mathbf{\Psi}_1|_{\mathbf{U}_{12}} = \mathbf{\Psi}_2|_{\mathbf{U}_{12}}$. Then we define $(\mathbf{\Psi} : \mathbf{U} \longrightarrow \mathbf{C}) \in \mathrm{Fun}(\mathbf{U}, \mathbf{C}) = \mathrm{Obj}(\mathbf{C}^{\mathbf{U}})$ as follows:

$$\begin{aligned} &\mathrm{Obj}(\mathbf{U}) \longrightarrow \mathrm{Obj}(\mathbf{C}) \\ &\mathrm{Obj}(\mathbf{U}_1) \cup \mathrm{Obj}(\mathbf{U}_2) \longrightarrow \mathrm{Obj}(\mathbf{C}) \\ &\qquad\qquad a \mapsto \mathbf{\Psi}_\alpha(a), \qquad \text{if } a \in \mathbf{U}_\alpha, \alpha \in \{1, 2\}, \end{aligned} \tag{4.27}$$

and,

$$\begin{aligned} &\mathrm{Mor}(\mathbf{U}) \longrightarrow \mathrm{Mor}(\mathbf{C}) \\ &\mathbf{Gen}\Big(\mathrm{Mor}(\mathbf{U}_1) \cup \mathrm{Mor}(\mathbf{U}_2)\Big) \longrightarrow \mathrm{Mor}(\mathbf{C}), \\ &\qquad\qquad f_2 \circ f_1 \mapsto \mathbf{\Psi}_2(f_2) \circ \mathbf{\Psi}_1(f_1), \\ &\qquad\qquad f_2 \in \mathrm{Mor}(\mathbf{U}_2), f_1 \in \mathrm{Mor}(\mathbf{U}_1). \end{aligned} \tag{4.28}$$

Since $\mathbf{\Psi}_1|_{\mathbf{U}_{12}} = \mathbf{\Psi}_2|_{\mathbf{U}_{12}}$, (4.27) is well defined. For the same reason, right hand side of (4.28) makes sense [$\because t(f_1) = s(f_2) \in \mathrm{Obj}(\mathbf{U}_{12})$. So, $s(\mathbf{\Psi}_2(f_2)) = t(\mathbf{\Psi}_1(f_1))$.]. But we have to check if (4.28) is well defined; that is, if $f_2' \circ f_1' = f_2 \circ f_1$, for some other $f_2' \in \mathrm{Mor}(\mathbf{U}_2), f_1' \in \mathrm{Mor}(\mathbf{U}_1)$, then we should have

$$\mathbf{\Psi}_2(f_2) \circ \mathbf{\Psi}_1(f_1) = \mathbf{\Psi}_2(f_2') \circ \mathbf{\Psi}_1(f_1').$$

Let

$$s(f_2) = b = t(f_1), s(f_1) = a = s(f_1'), t(f_2') = c = t(f_2), s(f_2') = b' = t(f_1'),$$

so $b, b' \in \mathrm{Obj}(\mathbf{U}_{12})$.

By Lemma 3.1 we have an isomorphism $b \xrightarrow{g_0} b' \in \mathrm{Mor}(\mathbf{U})$ such that:

$$\begin{aligned} g_0 \circ f_1 &= f_1', \\ f_2 \circ g_0^{-1} &= f_2'. \end{aligned}$$

In fact,

$$b \xrightarrow{g_0} b' \in \mathrm{Mor}(\mathbf{U}_{12}).$$

Then

$$\begin{aligned} &\mathbf{\Psi}_2(f_2') \circ \mathbf{\Psi}_1(f_1') \\ &= \mathbf{\Psi}_2(f_2 \circ g_0^{-1}) \circ \mathbf{\Psi}_1(g_0 \circ f_1) \\ &= \mathbf{\Psi}_2(f_2) \circ \mathbf{\Psi}_2(g_0^{-1}) \circ \mathbf{\Psi}_1(g_0) \circ \mathbf{\Psi}_1(f_1) \\ &= \mathbf{\Psi}_2(f_2) \circ \mathbf{\Psi}_1(f_1) \quad [\because g_0 \in \mathrm{Mor}(\mathbf{U}_{12}) \text{ and } \mathbf{\Psi}_1|_{\mathbf{U}_{12}} = \mathbf{\Psi}_2|_{\mathbf{U}_{12}}]. \end{aligned} \tag{4.29}$$

Hence, (4.28) is well defined, and we have $(\mathbf{\Psi} : \mathbf{U} \longrightarrow \mathbf{C}) \in \mathrm{Fun}(\mathbf{U}, \mathbf{C}) = \mathrm{Obj}(\mathbf{C}^{\mathbf{U}})$ satisfying

$$\mathbf{\Psi}|_{\mathbf{U}_\alpha} = \mathbf{\Psi}_\alpha, \qquad \alpha = \{1, 2\}$$

Next, suppose $\left(\mathcal{S}_\alpha : \mathbf{\Psi}_\alpha \Longrightarrow \mathbf{\Psi}_\alpha'\right) \in \mathrm{Mor}(\mathbf{C}^{\mathbf{U}_\alpha}) = \mathcal{N}(\mathbf{U}_\alpha, \mathbf{C})$ given for $\alpha \in \{1, 2\}$, such that, $\mathbf{\Psi}_1, \mathbf{\Psi}_2$ glue to form a $\mathbf{\Psi} \in \mathrm{Fun}(\mathbf{U}, \mathbf{C})$, similarly, $\mathbf{\Psi}_1', \mathbf{\Psi}_2'$ glue to form a $\mathbf{\Psi}' \in \mathrm{Fun}(\mathbf{U}, \mathbf{C})$, [as described in the previous part of the proof] and $\mathcal{S}_\alpha$ satisfy

$$\mathcal{S}_1|_{\mathbf{U}_{12}} = \mathcal{S}_2|_{\mathbf{U}_{12}}. \tag{4.30}$$

We define $\mathcal{S} \in \mathcal{N}(\mathbf{U}, \mathbf{C})$ by:

$$\mathcal{S}(a) = \mathcal{S}_\alpha(a), \quad \text{for } a \in \mathrm{Obj}(\mathbf{U}_\alpha), \alpha \in \{1, 2\}. \tag{4.31}$$

Above equation makes sense because of (4.30). We have to ensure that (4.31) is a well defined natural transformation between $\mathbf{\Psi}$ and $\mathbf{\Psi}'$. It is obvious that for any $a \xrightarrow{f} b \in \mathrm{Mor}(\mathbf{U}_1) \cup \mathrm{Mor}(\mathbf{U}_2) \subset \mathrm{Mor}(\mathbf{U}) = \mathrm{Mor}(\mathbf{U}_1 \cup \mathbf{U}_2)$, we have the commuting diagram:

$$\begin{array}{ccc} \mathbf{\Psi}(a) & \xrightarrow{\mathbf{\Psi}(f)} & \mathbf{\Psi}(b) \\ \Big\downarrow{\scriptstyle \mathcal{S}_\alpha(a)} & & \Big\downarrow{\scriptstyle \mathcal{S}_\alpha(b)} \\ \mathbf{\Psi}'(a) & \xrightarrow[\mathbf{\Psi}'(f)]{} & \mathbf{\Psi}'(b) \end{array}, \tag{4.32}$$

where $a \xrightarrow{f} b \in \mathrm{Mor}(\mathbf{U}_\alpha), \alpha = \{1, 2\}$. We have to verify that $\mathcal{S}$ is well defined (as a natural transformation) for all $(a \xrightarrow{g} c) \in \mathrm{Mor}(\mathbf{U})$. Again, by the previous argument, we have $g = g_2 \circ g_1$, for some $(a \xrightarrow{g_1} b) \in \mathrm{Mor}(\mathbf{U}_1), (b \xrightarrow{g_2} c) \in \mathrm{Mor}(\mathbf{U}_2)$. [Here also we are ignoring the possibility $g = \tilde{g}_2 \circ \tilde{g}_1$, where $(a \xrightarrow{\tilde{g}_1} b') \in \mathrm{Mor}(\mathbf{U}_2), (b' \xrightarrow{\tilde{g}_2} c) \in \mathrm{Mor}(\mathbf{U}_1)$, as it can be handled in an exactly same manner.] We have a pair of commuting diagrams respectively in $\mathbf{U}_1$ and $\mathbf{U}_2$,

$$\begin{array}{ccc} \mathbf{\Psi}(a) & \xrightarrow{\mathbf{\Psi}(g_1)} & \mathbf{\Psi}(b) \\ \Big\downarrow{\scriptstyle \mathcal{S}_1(a)} & & {\scriptstyle \mathcal{S}_1(b)}\Big\downarrow \\ \mathbf{\Psi}'(a) & \xrightarrow[\mathbf{\Psi}'(g_1)]{} & \mathbf{\Psi}'(b) \end{array}, \tag{4.33}$$

and,

$$\begin{array}{ccc} \mathbf{\Psi}(b) & \xrightarrow{\mathbf{\Psi}(g_2)} & \mathbf{\Psi}(c) \\ \Big\downarrow{\scriptstyle \mathcal{S}_2(b)} & & {\scriptstyle \mathcal{S}_2(c)}\Big\downarrow \\ \mathbf{\Psi}'(b) & \xrightarrow[\mathbf{\Psi}'(g_2)]{} & \mathbf{\Psi}'(c) \end{array}. \tag{4.34}$$

Since $b \in \mathrm{Obj}(\mathbf{U}_{12})$, and $\mathcal{S}_1|_{\mathbf{U}_{12}} = \mathcal{S}_2|_{\mathbf{U}_{12}}$, we have a commuting diagram

$$\begin{array}{ccc} \mathbf{\Psi}(b) & \xrightarrow{\mathbf{\Psi}(g)} & \mathbf{\Psi}(c) \\ \Big\downarrow{\scriptstyle \mathcal{S}_1(a)} & & {\scriptstyle \mathcal{S}_2(c)}\Big\downarrow \\ \mathbf{\Psi}'(b) & \xrightarrow[\mathbf{\Psi}'(g)]{} & \mathbf{\Psi}'(c) \end{array} \qquad [\text{putting } g_2 \circ g_1 = g]. \tag{4.35}$$

So we have a well defined natural transformation $(\mathcal{S} : \mathbf{\Psi} \Longrightarrow \mathbf{\Psi}') \in \mathcal{N}(\mathbf{U}, \mathbf{C})$ satisfying:

$$\mathcal{S}|_{\mathbf{U}_\alpha} = \mathcal{S}_\alpha, \qquad \alpha \in \{1, 2\}.$$

In conclusion, we have proven that, on $\mathbf{U} = \mathbf{U}_1 \cup \mathbf{U}_2, \mathcal{R}$ satisfies [**Locality**] and [**Gluing**] conditions. □

Now we turn to the proof of Theorem 4.2. The proof would be rather long, and we proceed by one step at a time. The verification of [**Locality**] is not very difficult. The main hardship is to ensure that the [**Gluing**] conditions are satisfied. Let us first explain what we are trying to achieve here, and give a brief description of the strategy of our proof.

Suppose $\{\mathbf{U}_\alpha\}_{\alpha \in I}$ is a categorical cover of an open subcategory $\mathbf{U} \subset \mathbf{B}$. Let $f \in \mathrm{Mor}(\mathbf{U}) = \mathrm{Mor}(\cup_{\alpha \in I} \mathbf{U}_\alpha)$ be an arbitrary morphism in $\mathrm{Mor}(\mathbf{U})$. That means, there exists $J := \{j_1, \cdots, j_m\}$

$\subset I$ such that

$$f = f_{j_m} \circ \cdots \circ f_{j_1}, \tag{4.36}$$

where $f_{j_r} \in \mathrm{Mor}(\mathbf{U}_{j_r}), r \in \{1, \cdots, m\}$. Note that $\mathbf{U}_{j_r} \cap \mathbf{U}_{j_{r-1}}, r \in \{2, \cdots, m\}$, is always non empty, because

$$s(f_{j_r}) = t(f_{j_{r-1}}).$$

If we are given a $\mathbf{\Psi}_\alpha \in \mathrm{Obj}(\mathbf{C}^{\mathbf{U}_\alpha}) = \mathrm{Fun}(\mathbf{U}_\alpha, \mathbf{C})$ for each $\alpha \in I$, satisfying

$$\mathbf{\Psi}_\alpha|_{\mathbf{U}_{\alpha\beta}} = \mathbf{\Psi}_\beta|_{\mathbf{U}_{\alpha\beta}} \qquad \text{for any non-empty } \mathbf{U}_{\alpha\beta},$$

we intend to find a $\mathbf{\Psi} \in \mathrm{Obj}(\mathbf{C}^{\mathbf{U}}) = \mathrm{Fun}(\mathbf{U}, \mathbf{C})$, such that

$$\mathbf{\Psi}|_{\mathbf{U}_\alpha} = \mathbf{\Psi}_\alpha.$$

We define $\mathbf{\Psi}$ as follows

$$\mathbf{\Psi}(f) := \mathbf{\Psi}_{j_m}(f_{j_m}) \circ \cdots \circ \mathbf{\Psi}_{j_1}(f_{j_1}). \tag{4.37}$$

[Here and afterwards we will focus on the gluing (respectively locality) condition only for morphisms. For the objects it is obvious due to the first part of (3.3)].

Now, suppose for some other $J' := \{j'_1, \cdots, j'_n\} \subset I$, we have

$$f = f'_{j'_n} \circ \cdots \circ f'_{j'_1}, \tag{4.38}$$

where $f'_{j'_k} \in \mathrm{Mor}(\mathbf{U}_{j'_k}), k \in \{1, \cdots, n\}$. Then $\mathbf{\Psi}$ to be well defined, we should have

$$\mathbf{\Psi}_{j_m}(f_{j_m}) \circ \cdots \circ \mathbf{\Psi}_{j_1}(f_{j_1}) = \mathbf{\Psi}_{j'_n}(f'_{j'_n}) \circ \cdots \circ \mathbf{\Psi}_{j'_1}(f'_{j'_1}). \tag{4.39}$$

Our primary concern is to prove (4.39). Rest of the proof is straightforward.

Strategy of the proof : The proof of (4.39) will be carried out by the method of induction. First we will prove that, if $J = \{j_1\}$ and $J' = \{j'_2, j'_1\}$, that is, $f \in \mathrm{Mor}(\mathbf{U}_{j_1})$ and $f = f'_{j'_2} \circ f'_{j'_1}$, then

$$\mathbf{\Psi}_{j_1}(f) = \mathbf{\Psi}_{j'_2}(f'_{j'_2}) \circ \mathbf{\Psi}_{j'_1}(f'_{j'_1}).$$

Next, we make the induction assumption:

[Ind assum 1]: "If $J = \{j_1\}$ and $J' = \{j'_1, \cdots, j'_k\}$, that is, $f \in \mathrm{Mor}(\mathbf{U}_{j_1})$ and $f = f'_{j'_k} \circ \cdots \circ f'_{j'_1}$, then

$$\mathbf{\Psi}_{j_1}(f) = \mathbf{\Psi}_{j'_k}(f'_{j'_k}) \circ \cdots \circ \mathbf{\Psi}_{j'_1}(f'_{j'_1})."$$

Then we show that it is true for $J = \{j_1\}$ and $J' = \{j'_1, \cdots, j'_{k+1}\}$. Thus, it holds for $J = \{j_1\}$ and $J' = \{j'_1, \cdots, j'_n\}$. We proceed with a second induction assumption.

[Ind assum 2]: "If $J = \{j_1, \cdots, j_k\}$ and $J' = \{j'_1, \cdots, j'_n\}$, that is, $f_{j_l} \in \mathrm{Mor}(\mathbf{U}_{j_l})$ and $f_{j_k} \circ \cdots \circ f_{j_1} = f'_{j'_n} \circ \cdots \circ f'_{j'_1}$, then

$$\mathbf{\Psi}_{j_k}(f_{j_k}) \circ \cdots \circ \mathbf{\Psi}_{j_1}(f_{j_1}) = \mathbf{\Psi}_{j'_n}(f'_{j'_n}) \circ \cdots \circ \mathbf{\Psi}_{j'_1}(f'_{j'_1})."$$

Then we show that it holds for $J = \{j_1, \cdots, j_{k+1}\}$ and $J' = \{j'_1, \cdots, j'_n\}$. Thus, we can conclude that it holds for $J = \{j_1, \cdots, j_m\}$ and $J' = \{j'_1, \cdots, j'_n\}$.

And, that would complete the proof of (4.39).

Proof of (4.39) : We implement our strategy described above.

Proposition 4.2 — With notations and conventions as above, let $\mathbf{U}$ be a subcategory of $\mathbf{B}$, with an open categorical cover $\{\mathbf{U}_\alpha\}_{\alpha \in I}$. Let $j_1, j'_1, j'_2 \in I$, and

$$f = f'_{j'_2} \circ f'_{j'_1} \in \mathrm{Mor}(\mathbf{U}_{j_1}) \cap \mathrm{Mor}(\mathbf{U}_{j'_1} \cup \mathbf{U}_{j'_2}),$$

where $f \in \mathrm{Mor}(\mathbf{U}_{j_1}), f'_{j'_2} \in \mathrm{Mor}(\mathbf{U}_{j'_2}), f'_{j'_1} \in \mathrm{Mor}(\mathbf{U}_{j'_1})$. Then,

$$\mathbf{\Psi}_{j_1}(f) = \mathbf{\Psi}_{j'_2}(f'_{j'_2}) \circ \mathbf{\Psi}_{j'_1}(f'_{j'_1}). \tag{4.40}$$

Proof : First we observe that, both $\mathbf{U}_{j_1} \cap \mathbf{U}_{j'_1}$ and $\mathbf{U}_{j_1} \cap \mathbf{U}_{j'_2}$ are non empty [$\because s(f) = s(f'_{j'_1}), t(f) = t(f'_{j'_2})$, and $s(f), t(f) \in \mathrm{Obj}(\mathbf{U}_{j_1})$]. Now, using (iii) of Proposition 3.1, we write

$$f \in \mathrm{Mor}\Big(\mathbf{U}_{j_1} \cap \big(\mathbf{U}_{j'_1} \cup \mathbf{U}_{j'_2}\big)\Big) = \mathrm{Mor}\Big(\big(\mathbf{U}_{j_1} \cap \mathbf{U}_{j'_1}\big) \cup \big(\mathbf{U}_{j_1} \cap \mathbf{U}_{j'_2}\big)\Big).$$

That means, there exists $f_2 \in \mathrm{Mor}\big(\mathbf{U}_{j_1} \cap \mathbf{U}_{j'_2}\big), f_1 \in \mathrm{Mor}\big(\mathbf{U}_{j_1} \cap \mathbf{U}_{j'_1}\big)$, such that

$$f = f_2 \circ f_1.$$

By (3.7)-(3.8), we have an isomorphism $g_0 \in \mathrm{Mor}(\mathbf{U}_{j'_1} \cap \mathbf{U}_{j'_2})$, such that

$$f'_{j_2} \circ g_0 = f_2,$$
$$g_0{}^{-1} \circ f'_{j_1} = f_1.$$

So,

$$\begin{aligned}\boldsymbol{\Psi}_{j_1}(f) &= \boldsymbol{\Psi}_{j_1}(f_2 \circ f_1) \\ &= \boldsymbol{\Psi}_{j_1}(f_2) \circ \boldsymbol{\Psi}_{j_1}(f_1) [\because \boldsymbol{\Psi}_{j_1} \text{ is a functor, and } f_2, f_1 \in \mathrm{Mor}(\mathbf{U}_{j_1})] \\ &= \boldsymbol{\Psi}_{j_2'}(f_2) \circ \boldsymbol{\Psi}_{j_1'}(f_1) \\ &[\because \boldsymbol{\Psi}_{j_1}|_{\mathbf{U}_{j_1 j_i'}} = \boldsymbol{\Psi}_{j_i'}|_{\mathbf{U}_{j_1 j_i'}}, i \in \{1,2\}, \text{ and } f_2 \in \mathrm{Mor}(\mathbf{U}_{j_2'}), f_1 \in \mathrm{Mor}(\mathbf{U}_{j_1'})] \\ &= \boldsymbol{\Psi}_{j_2'}(f'_{j_2} \circ g_0) \circ \boldsymbol{\Psi}_{j_1'}({g_0}^{-1} \circ f'_{j_1}) \\ &= \boldsymbol{\Psi}_{j_2'}(f'_{j_2'}) \circ \boldsymbol{\Psi}_{j_1'}(f'_{j_1'}) [\because \boldsymbol{\Psi}_{j_1'}|_{\mathbf{U}_{j_1' j_2'}} = \boldsymbol{\Psi}_{j_2'}|_{\mathbf{U}_{j_1' j_2'}}]\end{aligned} \tag{4.41}$$

This proves the proposition. □

We assume [**Ind assum 1**].

Proposition 4.3 — Suppose [**Ind assum 1**] is true. Let $j_1 \in I, \widetilde{J}' = \{j_1', \cdots, j_{k+1}'\} \subset I$, and

$$f = f'_{j'_{k+1}} \circ \cdots \circ f'_{j'_1} \in \mathrm{Mor}(\mathbf{U}_{j_1}) \bigcap \mathrm{Mor}\Big(\bigcup_{j_i' \in \widetilde{J}'} \mathbf{U}_{j_1'}\Big),$$

where $f \in \mathrm{Mor}(\mathbf{U}_{j_1}), f'_{j_i'} \in \mathrm{Mor}(\mathbf{U}_{j_i'}), i \in \{1, \cdots, k+1\}$. Then,

$$\boldsymbol{\Psi}_{j_1}(f) = \boldsymbol{\Psi}_{j'_{k+1}}(f'_{j'_{k+1}}) \circ \cdots \circ \boldsymbol{\Psi}_{j_1'}(f'_{j_1'}). \tag{4.42}$$

Proof : In spirit, the proof is similar to that of Proposition 4.2. Let us denote

$$\mathbf{U}_{j_1'} \cup \cdots \cup \mathbf{U}_{j_k'} \overset{\text{notation}}{=} \mathbf{U}(j_1', \cdots, j_k').$$

Again $\mathbf{U}_{j_1} \cap \mathbf{U}(j_1' \cdots j_k')$ and $\mathbf{U}_{j_1} \cap \mathbf{U}_{j'_{k+1}}$ are non empty [$\because s(f) = s(f'_{j_1'}), t(f) = t(f'_{j'_{k+1}})$, and $s(f), t(f) \in \mathrm{Obj}(\mathbf{U}_{j_1})$]. Then we have,

$$f \in \mathbf{U}_{j_1} \bigcap \Big(\mathbf{U}(j_1' \cdots j_k') \cup \mathbf{U}_{j'_{k+1}}\Big) = \Big(\mathbf{U}_{j_1} \cap \mathbf{U}(j_1', \cdots, j_k')\Big) \bigcup \Big(\mathbf{U}_{j_1} \cap \mathbf{U}_{j'_{k+1}}\Big).$$

Therefore there exists $g_1 \in \mathrm{Mor}\big(\mathbf{U}_{j_1} \cap \mathbf{U}(j_1', \cdots, j_k')\big)$, and $g_2 \in \mathrm{Mor}\big(\mathbf{U}_{j_1} \cap \mathbf{U}_{j'_{k+1}}\big)$, such that,

$$f = g_2 \circ g_1.$$

So, we have an isomorphism $g_0 \in \mathrm{Mor}(\mathbf{U}_{j_k'}) \cap \mathrm{Mor}(\mathbf{U}_{j'_{k+1}})$ satisfying,

$$\begin{aligned} g_1 &= g_0^{-1} \circ f'_{j_k'} \circ \cdots \circ f'_{j_1'}, \\ g_2 &= f'_{j'_{k+1}} \circ g_0. \end{aligned} \tag{4.43}$$

Since $g_1 \in \mathrm{Mor}\big(\mathbf{U}_{j_1} \cap \mathbf{U}(j'_1, \cdots, j'_k)\big)$, we can apply **[Ind assum 1]** to obtain,

$$\mathbf{\Psi}_{j_1}(g_1) = \mathbf{\Psi}_{j'_k}(g_0^{-1} \circ f'_{j'_k}) \circ \cdots \circ \mathbf{\Psi}_{j'_1}(f'_{j'_1}). \tag{4.44}$$

Similarly for g_2, we have

$$\mathbf{\Psi}_{j_1}(g_2) = \mathbf{\Psi}_{j'_{k+1}}(f'_{j'_{k+1}} \circ g_0). \tag{4.45}$$

Composing (4.44) and (4.45) we arrive at our desired result,

$$\begin{aligned}\mathbf{\Psi}_{j_1}(f) &= \mathbf{\Psi}_{j_1}(g_2) \circ \mathbf{\Psi}_{j_1}(g_1)\\ &\quad [\because \mathbf{\Psi}_{j_1} \text{ is a functor, and } g_2, g_1 \in \mathrm{Mor}(\mathbf{U}_{j_1})]\\ &= \mathbf{\Psi}_{j'_{k+1}}(f'_{j'_{k+1}} \circ g_0) \circ \mathbf{\Psi}_{j'_k}(g_0^{-1} \circ f'_{j'_k}) \circ \cdots \circ \mathbf{\Psi}_{j'_1}(f'_{j'_1})\\ &= \mathbf{\Psi}_{j'_{k+1}}(f'_{j'_{k+1}}) \circ \mathbf{\Psi}_{j'_k}(f'_{j'_k}) \circ \cdots \circ \mathbf{\Psi}_{j'_1}(f'_{j'_1})\\ &[\because \mathbf{\Psi}_{j'_{k+1}}|_{\mathbf{U}_{j'_{k+1}j'_k}} = \mathbf{\Psi}_{j'_k}|_{\mathbf{U}_{j'_{k+1}j'_k}}].\end{aligned} \tag{4.46}$$

Hence proved. □

That means, if for $J' = \{j'_1, \cdots, j'_n\} \subset I$, $f \in \mathrm{Mor}(\mathbf{U}_{j_1})$ satisfies

$$f = f'_{j'_n} \circ \cdots \circ f'_{j'_1},$$

where $f'_{j'_k} \in \mathrm{Mor}(\mathbf{U}_{j'_k}), k \in \{1, \cdots, n\}$, then,

$$\mathbf{\Psi}_{j_1}(f) = \mathbf{\Psi}_{j'_n}(f'_{j'_n}) \circ \cdots \circ \mathbf{\Psi}_{j'_1}(f'_{j'_1}). \tag{4.47}$$

Next we come to the second induction assumption **[Ind assum 2]**.

Proposition 4.4 — Suppose **[Ind assum 2]** is true. Let $\{j_1, \cdots, j_{k+1}\} \subset I, J' = \{j'_1, \cdots, j'_n\} \subset I$, and

$$f_{j_{k+1}} \circ \cdots \circ f_{j_1} = f'_{j'_n} \circ \cdots \circ f'_{j'_1} \in \mathrm{Mor}\Big(\bigcup_{j_l \in J} \mathbf{U}_{j_l}\Big) \bigcap \mathrm{Mor}\Big(\bigcup_{j'_i \in J'} \mathbf{U}_{j'_i}\Big),$$

where $f_{j_l} \in \mathrm{Mor}(\mathbf{U}_{j_l}), l \in \{1, \cdots, k+1\}, f'_{j'_i} \in \mathrm{Mor}(\mathbf{U}_{j'_i}), i \in \{1, \cdots, n\}$. Then,

$$\mathbf{\Psi}_{j_{k+1}}(f_{j_{k+1}}) \circ \cdots \circ \mathbf{\Psi}_{j_1}(f_{j_1}) = \mathbf{\Psi}_{j'_n}(f'_{j'_n}) \circ \cdots \circ \mathbf{\Psi}_{j'_1}(f'_{j'_1}). \tag{4.48}$$

PROOF : We follow our previous notation,

$$\mathbf{U}_{j_1} \cup \cdots \cup \mathbf{U}_{j_k} \overset{\text{notation}}{=} \mathbf{U}(j_1, \cdots, j_k),$$

and

$$\mathbf{U}_{j'_1} \cup \cdots \cup \mathbf{U}_{j'_n} \overset{\text{notation}}{=} \mathbf{U}(j'_1, \cdots, j'_n).$$

Again $\mathbf{U}(j_1, \cdots, j_k) \cap \mathbf{U}(j'_1, \cdots, j'_n)$ and $\mathbf{U}_{j_{k+1}} \cap \mathbf{U}_{j'_n}$ are non empty $[\because s(f_{j_1}) = s(f'_{j'_1})$, $t(f_{j_{k+1}}) = t(f'_{j'_n})$, and $s(f) \in \mathrm{Obj}(\mathbf{U}_{j_1}), t(f) \in \mathrm{Obj}(\mathbf{U}_{j_{k+1}})]$. Then we have,

$$\begin{aligned} & f_{j_{k+1}} \circ \cdots \circ f_{j_1} = f'_{j'_n} \circ \cdots \circ f'_{j'_1} \\ & \in \Big(\mathbf{U}(j_1, \cdots, j_k) \cup \mathbf{U}_{j_{k+1}}\Big) \bigcap \Big(\mathbf{U}(j'_1, \cdots, j'_n)\Big) = \\ & \Big(\mathbf{U}_{j_{k+1}} \cap \mathbf{U}(j'_1, \cdots, j'_n)\Big) \bigcup \Big(\mathbf{U}(j_1, \cdots, j_k) \cap \mathbf{U}(j'_1, \cdots, j'_n\Big). \end{aligned} \tag{4.49}$$

Therefore there exists $g_1 \in \mathrm{Mor}\big(\mathbf{U}(j_1 \cdots j_k) \cap \mathbf{U}(j'_1 \cdots j'_n)\big)$, and $g_2 \in \mathrm{Mor}\big(\mathbf{U}_{j_{k+1}} \cap \mathbf{U}(j'_1, \cdots, j'_n)\big)$, such that,

$$f'_{j'_n} \circ \cdots \circ f'_{j'_1} = f_{j_{k+1}} \circ \cdots \circ f_{j_1} = g_2 \circ g_1.$$

So, we have a pair of isomorphisms $g_0 \in \mathrm{Mor}(\mathbf{U}_{j_k}) \cap \mathrm{Mor}(\mathbf{U}_{j_{k+1}}), g'_0 \in \mathrm{Mor}(\mathbf{U}_{j'_{n-1}}) \cap \mathrm{Mor}(\mathbf{U}_{j'_n})$ satisfying,

$$\begin{aligned} g_1 &= g_0^{-1} \circ f_{j_k} \circ \cdots \circ f_{j_1}, \\ g_2 &= f_{j_{k+1}} \circ g_0, \end{aligned}$$

and,

$$\begin{aligned} g_1 &= {g'_0}^{-1} \circ f_{j'_{n-1}} \circ \cdots \circ f'_{j'_1}, \\ g_2 &= f_{j'_n} \circ g'_0. \end{aligned}$$

Now applying [**Ind assum 2**], and using same argument as previous proposition, we deduce ,

$$\mathbf{\Psi}_{j_{k+1}}(f_{j_{k+1}}) \circ \cdots \circ \mathbf{\Psi}_{j_1}(f_{j_1}) = \mathbf{\Psi}_{j'_n}(f'_{j'_n}) \circ \cdots \circ \mathbf{\Psi}_{j'_1}(f'_{j'_1}).$$

Hence proved. □

Proof : By induction, we conclude, if $f_{j_l} \in \mathrm{Mor}(\mathbf{U}_{j_l}), f'_{j'_i} \in \mathrm{Mor}(\mathbf{U}_{j'_i})$ and for $J = \{j_1, \cdots, j_m\}, J' = \{j'_1, \cdots, j'_n\} \subset I$, we have

$$f_{j_m} \circ \cdots \circ f_{j_1} = f'_{j'_n} \circ \cdots \circ f'_{j'_1},$$

then,

$$\mathbf{\Psi}_{j_m}(f_{j_m}) \circ \cdots \circ \mathbf{\Psi}_{j_1}(f_{j_1}) = \mathbf{\Psi}_{j'_n}(f'_{j'_n}) \circ \cdots \circ \mathbf{\Psi}_{j'_1}(f'_{j'_1}). \tag{4.50}$$

So, (4.39) is proven.

Proof of Theorem 4.2 : We conclude this subsection by presenting the proof of Theorem 4.2. We have already derived all the required results. We only have to collect and organize them. We do not reiterate the notation. They should be assumed to be as per with the Theorem 4.2 and subsequent part of this section.

Verification of [**Locality**] condition is virtually identical to the given in the proof of Proposition 4.1. For the sake of completeness, we only state the result here.

If, $\mathbf{\Psi}_1, \mathbf{\Psi}_2 \in \mathrm{Obj}(\mathbf{C}^{\mathbf{U}}) = \mathrm{Fun}(\mathbf{U}, \mathbf{C})$, such that,

$$\begin{aligned} \mathbf{\Psi}_1|_{\mathbf{U}_\alpha} &= \mathbf{\Psi}_2|_{\mathbf{U}_\alpha}, \qquad \forall \alpha \in I, \\ \text{then } \mathbf{\Psi}_1 &= \mathbf{\Psi}_2. \end{aligned} \tag{4.51}$$

Similarly, if $\big(\mathcal{S}_1 : \mathbf{\Psi}_1 \Longrightarrow \mathbf{\Psi}_2\big), \big(\mathcal{S}_2 : \mathbf{\Psi}_1 \Longrightarrow \mathbf{\Psi}_2\big) \in \mathrm{Mor}(\mathbf{C}^{\mathbf{U}}) = \mathcal{N}(\mathbf{U}, \mathbf{C})$, such that,

$$\begin{aligned} \mathcal{S}_1|_{\mathbf{U}_\alpha} &= \mathcal{S}_2|_{\mathbf{U}_\alpha}, \qquad \forall \alpha \in I, \\ \text{then } \mathcal{S}_1 &= \mathcal{S}_2. \end{aligned} \tag{4.52}$$

Let us verify the [**Gluing**] condition.

$\mathbf{\Psi}_\alpha \in \mathrm{Obj}(\mathbf{C}^{\mathbf{U}_\alpha}) = \mathrm{Fun}(\mathbf{U}_\alpha, \mathbf{C})$ given for each $\alpha \in I$, such that: $\mathbf{\Psi}_\alpha|_{\mathbf{U}_{\alpha\beta}} = \mathbf{\Psi}_\beta|_{\mathbf{U}_{\alpha\beta}}$, for every non-empty $\mathbf{U}_{\alpha\beta}$. Then we define $(\mathbf{\Psi} : \mathbf{U} \longrightarrow \mathbf{C}) \in \mathrm{Fun}(\mathbf{U}, \mathbf{C}) = \mathrm{Obj}(\mathbf{C}^{\mathbf{U}})$ as follows:

$$\begin{aligned} &\mathrm{Obj}(\mathbf{U}) \longrightarrow \mathrm{Obj}(\mathbf{C}) \\ &\mathrm{Obj}\Big(\bigcup_{\alpha \in I} \mathbf{U}_\alpha\Big) \longrightarrow \mathrm{Obj}(\mathbf{C}) \\ &\qquad a \mapsto \mathbf{\Psi}_\alpha(a), \qquad \text{if } a \in \mathbf{U}_\alpha, \alpha \in I, \end{aligned} \tag{4.53}$$

and,

$$\begin{aligned} &\mathrm{Mor}(\mathbf{U}) \longrightarrow \mathrm{Mor}(\mathbf{C}) \\ &\mathbf{Gen}\Big(\bigcup_{\alpha \in I} \mathrm{Mor}(\mathbf{U}_\alpha)\Big) \longrightarrow \mathrm{Mor}(\mathbf{C}), \\ &f_{j_m} \circ \cdots \circ f_{j_1} \mapsto \mathbf{\Psi}_{j_m}(f_{j_m}) \circ \cdots \circ \mathbf{\Psi}_1(f_{j_1}), \\ &\text{where } f_{j_k} \in \mathrm{Mor}(\mathbf{U}_{j_k}), k \in \{1, \cdots, m\}, \\ &\text{and, } J := \{j_1, \cdots j_m\} \subset I. \end{aligned} \tag{4.54}$$

Since $\boldsymbol{\Psi}_\alpha|_{\mathbf{U}_{\alpha\beta}} = \boldsymbol{\Psi}_\beta|_{\mathbf{U}_{\alpha\beta}}$, (4.53) is well defined. Also (4.39) ensures that (4.54) is well defined. Thus we have a $(\boldsymbol{\Psi} : \mathbf{U} \longrightarrow \mathbf{C}) \in \mathrm{Fun}(\mathbf{U}, \mathbf{C})$ satisfying,

$$\boldsymbol{\Psi}_\alpha = \boldsymbol{\Psi}|_\alpha, \alpha \in I.$$

Next, suppose $\left(\mathcal{S}_\alpha : \boldsymbol{\Psi}_\alpha \Longrightarrow \boldsymbol{\Psi}'_\alpha\right) \in \mathrm{Mor}(\mathbf{C}^{\mathbf{U}_\alpha}) = \mathcal{N}(\mathbf{U}_\alpha, \mathbf{C})$ is given for each $\alpha \in I$, such that, $\{\boldsymbol{\Psi}_\alpha\}$ glue to form a $\boldsymbol{\Psi} \in \mathrm{Fun}(\mathbf{U}, \mathbf{C})$, and similarly, $\{\boldsymbol{\Psi}'_\alpha\}$ glue to form a $\boldsymbol{\Psi}' \in \mathrm{Fun}(\mathbf{U}, \mathbf{C})$, [as described in the previous part of the proof] and $\mathcal{S}_\alpha$ satisfy

$$\mathcal{S}_\alpha|_{\mathbf{U}_{\alpha\beta}} = \mathcal{S}_\beta|_{\mathbf{U}_{\alpha\beta}}. \tag{4.55}$$

We define $\mathcal{S} \in \mathcal{N}(\mathbf{U}, \mathbf{C})$ by:

$$\mathcal{S}(a) = \mathcal{S}_\alpha(a), \quad \text{for } a \in \mathrm{Obj}(\mathbf{U}_\alpha), \alpha \in I. \tag{4.56}$$

Above equation is well defined because of (4.55). The proof that (4.56) defines a natural transformation from $\boldsymbol{\Psi}$ to $\boldsymbol{\Psi}'$, is exactly same as the counter part in Proposition 4.1.

And, that completes the proof of Theorem 4.2. □

5. Sheaves of Categorical Groups

Objective of this section is to define the (pre)sheaves of categorical groups, and construct an example for the same. In section 2 we made some passing remarks about categorical group valued presheaves. Here we will take a more formal approach. Before that let us briefly review categorical groups, and associated notions.

5.1 *Categorical groups*

There are many equivalent definitions of a categorical group available in literature [19, 22, 25, 27]. For our purpose we will mostly think of a categorical group in terms of a *crossed-module*. A *categorical group* $\mathcal{G}$ is given by a category $\mathcal{G}$ along with a functor

$$\mathcal{G} \times \mathcal{G} \longrightarrow \mathcal{G} \tag{5.1}$$

which makes both $\mathrm{Obj}(\mathcal{G})$ and $\mathrm{Mor}(\mathcal{G})$ groups. Some of the immediate consequences are as follows:

(i) the identity-assigning map

$$\mathrm{Obj}(\mathcal{G}) \longrightarrow \mathrm{Mor}(\mathcal{G}) : x \mapsto 1_x$$

is a homomorphism, and so, 1_e is the identity element in $\mathrm{Mor}(\mathcal{G})$, where e is the identity element in $\mathrm{Obj}(\mathcal{G})$,

(ii) the source and target maps

$$s, t : \mathrm{Mor}(\mathcal{G}) \longrightarrow \mathrm{Obj}(\mathcal{G}) \tag{5.2}$$

are both homomorphisms;

(iii) functoriality of (5.1) implies the following exchange law,

$$(\phi_2\psi_2) \circ (\phi_1\psi_1) = (\phi_2 \circ \phi_1)(\psi_2 \circ \psi_1), \tag{5.3}$$

whenever right hand side is well defined for $\phi_1, \phi_2, \psi_1, \psi_2 \in \mathrm{Mor}(\mathcal{G})$. Here and onwards $\circ$ will denote the composition of morphisms (as usual) and juxtaposition of two elements denote group product.

Specializing to the case, where both $\mathrm{Obj}(\mathcal{G})$ and $\mathrm{Mor}(\mathcal{G})$ are Lie groups and the maps s, t and $x \mapsto 1_x$ are smooth, we have a *categorical Lie group* $\mathcal{G}$.

A *crossed-module* is given by a pair of groups G and H, along with maps

$$\alpha : G \times H \longrightarrow H : (g, h) \mapsto \alpha_g(h) \quad \text{and} \quad \tau : H \longrightarrow G,$$

where τ is a homomorphism, α_g is an automorphism of H for each $g \in G$, and the map $g \mapsto \alpha_g \in \mathrm{Aut}(H)$ is a homomorphism. The map τ and the map α interrelated via following identities

$$\begin{aligned} &\tau(\alpha_g(h)) = g\tau(h)g^{-1}, \\ &\alpha_{\tau(h)}(h') = hh'h^{-1} \qquad \text{for all } g \in G, h, h' \in H. \end{aligned} \tag{5.4}$$

We write a crossed module as (G, H, α, τ). When G and H are Lie groups, and α and τ are smooth, (G, H, α, τ) is called a *Lie crossed module*. For us, most useful property of a (Lie) crossed module is the following.

It is well known that there is a one-to-one correspondence between categorical (Lie) groups and (Lie) crossed modules [5, 6, 18]. The bijection is given as follows.

Let $\mathcal{G}$ be a categorical group. We take $G := \mathrm{Obj}(\mathcal{G})$, $H := \ker s$, $\tau = t|_H$, and

$$\alpha_g(h) = 1_g h 1_g^{-1}$$

for all $g \in G$ and $h \in H$. Then we have a group isomorphism $\mathrm{Mor}(\mathcal{G}) \xrightarrow{\simeq} H \rtimes_\alpha G$ defined by the map

$$\mathrm{Mor}(\mathcal{G}) \longrightarrow H \rtimes_\alpha G$$
$$: \phi \mapsto \big(\phi 1_{s(\phi)^{-1}}, s(\phi)\big).$$

The target map t, viewed as a mapping $H \rtimes_\alpha G \longrightarrow G$, is given by

$$t(h,g) = \tau(h)g \qquad \text{for all } (h,g) \in H \rtimes_\alpha G. \tag{5.5}$$

We note here that the group product in $H \rtimes_\alpha G$ is given by the usual group multiplication law for a semi direct product

$$(h_2, g_2)(h_1, g_1) = \big(h_2 \alpha_{g_2}(h_1), g_2 g_1\big) \tag{5.6}$$

for all $(h_2, g_2), (h_1, g_1) \in H \rtimes_\alpha G$. It also easily follows that the composition of morphisms in $\mathrm{Mor}(\mathcal{G}) \simeq H \rtimes_\alpha G$ is given by

$$(h_2, g_2) \circ (h_1, g_1) = (h_2 h_1, g_1); \tag{5.7}$$

of course this composition is defined only when

$$\tau(h_1)g_1 = t(h_1, g_1) = s(h_2, g_2) = g_2.$$

A *morphism between a pair of categorical groups* $\mathcal{G}$ and $\mathcal{H}$ is a functor [5, 6].

$$\lambda : \mathcal{G} \longrightarrow \mathcal{H}, \tag{5.8}$$

such that, both $\lambda : \mathrm{Obj}(\mathcal{G}) \longrightarrow \mathrm{Obj}(\mathcal{H})$ and $\lambda : \mathrm{Mor}(\mathcal{G}) \longrightarrow \mathrm{Mor}(\mathcal{H})$ are homomorphisms of groups.

The category of all categorical groups; that is, the category whose objects are categorical groups and morphisms are morphisms between categorical groups (defined in (5.8)), will be denoted as

CatGrp.

Clearly there is a full, faithful, forgetful functor

$$\mathbf{CatGrp} \longrightarrow \mathbf{Cat},$$

and **CatGrp** is a full subcategory of **Cat**. Our next goal is to consider the **CatGrp** $\subset$ **Cat** valued presheaf over a category of a set of (small) categories, $\mathcal{C}$. We define a presheaf of categorical groups or a **CatGrp**-valued presheaf, over $\mathcal{C}$, to be a contravariant functor from $\mathcal{C}$ to **CatGrp** :

$$\rho : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{CatGrp}. \tag{5.9}$$

We shall denote the *category of presheaves of categorical groups* by

$$\mathbf{Prsh}(\mathcal{C}, \mathbf{CatGrp}). \tag{5.10}$$

Let $\mathbf{B}$ be a topological category and $\widetilde{\mathcal{O}}(\mathbf{B})$ be as defined in (4.1). A presheaf of categorical groups over $\widetilde{\mathcal{O}}(\mathbf{B})$ is a *sheaf of categorical groups* over $\widetilde{\mathcal{O}}(\mathbf{B})$ if it satisfies [**Locality**] and [**Gluing**] conditions listed between (4.6)-(4.13).

5.2 Functor category $\mathcal{G}^{\mathbf{U}}$

Let $\mathbf{U}$ be a category, and $\mathbf{\Phi}_1, \mathbf{\Phi}_2 : \mathbf{U} \longrightarrow \mathcal{G}$ be a pair of functors from $\mathbf{U}$ to a categorical group $\mathcal{G}$. Then the pointwise product $\mathbf{\Phi}_2\mathbf{\Phi}_1 : \mathbf{U} \longrightarrow \mathcal{G}$ is also a functor. In other words, we have following result [[17], Proposition 3.2].

Lemma 5.1 — The set of all functors from $\mathbf{U}$ to $\mathcal{G}$ form a group.

PROOF : $\mathbf{\Phi}_2\mathbf{\Phi}_1$ is defined as follows.

$$\big(\mathbf{\Phi}_2\mathbf{\Phi}_1\big)(x) = \mathbf{\Phi}_2(x)\mathbf{\Phi}_1(x), \tag{5.11}$$

where x is in $\mathrm{Obj}(\mathbf{U})$ or $\mathrm{Mor}(\mathbf{U})$. Thus, if $f_2, f_1 \in \mathrm{Mor}(\mathbf{U})$ and they are composable, then

$$\begin{aligned}
(\mathbf{\Phi}_2\mathbf{\Phi}_1)(f_2 \circ f_1) \quad &= \mathbf{\Phi}_2(f_2 \circ f_1)\mathbf{\Phi}_1(f_2 \circ f_1) \\
&= \big(\mathbf{\Phi}_2(f_2) \circ \mathbf{\Phi}_2(f_1)\big)\big(\mathbf{\Phi}_1(f_2) \circ \mathbf{\Phi}_2(f_1)\big) \\
&= \big(\mathbf{\Phi}_2(f_2)\mathbf{\Phi}_1(f_2)\big) \circ \big(\mathbf{\Phi}_2(f_1)\mathbf{\Phi}_1(f_1)\big)[\text{using (5.3)}] \\
&= \big(\mathbf{\Phi}_2\mathbf{\Phi}_1\big)(f_2) \circ \big(\mathbf{\Phi}_2\mathbf{\Phi}_1\big)(f_1).
\end{aligned}$$

So $\mathbf{\Phi}_2\mathbf{\Phi}_1$ is a functor. It is obvious that the constant functor

$$\begin{aligned}
\mathbf{\Phi}_0 : \quad & \mathbf{U} \longrightarrow \mathcal{G} \\
& a \mapsto e, \\
& f \mapsto 1_e
\end{aligned}$$

defines the multiplicative identity, where $a \in \mathrm{Obj}(\mathbf{U})$ and $f \in \mathrm{Mor}(\mathbf{U})$, and $e, 1_e$ respectively denote the group identity elements in $\mathrm{Obj}(\mathbf{U})$ and $\mathrm{Mor}(\mathbf{U})$. Group inverse $\mathbf{\Phi}^{\mathrm{inv}}$ is given by $\mathbf{\Phi}^{\mathrm{inv}}(x) = \mathbf{\Phi}(x)^{-1}$, where $x \in \mathrm{Obj}(\mathcal{G})$ or $\mathrm{Mor}(\mathcal{G})$.

Now suppose $\mathcal{T} : \mathbf{\Phi}_1 \Longrightarrow \mathbf{\Phi}_2$ and $\mathcal{T}' : \mathbf{\Phi}'_1 \Longrightarrow \mathbf{\Phi}'_2$ are a pair of natural transformations between respective functors in $\{\mathbf{U} \longrightarrow \mathcal{G}\}$; that is, the diagrams in (5.12) commute, where $a \xrightarrow{f} b$ is a morphism in category $\mathbf{U}$.

$$\begin{array}{ccc} \mathbf{\Phi}_1(a) & \xrightarrow{\mathbf{\Phi}_1(f)} & \mathbf{\Phi}_1(b) \\ \downarrow{\scriptstyle \mathcal{T}(a)} & & \downarrow{\scriptstyle \mathcal{T}(b)} \\ \mathbf{\Phi}_2(a) & \xrightarrow[\mathbf{\Phi}_2(f)]{} & \mathbf{\Phi}_2(b) \end{array} \qquad \begin{array}{ccc} \mathbf{\Phi}'_1(a) & \xrightarrow{\mathbf{\Phi}'_1(f)} & \mathbf{\Phi}'_1(b) \\ \downarrow{\scriptstyle \mathcal{T}'(a)} & & \downarrow{\scriptstyle \mathcal{T}'(b)} \\ \mathbf{\Phi}'_2(a) & \xrightarrow[\mathbf{\Phi}'_2(f)]{} & \mathbf{\Phi}'_2(b) \end{array} \tag{5.12}$$

We can define a product natural transformation, given by

$$\mathcal{T}'\mathcal{T}(a) := \mathcal{T}'(a)\mathcal{T}(a), \tag{5.13}$$

for any $a \in \mathrm{Obj}(\mathbf{U})$. Functoriality of the group product $\mathcal{G} \times \mathcal{G} \longrightarrow \mathcal{G}$ ensures that $\mathcal{T}'\mathcal{T}$ is a well defined natural transformation; that is, following diagram commutes:

$$\begin{array}{ccc} \mathbf{\Phi}_1(a)\mathbf{\Phi}'_1(a) & \xrightarrow{\mathbf{\Phi}_1(f)\mathbf{\Phi}'_1(f)} & \mathbf{\Phi}_1(b)\mathbf{\Phi}'_1(b) \\ \downarrow{\scriptstyle \mathcal{T}'(a)\mathcal{T}(a)} & & \downarrow{\scriptstyle \mathcal{T}'(b)\mathcal{T}(b)} \\ \mathbf{\Phi}_2(a)\mathbf{\Phi}'_2(a) & \xrightarrow[\mathbf{\Phi}_2(f)\mathbf{\Phi}'_2(f)]{} & \mathbf{\Phi}_2(b)\mathbf{\Phi}'_2(b) \end{array} . \tag{5.14}$$

For any $\mathcal{T}$ we have a corresponding (multiplicative) inverse given by

$$\mathcal{T}^{\mathrm{inv}}(a) := (\mathcal{T}(a))^{-1}.$$

Moreover, if $\mathcal{T}_1 : \mathbf{\Phi}_1 \Longrightarrow \mathbf{\Phi}_2$ and $\mathcal{T}_2 : \mathbf{\Phi}_2 \Longrightarrow \mathbf{\Phi}_3$ are natural transformations, then we have a composite natural transformation $\mathcal{T}_2 \circ \mathcal{T}_1 : \mathbf{\Phi}_1 \Longrightarrow \mathbf{\Phi}_3$,

$$(\mathcal{T}_2 \circ \mathcal{T}_1)(a) = \mathcal{T}_2(a) \circ \mathcal{T}_1(a), \qquad [\text{for all } a \in \mathrm{Obj}(\mathbf{U})].$$

Also, using functoriality of $\mathcal{G} \times \mathcal{G} \longrightarrow \mathcal{G}$, it is easy to show that,

$$(\mathcal{T}'_2\mathcal{T}_2) \circ (\mathcal{T}'_1\mathcal{T}_1) = (\mathcal{T}'_2 \circ \mathcal{T}'_1)(\mathcal{T}_2 \circ \mathcal{T}_1), \tag{5.15}$$

when $\mathcal{T}_2 \circ \mathcal{T}_1$ and $\mathcal{T}'_2 \circ \mathcal{T}'_1$ are defined. The natural transformation $\mathcal{T}_0 : \mathbf{\Phi}_0 \Longrightarrow \mathbf{\Phi}_0$ defined as

$$\mathcal{T}_0(a) = 1_e, \qquad \forall a \in \mathrm{Obj}(\mathbf{U})$$

is the multiplicative identity.

Thus we make following proposition [[17], Proposition 3.4]:

Proposition 5.1 — Let $\mathcal{G}^{\mathbf{U}}$ be the category of functors from $\mathbf{U}$ to $\mathcal{G}$:

$$\begin{aligned} \mathrm{Obj}(\mathcal{G}^{\mathbf{U}}) &:= \mathrm{Fun}(\mathbf{U}, \mathcal{G}) \\ \mathrm{Mor}(\mathcal{G}^{\mathbf{U}}) &:= \mathcal{N}(\mathbf{U}, \mathcal{G}). \end{aligned} \tag{5.16}$$

Then $\mathcal{G}^{\mathbf{U}}$ is a categorical group.

Now suppose, as in Theorem 4.2, $\widetilde{\mathcal{O}}(\mathbf{B})$ is the category of open subcategories of the topological groupoid $\mathbf{B}$, and $\{\mathbf{U}_\alpha\}$ is an open categorical cover of $\mathbf{U}$. Then by Lemma 4.1, for any functor $\left(\mathbf{U} \xrightarrow{\boldsymbol{\Theta}} \mathbf{V}\right) \in \mathrm{Mor}\left(\widetilde{\mathcal{O}}(\mathbf{B})\right)$, we have a functor $\rho(\boldsymbol{\Theta}) : \mathcal{G}^{\mathbf{V}} \longrightarrow \mathcal{G}^{\mathbf{U}}$:

$$\begin{aligned} \rho(\boldsymbol{\Theta}) :&\mathrm{Obj}(\mathcal{G}^{\mathbf{V}}) \longrightarrow \mathrm{Obj}(\mathcal{G}^{\mathbf{U}}), \\ &\boldsymbol{\Phi} \mapsto \boldsymbol{\Phi}\boldsymbol{\Theta}, \\ \rho(\boldsymbol{\Theta}) :&\mathrm{Mor}(\mathcal{G}^{\mathbf{V}}) \longrightarrow \mathrm{Mor}(\mathcal{G}^{\mathbf{U}}). \\ &\mathcal{T} \mapsto \mathcal{T}\boldsymbol{\Theta}. \end{aligned} \tag{5.17}$$

In fact, the functor in (5.17) is a morphism of categorical groups defined in (5.8); that is, the functor defines a pair of group homomorphisms,

$$\begin{aligned} &\mathrm{Obj}\big(\mathcal{G}^{\mathbf{V}}\big) \longrightarrow \mathrm{Obj}\big(\mathcal{G}^{\mathbf{U}}\big), \text{ and} \\ &\mathrm{Mor}\big(\mathcal{G}^{\mathbf{V}}\big) \longrightarrow \mathrm{Mor}\big(\mathcal{G}^{\mathbf{U}}\big). \end{aligned} \tag{5.18}$$

Lemma 5.2 — Let $\mathbf{U}, \mathbf{V} \subset \mathbf{B}$, and let $\mathcal{G}$ be a fixed categorical group. Let $\mathcal{G}^{\mathbf{U}}, \mathcal{G}^{\mathbf{V}}$ be the respective categorical groups of functors defined in Proposition 5.1. Then for any $\left(\mathbf{U} \xrightarrow{\boldsymbol{\Theta}} \mathbf{V}\right) \in \mathrm{Mor}\left(\widetilde{\mathcal{O}}(\mathbf{B})\right)$, the functor $\rho(\boldsymbol{\Theta})$, defined in (5.17), is a morphism from $\mathcal{G}^{\mathbf{V}}$ to $\mathcal{G}^{\mathbf{U}}$.

PROOF : The proof directly follows from the group product defined for objects and morphisms respectively in (5.11) and (5.13). Let $\boldsymbol{\Phi}, \boldsymbol{\Phi}' \in \mathrm{Obj}(\mathcal{G}^{\mathbf{V}})$. Then for any $x \in \mathrm{Obj}(\mathbf{U})$ or $\mathrm{Mor}(\mathbf{U})$, we have

$$
\begin{aligned}
\Big(\rho(\boldsymbol{\Theta})(\boldsymbol{\Phi}\boldsymbol{\Phi}')\Big)(x) &= \big(\boldsymbol{\Phi}\boldsymbol{\Phi}'\boldsymbol{\Theta}\big)(x) \ \text{[using (5.17)]}\\
&= \big(\boldsymbol{\Phi}\boldsymbol{\Phi}'\big)(\boldsymbol{\Theta}(x))\\
&= \boldsymbol{\Phi}(\boldsymbol{\Theta}(x))\boldsymbol{\Phi}'(\boldsymbol{\Theta}(x)) \ \text{[using (5.11)]}\\
&= \Big(\rho(\boldsymbol{\Theta})(\boldsymbol{\Phi})\Big)(x)\Big(\rho(\boldsymbol{\Theta})(\boldsymbol{\Phi}')\Big)(x) \ \text{[using (5.17)]}\\
&= \Big(\rho(\boldsymbol{\Theta})(\boldsymbol{\Phi})\rho(\boldsymbol{\Theta})(\boldsymbol{\Phi}')\Big)(x) \ \text{[using (5.11)]}.
\end{aligned} \tag{5.19}
$$

So,

$$\rho(\boldsymbol{\Theta})(\boldsymbol{\Phi})\rho(\boldsymbol{\Theta})(\boldsymbol{\Phi}') = \rho(\boldsymbol{\Theta})(\boldsymbol{\Phi}\boldsymbol{\Phi}').$$

Similarly, if $\mathcal{T}, \mathcal{T}' \in \mathrm{Mor}(\mathcal{G}^{\mathbf{V}})$, the using (5.13) we can show

$$\rho(\boldsymbol{\Theta})(\mathcal{T})\rho(\boldsymbol{\Theta})(\mathcal{T}') = \rho(\boldsymbol{\Theta})(\mathcal{T}\mathcal{T}').$$

Thus $\rho(\boldsymbol{\Theta})$ defines a pair of group homomorphisms

$$
\begin{aligned}
&\mathrm{Obj}\big(\mathcal{G}^{\mathbf{V}}\big) \longrightarrow \mathrm{Obj}\big(\mathcal{G}^{\mathbf{U}}\big), \text{ and}\\
&\mathrm{Mor}\big(\mathcal{G}^{\mathbf{V}}\big) \longrightarrow \mathrm{Mor}\big(\mathcal{G}^{\mathbf{U}}\big).
\end{aligned}
$$

And, the functor $\rho(\boldsymbol{\Theta})$, as per definition in (5.8), is a morphism between categorical groups. □

In other words

$$
\begin{aligned}
\rho : &\big(\widetilde{\mathcal{O}}(\mathbf{B})\big)^{\mathrm{op}} \longrightarrow \mathbf{CatGrp},\\
&\mathrm{Obj}\Big(\widetilde{\mathcal{O}}(\mathbf{B})\Big) \longrightarrow \mathrm{Obj}(\mathbf{CatGrp}),\\
&\mathbf{U} \mapsto \mathcal{G}^{\mathbf{U}},\\
&\mathrm{Mor}\Big(\widetilde{\mathcal{O}}(\mathbf{B})\Big) \longrightarrow \mathrm{Mor}(\mathbf{CatGrp}),\\
&\Big(\mathbf{U} \xrightarrow{\boldsymbol{\Theta}} \mathbf{V}\Big) \mapsto \Big(\mathcal{G}^{\mathbf{V}} \xrightarrow{\rho(\boldsymbol{\Theta})} \mathcal{G}^{\mathbf{U}}\Big),
\end{aligned} \tag{5.20}
$$

is an element of $\mathbf{Prsh}(\mathcal{C}, \mathbf{CatGrp})$.

Theorem 4.2 implies that **CatGrp**-valued presheaf ρ in (5.20) is actually a **CatGrp**-valued sheaf. Hence we conclude,

Proposition 5.2 — Let $\widetilde{\mathcal{O}}(\mathbf{B})$ be the category of open subcategories for a topological groupoid $\mathbf{B}$. Let ρ be as defined in (5.20). Then ρ is a **CatGrp**-valued sheaf over $\widetilde{\mathcal{O}}(\mathbf{B})$.

6. A Comparison with Stacks

In this section we give a brief comparison between the notion of *stacks* and framework developed in this article. A stack over a category $\mathcal{D}$ (equipped with a Grothendieck topology) is the conventional description of a "sheaf of categories" over $\mathcal{D}$. Before we describe the general program of stacks to distinguish and relate with our framework, let us discuss what has been called *sheaf with Grothendieck topologies* [36]. Let $(\mathcal{D}, \mathcal{J})$ be a *site*; that is a category $\mathcal{D}$ equipped with a Grothendieck topology $\mathcal{J}$. Explicitly this means the following. With each object U of $\mathcal{D}$ a collection of set of morphisms $\mathcal{J}(U) = \{\mathcal{U}\}$ is given, where each $\mathcal{U} = \{U_\alpha \longrightarrow U\}_{\alpha \in I}$ is a collection of arrows called a *covering* of U, such that following conditions are satisfied:

(i) If $V \xrightarrow{\simeq} U$ is an isomorphism, then $\{V \xrightarrow{\simeq} U\} \in \mathcal{J}(U)$.

(ii) Pull-backs $U_\alpha \times_U V$ exist for any $f \in \{U_\alpha \longrightarrow U\}_{\alpha \in I}$ and any arrow $V \xrightarrow{f} U$, where $\{U_\alpha \longrightarrow U\}_{\alpha \in I}$ is a covering. Moreover collection of projections $\{U_\alpha \times_U V \longrightarrow V\} \in \mathcal{J}(V)$.

(iii) If $\{U_\alpha \longrightarrow U\} \in \mathcal{J}(U)$, and for each α, $\{U_{\alpha i} \longrightarrow U_\alpha\}_i \in \mathcal{J}(U_\alpha)$, then the collection of composed morphisms $\{g \circ f | f \in \{U_{\alpha i} \longrightarrow U_\alpha\}, g \in \{U_\alpha \longrightarrow U\}\} \in \mathcal{J}(U)$.

Now suppose B is a topological space and $\widetilde{\mathcal{O}}(B)$ is the category of open subsets of B. Then a sheaf over $\widetilde{\mathcal{O}}(B)$ is a functor

$$R : \widetilde{\mathcal{O}}(B)^{\mathrm{op}} \longrightarrow \mathbf{Set}$$

satisfying following properties. Let $\bigcup_\alpha U_\alpha$ be an open cover (in the classical topological sense) of $U \in \mathrm{Obj}\Big(\widetilde{\mathcal{O}}(B)\Big)$ and $i_\alpha : U_\alpha \hookrightarrow U, i_{\alpha\beta} : U_{\alpha\beta} \hookrightarrow U_\alpha, i_{\alpha\beta} : U_{\alpha\beta} \hookrightarrow U_\beta$ be inclusion maps.

(i) If $\psi_1, \psi_2 \in R(U)$, such that,

$$R(i_\alpha)(\psi_1) = R(i_\alpha)(\psi_2), \tag{6.1}$$

for each $\alpha \in I$, then,

$$\psi_1 = \psi_2.$$

(ii) If for each $\alpha \in I$, a $\psi_\alpha \in R(U_\alpha)$ is given, such that, for any non empty $U_{\alpha\beta}$

$$R(i_{\alpha\beta})(\psi_\alpha) = R(i_{\alpha\beta})(\psi_\beta), \tag{6.2}$$

then there exists a unique $\psi \in U$, such that

$$R(i_\alpha)(\psi) = \psi_\alpha, \qquad \forall \alpha \in I.$$

It is not difficult to generalize the definition of sheaves over a topological space to sheaves over a site. Of course for a site unions (and thus cover in classical topological sense) or intersections do not make any sense. Coverings (in the sense of Grothendieck topology) and pull-backs are respectively the appropriate substitutes for a topological cover and intersections. A *sheaf with Grothendieck topology* or *sheaf over a site* $\mathcal{D}$ is defined [36] as a functor

$$\mathbf{R} : \mathcal{D}^{\mathrm{op}} \longrightarrow \mathbf{Set},$$

which has the following property. Let $\{U_\alpha \xrightarrow{f_\alpha} U\}$ be a covering in $\mathcal{D}$. Let $\mathrm{pr}_1 : U_\alpha \times_U U_\beta \longrightarrow U_\alpha$ and $\mathrm{pr}_2 : U_\alpha \times_U U_\beta \longrightarrow U_\beta$ be respectively first and second projections. Now if for each α a $\psi_\alpha \in \mathbf{R}(U_\alpha)$ is given such that $\Big(\mathbf{R}(\mathrm{pr}_1)\Big)(\psi_\alpha) = \Big(\mathbf{R}(\mathrm{pr}_2)\Big)(\psi_\beta) \in \mathbf{R}(U_\alpha \times_U U_\beta)$, then there exists a unique $\psi \in \mathbf{R}(U)$ such that

$$\Big(\mathbf{R}(f_\alpha)\Big)(\psi) = \psi_\alpha, \qquad \text{for all } \alpha.$$

It is a straightforward observation that, if $\{U_\alpha\}_\alpha$ is an open topological cover of $U \in \mathrm{Obj}\Big(\widetilde{\mathcal{O}}(B)\Big)$ then $\{i_\alpha : U_\alpha \longrightarrow U\}$ is a covering (in the sense of Grothendieck topology) of U, and thus all such collection of open covers define a Grothendieck topology. In turn, $U_\alpha \times_U U_\beta = U_\alpha \cap U_\beta$ and thus sheaves over the category of open subsets of a topological space can be considered as a specific example of sheaves over a site.

At this stage it would be pertinent to discuss some of the ideas behind the construction in this paper. Main object of interest in this paper is a type of groupoids, namely topological groupoids (see subsection 3.1), which is a groupoid with both object and morphism spaces are topological spaces. The path-space groupoid can be considered to be a prototype example of such groupoids. If $\mathbf{B}$ is such a groupoid, we can talk about open subsets of morphisms and open subsets of objects. However still "open subcategories" do not make sense. Because, union of subcategories, considered as union of objects and morphisms is not a subcategory. In this paper we first introduce a new notion of "categorical union" of subcategories, defined in (3.3), such that the union (in the sense of (3.3)) of subcategories is itself a subcategory. The intersection (defined in (3.1)) of subcategories is same as the subcategory obtained by taking intersection of object sets and morphism sets. The categorical union and intersection are consistent with the usual set theoretic Boolean relations (Proposition 3.1). Thus we can define "open subcategories" of a topological groupoid (see between Proposition 3.1 and Lemma 3.1) and define "open categorical cover" of $\mathbf{B}$ (section 3). With the aid of this construction of open subcategories, we define the category of open subcategories of $\mathbf{B}$, denoted as $\widetilde{\mathcal{O}}(\mathbf{B})$. Of course $\widetilde{\mathcal{O}}(\mathbf{B})$ is a subcategory of $\mathbf{Cat}$,

$$\widetilde{\mathcal{O}}(\mathbf{B}) \subset \mathbf{Cat}.$$

In this paper instead of assuming a Grothendieck topology, we generalize the definition of sheaf over a topological space in (6.1)-(6.2) to a *sheaf over the topological groupoid* $\mathbf{B}$ in subsection 4.1 as a functor

$$\mathcal{R} : \widetilde{\mathcal{O}}(\mathbf{B})^{\mathrm{op}} \longrightarrow \mathbf{Cat}$$

satisfying properties listed between (4.2)-(4.3), where open covers and intersections are respectively replaced by open categorical cover and categorical intersections. The primary example of a sheaf over a topological space B is the "sheaf of sections"; that is, for another given topological space M, the functor $R : \widetilde{\mathcal{O}}(B)^{\mathrm{op}} \longrightarrow \mathbf{Set}$ is given as follows:

$$\begin{aligned} &R(U) = \{f : U \longrightarrow M | f - \text{Continuous}\}, \qquad U \in \mathrm{Obj}\Big(\widetilde{\mathcal{O}}(B)\Big) \\ &\Big(R(\psi)\Big)(f) = f \circ \psi \in R(U), \qquad \psi \in \mathrm{Hom}(U, V), f \in R(V). \end{aligned} \tag{6.3}$$

Now suppose $\mathbf{C}$ is another topological groupoid. We can imitate and define a functor $\mathcal{R} : \widetilde{\mathcal{O}}(\mathbf{B})^{\mathrm{op}} \longrightarrow \mathbf{Cat}$ for the topological groupoid $\mathbf{B}$, where an open subcategory $\mathbf{U}$ of $\mathbf{B}$ is sent to the category of functors $\mathcal{F}(\mathbf{U}, \mathbf{C}) := \mathbf{C}^{\mathbf{U}}$ and a functor $\mathbf{\Theta} : \mathbf{V} \longrightarrow \mathbf{U}$ is sent to a functor $\mathbf{C}^{\mathbf{U}} \longrightarrow \mathbf{C}^{\mathbf{V}}$ (Lemma 4.1 confirms this functor exists). We call it "$\mathbf{Cat}$-valued presheaf of functorial sections". But the question is

whether this $\mathbf{Cat}$*-valued presheaf of functorial sections is also a* $\mathbf{Cat}$*-valued sheaf according to the definition given in subsection 4.1.*

In other words, whether $\mathcal{R}$ satisfies conditions listed between (4.2)-(4.13). Given the nature of categorical unions, answer to this question is not at all straightforward (see the paragraph preceding (4.19) for a more detailed explanation). The main result of this paper is Theorem 4.2, which gives an affirmative answer to the above question; that is a $\mathbf{Cat}$-valued presheaf of functorial sections is in fact a $\mathbf{Cat}$-valued sheaf and we call this sheaf the "sheaf of functorial sections".

Though we have not explicitly used the notion of stacks in this paper, it is not very difficult to interpret the result of Theorem 4.2 in the language of stacks. For that let us recall the definition of stacks. A stack is described in terms of a *lax 2-functor*, which naturally takes into account the fact that **Cat** is a 2-category. Let $\mathcal{D}$ be a category. Then a lax functor $\mathbf{F}$ on $\mathcal{D}$ is given by following data [36].

(i) For each $U \in \mathrm{Obj}\big(\mathcal{D}\big)$ a category $\mathbf{F}(U) \in \mathrm{Obj}\big(\mathbf{Cat}\big)$.

(ii) For each morphism $U \xrightarrow{f} V \in \mathrm{Mor}\big(\mathcal{D}\big)$ a functor $\mathbf{F}(V) \xrightarrow{\mathbf{F}(f)} \mathbf{F}(U) \in \mathrm{Mor}\big(\mathbf{Cat}\big)$.

(iii) For each $U \in \mathrm{Obj}\big(\mathcal{D}\big)$ a natural isomorphism $\alpha_U : \mathbf{F}(\mathrm{Id}_U) \Longrightarrow \mathrm{Id}_{\mathbf{F}(U)}$ between functors $\mathbf{F}(\mathrm{Id}_U)$ and $\mathrm{Id}_{\mathbf{F}(U)}$.

(iv) For any pair of composable morphisms $U \xrightarrow{f_1} V, V \xrightarrow{f_2} W \in \mathrm{Mor}\big(\mathcal{D}\big)$ a natural isomorphism $\beta_{f_1,f_2} : \mathbf{F}(f_1)\circ\mathbf{F}(f_2) \Longrightarrow \mathbf{F}(f_2 \circ f_1)$ between functors $\mathbf{F}(f_1)\circ\mathbf{F}(f_2), \mathbf{F}(f_2 \circ f_1)$.

Such that following coherence conditions are satisfied. For any $U \xrightarrow{f} V \in \mathrm{Mor}\big(\mathcal{D}\big)$ and $\mu \in \mathrm{Obj}\big(\mathbf{F}(V)\big)$ we should have

$$\begin{aligned} \beta_{\mathrm{Id}_U,f}(\mu) &= \alpha_U\Bigg(\big(\mathbf{F}(f)\big)(\mu)\Bigg), \\ \beta_{f,\mathrm{Id}_V}(\mu) &= \Bigg(\mathbf{F}(f)\Bigg)\big(\alpha_V(\mu)\big), \end{aligned} \tag{6.4}$$

and for composable morphisms $U \xrightarrow{f_1} V, V \xrightarrow{f_2} W, W \xrightarrow{f_3} X \in \mathrm{Mor}\big(\mathcal{D}\big)$ and $\mu \in \mathrm{Obj}\big(\mathbf{F}(X)\big)$, the diagram

$$\begin{array}{ccc} \Big(\mathbf{F}(f_1)\mathbf{F}(f_2)\mathbf{F}(f_3)\Big)(\mu) & \xrightarrow{\beta_{f_1,f_2}\left(\mathbf{F}(f_3)(\mu)\right)} & \Big(\mathbf{F}(f_2\circ f_1)\mathbf{F}(f_3)\Big)(\mu)\,, \\ \Big\downarrow{\scriptstyle \left(\mathbf{F}(f_1)\right)(\beta_{f_2,f_3}(\mu))} & & \Big\downarrow{\scriptstyle \beta_{f_2\circ f_1,f_3}(\mu)} \\ \Big(\mathbf{F}(f_1)\mathbf{F}(f_3\circ f_2)\Big)(\mu) & \xrightarrow[\beta_{f_1,f_3\circ f_2}(\mu)]{} & \Big(\mathbf{F}(f_3\circ f_2\circ f_1)\Big)(\mu) \end{array} \tag{6.5}$$

should commute. So for a lax 2-functor functoriality conditions hold up to natural isomorphisms. It is obvious that if $\mathbf{F} : \mathcal{D}^{\mathrm{op}} \longrightarrow \mathbf{Cat}$ is an ordinary functor, then it can also be considered as a lax 2-functor with $\alpha_U, \beta_{f_1,f_2}$ to be identities. Now suppose $\mathcal{D}$ is a site, with projections as follows

$$\begin{aligned}
&\mathrm{pr}_1 : U_\alpha \times_U U_\beta \longrightarrow U_\alpha,\\
&\mathrm{pr}_2 : U_\alpha \times_U U_\beta \longrightarrow U_\beta,\\
&\mathrm{pr}_{12} : U_\alpha \times_U U_\beta \times_U U_\delta \longrightarrow U_\alpha \times_U U_\beta,\\
&\mathrm{pr}_{13} : U_\alpha \times_U U_\beta \times_U U_\delta \longrightarrow U_\alpha \times_U U_\delta,\\
&\mathrm{pr}_{23} : U_\alpha \times_U U_\beta \times_U U_\delta \longrightarrow U_\beta \times_U U_\delta.
\end{aligned} \tag{6.6}$$

Let $\mathbf{F}$ be a lax 2-functor on $\mathcal{D}$. Then with any covering $\{U_\alpha \longrightarrow U\}_\alpha$ on $U \in \mathrm{Obj}\big(\mathcal{D}\big)$ we associate a category $\mathbf{F}(\{U_\alpha \longrightarrow U\})$ as described below. An object of $\mathbf{F}(\{U_\alpha \longrightarrow U\})$ is a collection

$$\big(\{\Psi_\alpha\}, \{\phi_{\alpha\beta}\}\big),$$

where $\Psi_\alpha \in \mathrm{Obj}\big(F(U_\alpha)\big)$ and each $\Big(\phi_{\alpha\beta} : \big(\mathbf{F}(\mathrm{pr}_2)\big)(\Psi_\beta) \xrightarrow{\simeq} \big(\mathbf{F}(\mathrm{pr}_1)\big)(\Psi_\alpha)\Big) \in \mathrm{Mor}\big(\mathbf{F}(\mathrm{U}_\alpha \times_\mathrm{U} \mathrm{U}_\beta)\big)$ is an isomorphism satisfying co-cycle condition:

$$\mathbf{F}(\mathrm{pr}_{13})(\phi_{\alpha\delta}) = \mathbf{F}(\mathrm{pr}_{12})(\phi_{\alpha\beta}) \circ \mathbf{F}(\mathrm{pr}_{23})(\phi_{\beta\delta}) : \big(\mathbf{F}(\mathrm{pr}_3)\big)(\Psi_\delta) \longrightarrow \big(\mathbf{F}(\mathrm{pr}_1)\big)(\Psi_\alpha). \tag{6.7}$$

A morphism $\big(\{\Psi_\alpha\}, \{\phi_{\alpha\beta}\}\big) \longrightarrow \big(\{\widetilde{\Psi}_\alpha\}, \{\widetilde{\phi}_{\alpha\beta}\}\big)$ in $\mathbf{F}(\{U_\alpha \longrightarrow U\})$ is a collection of morphisms

$$\{f_\alpha \in \mathrm{Mor}\big(\mathbf{F}(U_\alpha)\big) | f_\alpha : \Psi_\alpha \longrightarrow \widetilde{\Psi}_\alpha\}$$

such that the diagram

$$\begin{array}{ccc}
\big(\mathbf{F}(\mathrm{pr}_2)\big)(\Psi_\beta) & \xrightarrow{\big(\mathbf{F}(\mathrm{pr}_2)\big)(f_\beta)} & \big(\mathbf{F}(\mathrm{pr}_2)\big)(\widetilde{\Psi}_\beta)\\
\Big\downarrow{\scriptstyle \phi_{\alpha\beta}} & & {\scriptstyle \widetilde{\phi}_{\alpha\beta}}\Big\downarrow\\
\big(\mathbf{F}(\mathrm{pr}_1)\big)(\Psi_\alpha) & \xrightarrow[\big(\mathbf{F}(\mathrm{pr}_1)\big)(f_\alpha)]{} & \big(\mathbf{F}(\mathrm{pr}_1)\big)(\widetilde{\Psi}_\alpha)
\end{array} \tag{6.8}$$

commutes for all α, β, δ. Now, for a given lax 2-functor $\mathbf{F}$ on $\mathcal{D}$ and a covering $\mathcal{U} := \{k_\alpha : U_\alpha \longrightarrow U\}_\alpha$ on $U \in \mathrm{Obj}\big(\mathcal{D}\big)$ we have a functor $\mathbf{F}_\mathcal{U} : \mathbf{F}(U) \longrightarrow \mathbf{F}(\{U_\alpha \longrightarrow U\})$ as follows [36]. For any $\Psi \in \mathrm{Obj}(\mathbf{F}(U))$, we define

$$\mathbf{F}_\mathcal{U}(\Psi) := \Bigg(\big\{\big(\mathbf{F}(k_\alpha)\big)(\Psi)\big\}, \{\phi_{\alpha\beta}\}\Bigg), \tag{6.9}$$

where $\Bigg(\phi_{\alpha\beta} : \Big(\mathbf{F}(\mathrm{Pr}_1)\Big)\big(\big(\mathbf{F}(k_\alpha)\big)(\Psi)\big) \longrightarrow \Big(\mathbf{F}(\mathrm{Pr}_2)\Big)\big(\big(\mathbf{F}(k_\beta)\big)(\Psi)\big)\Bigg) \in \mathrm{Mor}\big(U_\alpha \times_U U_\beta\big)$ is the isomorphism obtained from the natural isomorphisms $\mathbf{F}(\mathrm{Pr}_1) \circ \mathbf{F}(k_\alpha) \simeq \mathbf{F}(k_\alpha \circ \mathrm{Pr}_1) =$

$\mathbf{F}(k_\beta \circ \mathrm{Pr}_2) \simeq \mathbf{F}(\mathrm{Pr}_2) \circ \mathbf{F}(k_\beta)$. It is customary to identify $\Big(\mathbf{F}(\mathrm{Pr}_1)\Big)\big(\big(\mathbf{F}(k_\alpha)\big)(\Psi)\big)$ and $\Big(\mathbf{F}(\mathrm{Pr}_2)\Big)\big(\big(\mathbf{F}(k_\beta)\big)(\Psi)\big)$. So $\phi_{\alpha\beta}$ become identities. Given a morphism $\Big(\Psi \xrightarrow{f} \widetilde{\Psi}\Big) \in \mathrm{Mor}\big(\mathbf{F}(U)\big)$ we obtain a morphism

$$\mathbf{F}_{\mathcal{U}}(f) := \Big\{\big(\mathbf{F}(k_\alpha)\big)(f) : \big(\mathbf{F}(k_\alpha)\big)(\Psi) \longrightarrow \big(\mathbf{F}(k_\alpha)\big)(\widetilde{\Psi})\Big\} \tag{6.10}$$

in $\mathbf{F}(\{U_\alpha \longrightarrow U\})$. Now we may state the definition of a stack.

Let $\mathcal{D}$ be a site and $\mathbf{F}$ be a lax functor on $\mathcal{D}$. If for each covering $\mathcal{U} = \{U_\alpha \longrightarrow U\}$, the corresponding functor $\mathbf{F}_{\mathcal{U}} : \mathbf{F}(U) \longrightarrow \mathbf{F}(\{U_\alpha \longrightarrow U\})$ is an equivalence of categories, then the lax functor $\mathbf{F}$ is called a *stack over the site* $\mathcal{D}$.

Let us turn to the content of this paper. Suppose $\mathbf{B}$ is a topological groupoid and $\widetilde{\mathcal{O}}(\mathbf{B})$ is the category of open subgroupoids (see between Proposition 3.1 and Lemma 3.1 for the definition of open subgroupoids) of $\mathbf{B}$, as defined in (4.1). Let $\{\mathbf{U}_\alpha\}$ be an open categorical cover of $\mathbf{U} \in \mathrm{Obj}\big(\widetilde{\mathcal{O}}(\mathbf{B})\big)$. As per definition given in (3.11), that means each $\mathbf{U}_\alpha$ is an open subgroupoid of $\mathbf{U}$, and

$$\mathbf{U} = \bigcup_\alpha \mathbf{U}_\alpha,$$

where the union on the right hand side is the categorical union defined in (3.3). It is easy to see that with such categorical covers for objects of $\widetilde{\mathcal{O}}(\mathbf{B})$, we can treat $\widetilde{\mathcal{O}}(\mathbf{B})$ as a site with coverings (in Grothendieck topology sense) $\mathcal{U} := \{\mathbf{i}_\alpha : \mathbf{U}_\alpha \longrightarrow \mathbf{U}\}$, where $\mathbf{i}_\alpha$ are the inclusion functors. The pull-backs are simply intersections of categories defined in (3.1)

$$\mathbf{U}_\alpha \times_{\mathbf{U}} \mathbf{U}_\beta = \mathbf{U}_\alpha \bigcap \mathbf{U}_\beta := \mathbf{U}_{\alpha\beta}$$

and the projections are inclusion functors of (4.4)

$$\mathrm{Pr}_1 = \mathbf{i}_{\alpha\beta} : \mathbf{U}_{\alpha\beta} \hookrightarrow \mathbf{U}_\alpha,$$
$$\mathrm{Pr}_2 = \mathbf{i}_{\alpha\beta} : \mathbf{U}_{\alpha\beta} \hookrightarrow \mathbf{U}_\beta.$$

Now suppose $\mathbf{C}$ is another topological groupoid. Then we have a functor

$$\mathcal{R} : \widetilde{\mathcal{O}}(\mathbf{B})^{\mathrm{op}} \longrightarrow \mathbf{Cat}$$

which sends any $\mathbf{U} \in \mathrm{Obj}\big(\widetilde{\mathcal{O}}(\mathbf{B})\big)$ to the category of functors from $\mathbf{U}$ to $\mathbf{C}$,

$$\mathbf{U} \mapsto \mathcal{F}(\mathbf{U}, \mathbf{C}) := \mathbf{C}^{\mathbf{U}},$$

and by Lemma 4.1 any functor $\mathbf{U} \xrightarrow{\mathbf{\Theta}} \mathbf{V}$ is sent to a functor $\left(\mathbf{C}^{\mathbf{V}} \xrightarrow{\mathcal{R}(\mathbf{\Theta})} \mathbf{C}^{\mathbf{U}}\right)$. In an obvious manner we view $\mathcal{R}$ as a lax 2-functor (see the comment following (6.5)). Now, an object of $\mathcal{R}(\mathbf{U}_\alpha)$ is a functor

$$\mathbf{\Psi}_\alpha : \mathbf{U}_\alpha \longrightarrow \mathbf{C},$$

and

$$\Big(\mathcal{R}(\mathrm{Pr}_1)\Big)(\mathbf{\Psi}_\alpha) = \Big(\mathcal{R}(\mathbf{i}_{\alpha\beta})\Big)(\mathbf{\Psi}_\alpha) = \mathbf{\Psi}_\alpha|_{\mathbf{U}_{\alpha\beta}} \in \mathrm{Obj}\big(\mathbf{U}_{\alpha\beta}\big),$$

where we have used the notation of (4.21). Similarly a morphism in $\mathcal{R}(\mathbf{U}_\alpha)$ is a natural transformation

$$\mathcal{S}_\alpha : \Big(\mathbf{\Psi}_\alpha : \mathbf{U}_\alpha \longrightarrow \mathbf{C}\Big) \Longrightarrow \Big(\widetilde{\mathbf{\Psi}}_\alpha : \mathbf{U}_\alpha \longrightarrow \mathbf{C}\Big)$$

and

$$\Big(\mathcal{R}(\mathbf{i}_{\alpha\beta})\Big)(\mathcal{S}_\alpha) = \mathcal{S}_\alpha|_{\mathbf{U}_{\alpha\beta}} : \mathbf{\Psi}_\alpha|_{\mathbf{U}_{\alpha\beta}} \Longrightarrow \widetilde{\mathbf{\Psi}}_\alpha|_{\mathbf{U}_{\alpha\beta}}.$$

Then we have the category $\mathcal{R}\big(\{\mathbf{i}_\alpha : \mathbf{U}_\alpha \longrightarrow \mathbf{U}\}\big)$ associated to the covering $\{\mathbf{i}_\alpha : \mathbf{U}_\alpha \longrightarrow \mathbf{U}\}$. An object of $\mathcal{R}\big(\{\mathbf{i}_\alpha : \mathbf{U}_\alpha \longrightarrow \mathbf{U}\}\big)$ is a collection $\left(\left\{\mathbf{\Psi}_\alpha \in \mathrm{Obj}\big(\mathcal{R}(\mathbf{U}_\alpha)\big)\right\}_\alpha\right)$ such that for non-empty $\mathbf{U}_{\alpha\beta}$, restrictions coincide

$$\mathbf{\Psi}_\alpha|_{\mathbf{U}_{\alpha\beta}} = \mathbf{\Psi}_\beta|_{\mathbf{U}_{\alpha\beta}},$$

and a morphism of $\mathcal{R}\big(\{\mathbf{i}_\alpha : \mathbf{U}_\alpha \longrightarrow \mathbf{U}\}\big)$ is a collection $\left(\left\{\mathcal{S}_\alpha \in \mathrm{Mor}\big(\mathcal{R}(\mathbf{U}_\alpha)\big)\right\}_\alpha\right)$ such that for non-empty $\mathbf{U}_{\alpha\beta}$, restrictions coincide

$$\mathcal{S}_\alpha|_{\mathbf{U}_{\alpha\beta}} = \mathcal{S}_\beta|_{\mathbf{U}_{\alpha\beta}}. \qquad \text{(See the diagram in (6.8)).}$$

By construction described in (6.9)-(6.10) we have a functor

$$\mathcal{R}_{\mathcal{U}} : \mathcal{R}(\mathbf{U}) \longrightarrow \mathcal{R}\big(\{\mathbf{i}_\alpha : \mathbf{U}_\alpha \longrightarrow \mathbf{U}\}\big).$$

In fact $\mathcal{R}_{\mathcal{U}}$ is an isomorphism of categories and the Theorem 4.2 can be equivalently stated as follows.

Proposition 6.1 — Let $\mathbf{B}$ be a topological groupoid. Let $\widetilde{\mathcal{O}}(\mathbf{B})$ be the category of open subcategories. Treat $\widetilde{\mathcal{O}}(\mathbf{B})$ as a site with coverings $\{\mathbf{i}_\alpha : \mathbf{U}_\alpha \longrightarrow \mathbf{U}\}$, for a categorical cover $\{\mathbf{U}_\alpha\}$ of $\mathbf{U} \in \mathrm{Obj}\big(\widetilde{\mathcal{O}}(\mathbf{B})\big)$, and also treat $\mathcal{R}$ in Lemma 4.1 as a lax 2-functor. Then we have an isomorphism of categories

$$\mathcal{R}_{\mathcal{U}} : \mathcal{R}(\mathbf{U}) \longrightarrow \mathcal{R}\big(\{\mathbf{i}_\alpha : \mathbf{U}_\alpha \longrightarrow \mathbf{U}\}\big).$$

To see this isomorphism we note that given an object $\mathbf{\Psi}$ of $\mathcal{R}(\mathbf{U})$, i.e. a functor $\mathbf{\Psi}: \mathbf{U} \longrightarrow \mathbf{C}$, $\mathcal{R}_{\mathcal{U}}$ sends it to a collection of functors $\{(\mathcal{R}(\mathbf{i}_\alpha))(\mathbf{\Psi}) = \mathbf{\Psi}|_{\mathbf{U}_\alpha} : \mathbf{U}_\alpha \longrightarrow \mathbf{C}\}$ and a morphism $\mathcal{S} : \mathbf{\Psi} \Longrightarrow \widetilde{\mathbf{\Psi}}$ in $\mathcal{R}(\mathbf{U})$ is sent to a collection of natural transformations $\Big\{(\mathcal{R}(\mathbf{i}_\alpha))(\mathcal{S}) = \mathcal{S}|_{\mathbf{U}_\alpha} : \mathbf{\Psi}|_{\mathbf{U}_\alpha} \longrightarrow \widetilde{\mathbf{\Psi}}|_{\mathbf{U}_\alpha}\Big\}$. Then the injectivity of $\mathcal{R}_{\mathcal{U}}$ on objects follows from (4.51). Whereas the injectivity

$$\begin{gathered}\operatorname{Hom}(\mathbf{\Psi}, \widetilde{\mathbf{\Psi}}) \longrightarrow \operatorname{Hom}\Big(\mathcal{R}_{\mathcal{U}}(\mathbf{\Psi}), \mathcal{R}_{\mathcal{U}}(\widetilde{\mathbf{\Psi}})\Big) = \operatorname{Hom}\Big(\{\mathbf{\Psi}|_{\mathbf{U}_\alpha}\}, \{\widetilde{\mathbf{\Psi}}|_{\mathbf{U}_\alpha}\}\Big), \\ \mathcal{S} \mapsto \{(\mathcal{R}(\mathbf{i}_\alpha))(\mathcal{S})\} = \{\mathcal{S}|_{\mathbf{U}_\alpha}\}.\end{gathered} \tag{6.11}$$

follows from (4.52). Now suppose $\{\mathbf{\Psi}_\alpha\}$ is an object of $\mathcal{R}(\{\mathbf{i}_\alpha : \mathbf{U}_\alpha \longrightarrow \mathbf{U}\})$; that means each $\mathbf{\Psi}_\alpha : \mathbf{U}_\alpha \longrightarrow \mathbf{C}$ is a functor such that

$$\mathbf{\Psi}_\alpha|_{\mathbf{U}_{\alpha\beta}} = \mathbf{\Psi}_\beta|_{\mathbf{U}_{\alpha\beta}} : \mathbf{U}_{\alpha\beta} \longrightarrow \mathbf{C}$$

for non-empty $\mathbf{U}_{\alpha\beta}$. Then by (4.53) and (4.54) we have a functor $\mathbf{\Psi} : \mathbf{U} \longrightarrow \mathbf{C}$ such that $\mathbf{\Psi}|_{\mathbf{U}_\alpha} = \mathbf{\Psi}_\alpha$, which establishes the surjectivity of $\mathcal{R}_{\mathcal{U}}$ on objects. We would like to emphasize here that due to the "twist" in the categorical union of subcategories, the most difficult part of the construction in this paper was to prove that (4.54) indeed makes sense and a major portion of this paper is devoted for the purpose. On the other hand if $\{\mathcal{S}_\alpha : \mathbf{\Psi} \Longrightarrow \widetilde{\mathbf{\Psi}}_\alpha\}$ is a morphism in $\mathcal{R}(\{\mathbf{i}_\alpha : \mathbf{U}_\alpha \longrightarrow \mathbf{U}\})$; that is each

$$\mathcal{S}_\alpha : \mathbf{\Psi}_\alpha \Longrightarrow \widetilde{\mathbf{\Psi}}_\alpha$$

is a natural transformation satisfying

$$\mathcal{S}_\alpha|_{\mathbf{U}_{\alpha\beta}} : \mathbf{\Psi}|_{\mathbf{U}_{\alpha\beta}} \Longrightarrow \widetilde{\mathbf{\Psi}}_\alpha|_{\mathbf{U}_{\alpha\beta}},$$

for all nonempty $\mathbf{U}_{\alpha\beta}$, then by (4.55) and (4.56) we have a natural transformation

$$\mathcal{S} : \mathbf{U} \longrightarrow \mathbf{C},$$

such that $\mathcal{S}|_{\mathbf{U}_\alpha} = \mathcal{S}_\alpha$, which establishes the surjectivity of

$$\operatorname{Hom}(\mathbf{\Psi}, \widetilde{\mathbf{\Psi}}) \longrightarrow \operatorname{Hom}\Big(\mathcal{R}_{\mathcal{U}}(\mathbf{\Psi}), \mathcal{R}_{\mathcal{U}}(\widetilde{\mathbf{\Psi}})\Big) = \operatorname{Hom}\Big(\{\mathbf{\Psi}|_{\mathbf{U}_\alpha}\}, \{\widetilde{\mathbf{\Psi}}|_{\mathbf{U}_\alpha}\}\Big).$$

Thus $\mathcal{R}_{\mathcal{U}}$ is an isomorphism of categories.

Appendix

Proof of Proposition 3.1 : Let $\mathbf{U}, \mathbf{V}, \mathbf{W}$ be subcategories of a category $\mathbf{B}$. Let intersection and union of a pair of subcategories be respectively as defined in (3.1), (3.3). If $\mathbf{U}$, $\mathbf{V}$ and $\mathbf{W}$ are mutually disjoint, then the statements in Proposition 3.1 are same as standard set theoretic statements, because

$$\mathrm{Mor}(\mathbf{U} \cup \mathbf{V}) = \mathrm{Mor}(\mathbf{U}) \cup \mathrm{Mor}(\mathbf{V}), \qquad \text{if } \mathbf{U} \cap \mathbf{V} = \emptyset.$$

If one of the $\mathbf{U}, \mathbf{V}, \mathbf{W}$ is disjoint with other two, then also the proof is straightforward. In what follows, we will assume

$$\mathbf{U} \cap \mathbf{V} \neq \emptyset,$$
$$\mathbf{U} \cap \mathbf{W} \neq \emptyset,$$
$$\mathbf{W} \cap \mathbf{V} \neq \emptyset.$$

Proof of identity (i) : The second equation of (i),

$$\mathbf{U} \cap (\mathbf{V} \cap \mathbf{W}) = (\mathbf{U} \cap \mathbf{V}) \cap \mathbf{W},$$

is an obvious consequence of the definition of intersection in (3.1). Let us prove the first equation of (i).

We first classify morphisms in $\mathbf{U} \cup (\mathbf{V} \cup \mathbf{W})$. If $f \in \mathrm{Mor}\Big(\mathbf{U} \cup (\mathbf{V} \cup \mathbf{W})\Big)$, then (3.3) implies the following, not necessarily mutually exclusive, classification (see left hand side of Figure 3) :

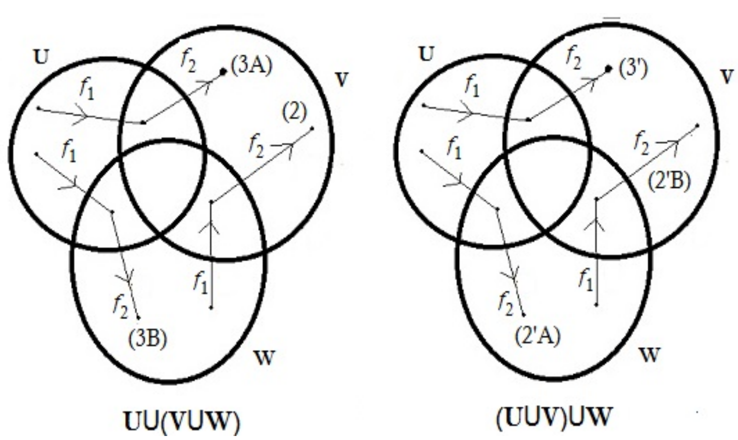


Figure 3: Comparison between morphisms in $\mathbf{U} \cup (\mathbf{V} \cup \mathbf{W})$ and $(\mathbf{U} \cup \mathbf{V}) \cup \mathbf{W}$

(1) $f \in \mathrm{Mor}(\mathbf{U}) \cup \mathrm{Mor}(\mathbf{V}) \cup \mathrm{Mor}(\mathbf{W})$.

(2) f is of the form $f_2 \circ f_1$, where $f_2, f_1 \in \mathrm{Mor}(\mathbf{V}) \cup \mathrm{Mor}(\mathbf{W})$, and $s(f_2) = t(f_1) \in \mathrm{Obj}(\mathbf{V} \cap \mathbf{W}) = \mathrm{Obj}(\mathbf{V}) \cap \mathrm{Obj}(\mathbf{W})$.

(3) f is of the form $f_2 \circ f_1$, where $f_2, f_1 \in \mathrm{Mor}(\mathbf{U}) \cup \mathrm{Mor}(\mathbf{V} \cup \mathbf{W})$, and $s(f_2) = t(f_1) \in \mathrm{Obj}\big(\mathbf{U} \cap (\mathbf{V} \cup \mathbf{W})\big)$. By (3.1), (3.3), for the objects, we have following identity $\mathrm{Obj}\big(\mathbf{U} \cap (\mathbf{V} \cup \mathbf{W})\big) = \mathrm{Obj}(\mathbf{U}) \cap \Big(\mathrm{Obj}(\mathbf{V}) \cup \mathrm{Obj}(\mathbf{W})\Big) = \Big(\mathrm{Obj}(\mathbf{U}) \cap \mathrm{Obj}(\mathbf{V})\Big) \cup \Big(\mathrm{Obj}(\mathbf{U}) \cap \mathrm{Obj}(\mathbf{W})\Big)$. So, we may further classify the morphisms in (3) as,

(3A) $f = f_2 \circ f_1$ such that $f_2, f_1 \in \mathrm{Mor}(\mathbf{U}) \cup \mathrm{Mor}(\mathbf{V})$ and $s(f_2) = t(f_1) \in \Big(\mathrm{Obj}(\mathbf{U}) \cap \mathrm{Obj}(\mathbf{V})\Big)$.

(3B) $f = f_2 \circ f_1$ such that $f_2, f_1 \in \mathrm{Mor}(\mathbf{U}) \cup \mathrm{Mor}(\mathbf{W})$ and $s(f_2) = t(f_1) \in \Big(\mathrm{Obj}(\mathbf{U}) \cap \mathrm{Obj}(\mathbf{W})\Big)$.

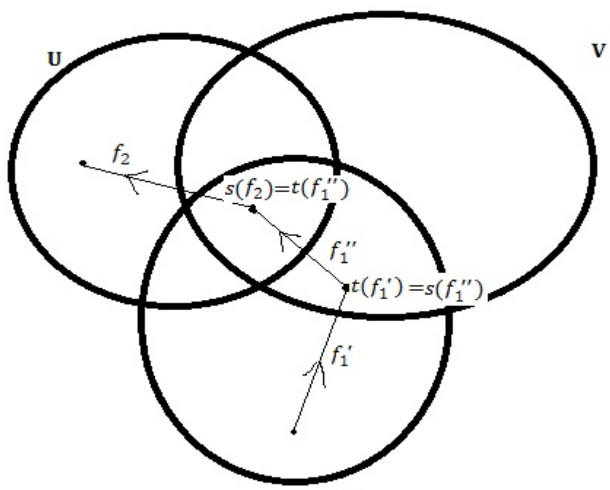


Figure 4:

[It may appear that in (3) we have overlooked the case when $f_1'', f_1' \in \mathrm{Mor}(\mathbf{V}) \cup \mathrm{Mor}(\mathbf{W})$ and $f_1 = f_1'' \circ f_1', s(f_1'') = t(f_1') \in \mathrm{Obj}(\mathbf{V} \cap \mathbf{W})$. But, note that in that case $t(f_1'') = s(f_2) \in \mathrm{Obj}\Big(\mathbf{U} \cap \mathbf{V}\Big) \cap \mathrm{Obj}\Big(\mathbf{V} \cap \mathbf{W}\Big) = \mathrm{Obj}\Big(\mathbf{U} \cap \mathbf{W}\Big) \cap \mathrm{Obj}\Big(\mathbf{V} \cap \mathbf{W}\Big) = \mathrm{Obj}(\mathbf{U} \cap \mathbf{V} \cap \mathbf{W})$, and therefore $f_1'' \circ f_1' = f_1 \in \mathrm{Mor}(\mathbf{W})$ or $f_1'' \circ f_1' = f_1 \in \mathrm{Mor}(\mathbf{V})$. Such scenarios have already been taken care of respectively by (3B) and (3A). Figure 4 illustrates the case when $f_1'' \circ f_1' = f_1 \in \mathrm{Mor}(\mathbf{W})$.]

We proceed with the classification of morphisms in $(\mathbf{U} \cup \mathbf{V}) \cup \mathbf{W}$. If $f \in \mathrm{Mor}\Big((\mathbf{U} \cup \mathbf{V}) \cup \mathbf{W}\Big)$, then we obtain following classification (see right hand side of Figure 3) :

$(\acute{1})$ $f \in \mathrm{Mor}(\mathbf{U}) \cup \mathrm{Mor}(\mathbf{V}) \cup \mathrm{Mor}(\mathbf{W})$.

$(\acute{2})$ f is of the form $f_2 \circ f_1$, where $f_2, f_1 \in \mathrm{Mor}\Big((\mathbf{U} \cup \mathbf{V}) \cap \mathbf{W}\Big)$ and $s(f_2) = t(f_1) \in \mathrm{Obj}\Big((\mathbf{U} \cup \mathbf{V}) \cap \mathbf{W}\Big)$. Using the identity $\mathrm{Obj}\Big((\mathbf{U} \cup \mathbf{V}) \cap \mathbf{W}\Big) = \Big(\mathrm{Obj}(\mathbf{U}) \cap \mathrm{Obj}(\mathbf{W})\Big) \cup \Big(\mathrm{Obj}(\mathbf{V}) \cap \mathrm{Obj}(\mathbf{W})\Big)$, we further classify morphisms in $(\acute{2})$ as,

$(\acute{2}\mathrm{A})$ $f = f_2 \circ f_1$ such that $f_2, f_1 \in \mathrm{Mor}(\mathbf{U}) \cup \mathrm{Mor}(\mathbf{W})$ and $s(f_2) = t(f_1) \in \Big(\mathrm{Obj}(\mathbf{U}) \cap \mathrm{Obj}(\mathbf{W})\Big)$.

$(\acute{2}\mathrm{B})$ $f = f_2 \circ f_1$ such that $f_2, f_1 \in \mathrm{Mor}(\mathbf{V}) \cup \mathrm{Mor}(\mathbf{W})$ and $s(f_2) = t(f_1) \in \Big(\mathrm{Obj}(\mathbf{V}) \cap \mathrm{Obj}(\mathbf{W})\Big)$.

$(\acute{3})$ $f = f_2 \circ f_1$, where $f_2, f_1 \in \mathrm{Mor}(\mathbf{U}) \cup \mathrm{Mor}(\mathbf{V})$ and $s(f_2) = t(f_1) \in \Big(\mathrm{Obj}(\mathbf{U}) \cap \mathrm{Obj}(\mathbf{V})\Big)$.

We make following correspondence between two classifications:

$$(1) \Longleftrightarrow (\acute{1})$$
$$(2) \Longleftrightarrow (\acute{2}\mathrm{B})$$
$$(3\mathrm{A}) \Longleftrightarrow (\acute{3})$$
$$(3\mathrm{B}) \Longleftrightarrow (\acute{2}\mathrm{A}).$$

Hence we conclude

$$\mathbf{U} \cup (\mathbf{V} \cup \mathbf{W}) = (\mathbf{U} \cup \mathbf{V}) \cup \mathbf{W}.$$

Proof of Identity (ii) :

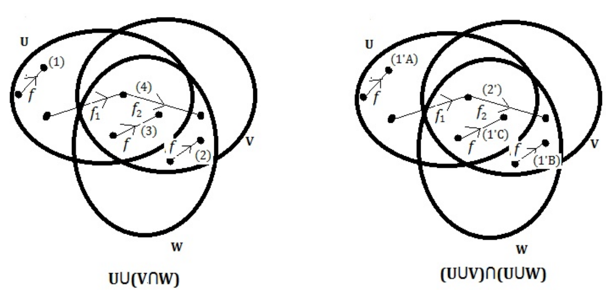


Figure 5: Comparison between morphisms in $\mathbf{U} \cup (\mathbf{V} \cap \mathbf{W})$ and $(\mathbf{U} \cup \mathbf{V}) \cap (\mathbf{U} \cup \mathbf{W})$

As before, we classify the morphisms in the left hand and right hand sides of (ii) and compare.

Let $f \in \mathrm{Mor}\Big(\mathbf{U} \cup (\mathbf{V} \cap \mathbf{W})\Big)$. Then following are the possibilities (see the left hand side of Figure 5).

(1) $f \in \mathrm{Mor}(\mathbf{U})$.

(2) $f \in \mathrm{Mor}(\mathbf{V} \cap \mathbf{W}) = \mathrm{Mor}(\mathbf{V}) \cap \mathrm{Mor}(\mathbf{W})$.

(3) $f \in \mathrm{Mor}(\mathbf{U}) \cap \mathrm{Mor}(\mathbf{V} \cap \mathbf{W}) = \mathrm{Mor}(\mathbf{U}) \cap \mathrm{Mor}(\mathbf{V}) \cap \mathrm{Mor}(\mathbf{W}) = \mathrm{Mor}(\mathbf{U} \cap \mathbf{V} \cap \mathbf{W})$.

(4) $f = f_2 \circ f_1$, where $f_2, f_1 \in \mathrm{Mor}(\mathbf{U}) \cup \mathrm{Mor}(\mathbf{V} \cap \mathbf{W})$ and $s(f_2) = t(f_1) \in \mathrm{Obj}(\mathbf{U}) \cap \mathrm{Obj}(\mathbf{V} \cap \mathbf{W}) = \mathrm{Obj}(\mathbf{U}) \cap \mathrm{Obj}(\mathbf{V}) \cap \mathrm{Obj}(\mathbf{W}) = \mathrm{Obj}(\mathbf{U} \cap \mathbf{V} \cap \mathbf{W})$.

On the other hand, if
$f \in \mathrm{Mor}\Big((\mathbf{U} \cup \mathbf{V}) \cap (\mathbf{U} \cup \mathbf{W})\Big) = \mathrm{Mor}\Big(\mathbf{U} \cup \mathbf{V}\Big) \bigcap \mathrm{Mor}\Big(\mathbf{U} \cup \mathbf{W}\Big)$, then possibilities are as follows (see the right hand side of Figure 5).

(1́) $f \in \Big(\mathrm{Mor}(\mathbf{U}) \cup \mathrm{Mor}(\mathbf{V})\Big) \cap \Big(\mathrm{Mor}(\mathbf{U}) \cup \mathrm{Mor}(\mathbf{W})\Big)$. Since

$$\mathrm{Mor}(\mathbf{U}) \cup \mathrm{Mor}(\mathbf{V}) \subset \mathrm{Mor}\Big(\mathbf{U} \cup \mathbf{V}\Big),$$
$$\mathrm{Mor}(\mathbf{U}) \cup \mathrm{Mor}(\mathbf{W}) \subset \mathrm{Mor}\Big(\mathbf{U} \cup \mathbf{W}\Big) [\text{by } (3.5)],$$

we have

$$\begin{aligned}&\Big(\mathrm{Mor}(\mathbf{U}) \cup \mathrm{Mor}(\mathbf{V})\Big) \bigcap \Big(\mathrm{Mor}(\mathbf{U}) \cup \mathrm{Mor}(\mathbf{W})\Big) \subset \mathrm{Mor}\Big(\mathbf{U} \cup \mathbf{V}\Big) \bigcap \mathrm{Mor}\Big(\mathbf{U} \cup \mathbf{W}\Big), \\ &\Rightarrow \mathrm{Mor}(\mathbf{U}) \cup \Big(\mathrm{Mor}(\mathbf{V}) \cap \mathrm{Mor}(\mathbf{W})\Big) \subset \mathrm{Mor}\Big(\mathbf{U} \cup \mathbf{V}\Big) \bigcap \mathrm{Mor}\Big(\mathbf{U} \cup \mathbf{W}\Big)\end{aligned}.$$

So we further classify them as

(1́A) $f \in \mathrm{Mor}(\mathbf{U})$.
(1́B) $f \in \Big(\mathrm{Mor}(\mathbf{V}) \cap \mathrm{Mor}(\mathbf{W})\Big) = \mathrm{Mor}\Big(\mathbf{V} \cap \mathbf{W}\Big)$.

(1́C) $f \in \mathrm{Mor}(\mathbf{U}) \cap \Big(\mathrm{Mor}(\mathbf{V}) \cap \mathrm{Mor}(\mathbf{W})\Big) = \mathrm{Mor}\Big(\mathbf{U} \cap \mathbf{V} \cap \mathbf{W}\Big)$.

($\acute{2}$) $f = f_2 \circ f_1$, where $f_2, f_1 \in \mathrm{Mor}(\mathbf{U}) \cup \mathrm{Mor}(\mathbf{V} \cap \mathbf{W})$ and $s(f_2) = t(f_1) \in \Big(\mathrm{Obj}(\mathbf{U}) \cap \mathrm{Obj}(\mathbf{V})\Big) \bigcap \Big(\mathrm{Obj}(\mathbf{V}) \cap \mathrm{Obj}(\mathbf{W})\Big) = \mathrm{Obj}(\mathbf{U} \cap \mathbf{V} \cap \mathbf{W})$.

We make following correspondence between two sets of classifications:

$$(1) \Longleftrightarrow (\acute{1}\mathrm{A})$$
$$(2) \Longleftrightarrow (\acute{1}\mathrm{B})$$
$$(3) \Longleftrightarrow (\acute{1}\mathrm{C})$$
$$(4) \Longleftrightarrow (\acute{2})$$

and conclude

$$\mathbf{U} \cup (\mathbf{V} \cap \mathbf{W}) = (\mathbf{U} \cup \mathbf{V}) \cap (\mathbf{U} \cup \mathbf{W}).$$

Proof of Identity (iii) : Same methodology can be adopted to prove

$$\mathbf{U} \cap (\mathbf{V} \cup \mathbf{W}) = (\mathbf{U} \cap \mathbf{V}) \cup (\mathbf{U} \cap \mathbf{W}). \square$$

Concluding Remarks

In this paper we have developed a framework of **Cat**-valued sheaves over a category $\widetilde{\mathcal{O}}(\mathbf{B})$ of subcategories of a topological groupoid **B**. Our starting point was the definition of **Cat**-valued presheaf introduced in [15]. We have constructed the **Cat**-valued sheaf of local functorial sections on **B** for a fixed category **C**. We have further shown that if we replace **C** by a categorical group $\mathcal{G}$, we obtain a **CatGrp**-valued sheaf. In traditional sheaf theory, sheaf of sections on a given topological space is the basic object of interest. In fact, every sheaf defined on a topological space can be realized as a sheaf of sections on the so called "étalé space" corresponding to the given topological space [28]. It would be interesting to see if similar consideration also arises in the context of this paper; that is, whether there exists an "étalé category", such that any **Cat**-valued sheaf on **B** can be realized as a sheaf of functorial sections on the "étalé category".

Lastly, we note that it would have been more natural if we had defined **Cat**-valued presheaf to be a 2-functor

$$\mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Cat}.$$

Because, $\mathbf{Cat}, \mathcal{C}, \widetilde{\mathcal{O}}(\mathbf{B})$ are all 2-categories. To reduce the complexity we have purposefully ignored the natural higher structures which aforementioned categories posses. However, it is not very difficult task to extend our framework to the higher level of enrichment.

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References

1. H. Abbaspour and F. Wagemann, *On 2-holonomy* at http://arxiv.org/abs/1202.2292 (2012).
2. P. Aschieri, L. Cantini and B. Jurco, Nonabelian bundle gerbes, their differential geometry and gauge theory, *Commun. Math. Phys.*, **254** (2005) 367-400.
3. R. Attal, *Two-dimensional parallel transport : combinatorics and functoriality*, http://arxiv.org/abs/math-ph/0105050 (2001).
4. R. Attal, *Combinatorics of non-abelian gerbes with connection and curvature*, *Ann. Fond. Louis de Broglie*, **29**(4) (2004), 609-633.
5. J. C. Baez and U. Schreiber, Higher gauge theory, categories in algebra, geometry and mathematical physics, (eds. A. Davydov *et al*)., *Contemp. Math., Amer. Math. Soc.*, **431** (2007), 7-30.
6. J. C. Baez and D. K. Wise, *Teleparallel gravity as a higher gauge theory*, http://arxiv.org/abs/1204.4339 (2012).
7. J. W. Barrett, Holonomy and path structures in general relativity and Yang-Mills Theory, *International Journal of Theoretical Physics*, **30** (1991), 1171-1215.
8. T Bartels, *Higher Gauge Theory I: 2-bundles*, at http://arxiv.org/abs/math/0410328 (2006).
9. G. E. Bredon, Sheaf theory, *Graduate texts in mathematics*, **170** (2nd ed.), Berlin, New York: Springer-Verlag (1997).
10. L. Breen and W. Messing, Differential geometry of gerbes, *Adv. Math.*, **198**(2) (2005), 732-846.
11. G. C. L. Brümmer, Topological categories, *Topology and its applications*, **18** (1984), 27-41.
12. M. Bunge, Stack completions and morita equivalence for category objects in a topos, *Cahiers de Top et Geo Diff Cat.*, **20** (1979), 401-436, online at http://archive.numdam.org/ARCHIVE/CTGDC/CTGDC_1979__20_4/CTGDC_1979__20_4_401_0/CTGDC_1979__20_4_401_0.pdf.
13. M. Bunge and R. Pare, Stacks and equivalence of indexed categories, *Cahiers de Top et Geo Diff Cat.*, **20** (1979), 373-399, online at http://archive.numdam.org/ ARCHIVE/CTGDC/CTGDC_1979__20_4/ CTGDC_1979__20_4_373_0/CTGDC_1979__20_4_373_0.pdf.

14. A. Caetano and R. F. Picken, An axiomatic definition of holonomy, *Internat. J. Math.*, **5**(6) (1994), 835-848.

15. S. Chatterjee, *Categorified presheaves and sieves* at http://arxiv.org/abs/1603.00807, (2016).

16. S. Chatterjee, A. Lahiri and A. N. Sengupta, Parallel transport over path spaces, *Reviews in Math. Phys.*, **9** (2010), 1033-1059.

17. S. Chatterjee, A. Lahiri and A. N. Sengupta, Twisted categorical bundles, *Journal of Geom. Phys.*, **98** (2015), 128-149.

18. S. Chatterjee, A. Lahiri and A. N. Sengupta, Path space connections and categorical geometry, *Journal of Geom. Phys.*, **75** (2014), 1033-1059.

19. S. Chatterjee, A. Lahiri and A. N. Sengupta, Twisted actions of categorical groups, *Theory and applications of categories*, **29** (2014), 215-255.

20. S. Chatterjee, A. Lahiri and A. N. Sengupta, *Connections on Decorated Path Bundles* at http://arxiv.org/abs/1502.05419 (2015).

21. S. Chatterjee, A. Lahiri and A. N. Sengupta, *Construction of categorical bundles from local data* at http://arxiv.org/abs/1602.04163, (2016).

22. M Forrester-Barker, *Group Objects and Internal Categories*, at http://arxiv.org/abs/math/0212065, (2002).

23. K. Gawędzki and Nuno Reis, WZW branes and gerbes, *Rev. Math. Phys.*, **14**(12) (2002), 1281-1334.

24. F. Hirzebruch, *Topological methods in algebraic geometry*, Classics in Mathematics, Berlin, New York: Springer-Verlag (1995).

25. G. M. Kelly and R. Street, Reviews of the elements of 2-categories, *Springer lecture notes in mathematics*, **420**, Berlin, (1974), 75-103.

26. T. Lévy, *Two-dimensional Markovian holonomy fields*, Astérisque, **329** (2010).

27. S. Mac Lane, *Categories for the Working Mathematician*, Springer-Verlag (1971).

28. S. Mac Lane and I. Moerdijk, *Sheaves in Geometry and Logic: A first introduction to topos theory*, Springer-Verlag (1992).

29. J. F. Martins and R. Picken, On two-dimensional holonomy, *Trans. Amer. Math. Soc.*, **362**(11) (2010), 5657-5695, http://arxiv.org/abs/0710.4310.

30. J. F. Martins and R. Picken, Surface holonomy for non-abelian 2-bundles via double groupoids, *Advances in Mathematics*, **226**(4) (2011), 3309-3366.

31. A. Parzygnat, *Gauge invariant surface holonomy and monopoles*, at http://arxiv.org/pdf/1410.6938v1.pdf, (2014).

32. U. Schreiber and K. Waldorf, Parallel transport and functors, *J. Homotopy Relat. Struct.*, **4**(1), (2009), 187-244.

33. U. Schreiber and K. Waldorf, Connections on non-abelian Gerbes and their Holonomy, *Theory and applications of categories*, **28** (2013), 476-540.

34. E. Soncini and R. Zucchini, *A new formulation of higher parallel transport in higher gauge theory* at http://arxiv.org/abs/1410.0775, (2014).

35. D. Viennot, *Non-abelian higher gauge theory and categorical bundle*, at http://arxiv.org/abs/1202.2280, (2012).

36. A. Vistolli, Grothendieck topologies, fibered categories and descent theory, *Fundamental algebraic geometry, Math. Surveys Monogr, 123, Amer. Math. Soc., Providence, RI*, (2005), 1-104.