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RATIONAL CHEBYSHEV COLLOCATION METHOD FOR THE SIMILARITY SOLUTION OF TWO DIMENSIONAL STAGNATION POINT FLOW

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In this study, we propose an efficient and accurate numerical technique that is called the rational Chebyshev collocation (RCC) method to solve the two dimensional flow of a viscous fluid in the vicinity of a stagnation point named Hiemenz flow. The Navier-Stokes equations governing the flow, are reduced to a third-order ordinary differential equation of a boundary value problem with a semi-infinite domain by using similarity transformation. The rational Chebyshev method reduces this nonlinear ordinary differential equation to a system of algebraic equations. This technique is a powerful type of the collocation methods for solving the boundary value problems over a semi-infinite interval without truncating it to a finite domain. We also present the comparison of this work with others and show that the present method is more accurate and efficient.

Key words : Hiemenz flow; stagnation point; collocation method; rational Chebyshev; boundary value problem.

1. Introduction

The two-dimensional stagnation point flow is one of the oldest problems in fluid dynamics. The fluid flow near a stagnation point is named stagnation flow or stagnation point flow and the stagnation area is where fluid pressure and the rates of heat and mass transfer are highest.

The stagnation flow has been studied during past decades because of technical importance in many industrial applications, such as the cooling of electronic components and gas turbine blades, the drying of papers and films, the tempering of glass and metal during processing and surface painting.

The two-dimensional stagnation point flow was first investigated by Hiemenz [19] and thus the plane stagnation point flow is widely known as the Hiemenz flow. He illustrated that the Navier-Stokes equations of this problem can be simplified to a third-order ordinary differential equation using similarity transformation. Hiemenz's solution was later improved by Howarth [21]. Subsequently, three-dimensional axisymmetric stagnation point flow was studied by Homann [20]. Goldstein [15] shows that Hiemenz's solution can be obtained without applying the simplifications of boundary layer theory. Howarth [22] and Davey [11] extended the two-dimensional and axisymmetric flows to three dimensions. Later, Cheng *et al*. [10] investigated the three-dimensional unsteady stagnation flow.

Because of the absence or the complexity of analytical solutions, the reduced differential equation is usually solved numerically with two point boundary conditions, that one of which is defined on infinity. Some attentions are needed in the solution of the differential boundary value problem (BVP) because of the asymptotic boundary condition [26]. Recently, spectral collocation methods have been successfully used to solve the boundary value problems defined on unbounded domains [24].

Spectral methods are very efficient and applicable methods for solving differential equations and generally are a member of the family of weighted residual methods. Spectral methods exhibit a special group of approximation techniques, that the residuals (or errors) are minimized in a particular way and hence generate the specific methods such as the Galerkin, collocation and Tau formulations [2]. In many studies, various kinds of spectral methods are investigated for solving problems in bounded domains or with special boundary conditions [4, 9, 14, 29, 30] but, many problems exist in science and engineering that are defined in the unbounded intervals. We can apply various spectral methods to solve problems in infinite domains and semi-infinite intervals. The different options for dealing with unbounded domains are categorized into three major groups:

The first approach is the applying of polynomials that are orthogonal over unbounded domains with respect to a weight function, such as the Hermite and Laguerre spectral methods [12, 13, 16, 17, 23, 28, 31].

The second approach is truncating infinite domain (-∞,∞) to $[-L, L]$ interval and semi-infinite domain $[0, \infty)$ to $[0, L]$ interval by choosing L large enough. This method is called domain truncation [4].

The third approach is established upon rational orthogonal functions [4, 30]. Boyd [5] introduce a new spectral basis on the semi-infinite interval, called rational Chebyshev functions, by mapping to the Chebyshev polynomials.

In this investigation, we use the RCC method to analyze the Hiemenz flow and then the nonlinear equations governing the two-dimensional Hiemenz flow are solved and analyzed.

2. Problem Formulation

We consider the two-dimensional, laminar, steady, incompressible flow of a viscous fluid impinging normal to an infinite plane situated at $y = 0$. A model of the flow is shown in Fig. 1 in Cartesian coordinates (x, y) with corresponding velocity components (u, v). For the steady, two-dimensional stagnation point flow, the velocity (U, V) in the potential flow is given by:

$$U = a\,x\,, \quad V = -a\,y \tag{1}$$

where a being a constant. By considering the boundary layer approximations, the equations of continuity and momentum become:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{2}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \upsilon\left(\frac{\partial^2 u}{\partial x^2} + \frac{d^2 u}{dy^2}\right) \tag{3}$$

where p, ρ and υ are the fluid pressure, density, and kinematic viscosity respectively. The boundary conditions of the velocity field are:

$$y = 0 \qquad : \qquad u = 0\,,\, v = 0 \tag{4}$$

$$y \to \infty \qquad : \qquad u = U = ax \tag{5}$$

Here, equations (4) are no-slip conditions on the plane and the relation (5) show that the viscous flow solution approaches the potential flow solution, as $y \to \infty$.

By introducing the similarity transformations in the form:

$$u = axf'(\eta) \quad , \quad v = -\sqrt{a\upsilon}\,f(\eta) \quad , \quad \eta = \sqrt{\frac{a}{\upsilon}}y \tag{6}$$

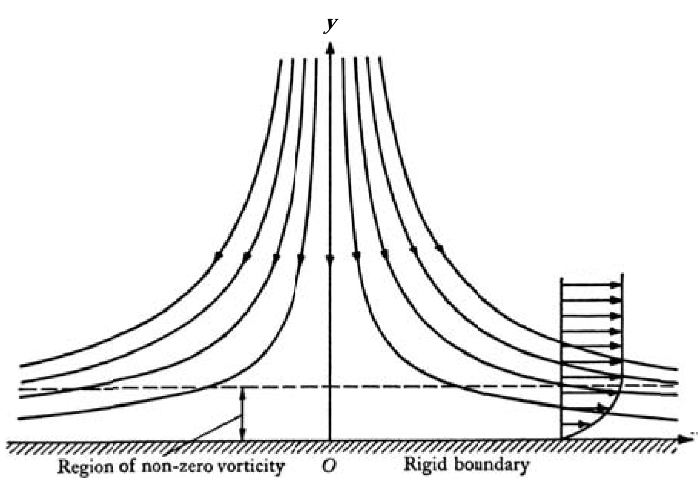

Figure 1 : Schematic diagram of a stationary plane situated at $y = 0$ under Hiemenz flow

The momentum equation (3) reduces to a nonlinear ordinary differential equation as the following:

$$f''' + ff'' - (f')^2 + 1 = 0 \tag{7}$$

and boundary conditions (4), (5) become:

$$\begin{cases} \eta = 0 & : \quad f = 0\,,\ f' = 0 \\ \eta \to \infty & : \quad f' = 1 \end{cases} \tag{8}$$

Equation (7) is the same as the one obtained by Hiemenz [19, 27] and have been solved by using the fourth-order Runge-Kutta method of numerical integration. Here, the equation (7) is solved by applying the RCC method.

3. Shear Stress

For boundary layer flow, the shear stress at the wall surface or the wall skin friction τ_w is calculated from:

$$\tau_w = \mu \frac{\partial u}{\partial y}\bigg|_{y=0} \tag{9}$$

where μ is the fluid viscosity. Using the definition (9), the surface shear stress becomes:

$$\tau_w = \mu\, a \sqrt{\frac{a}{\upsilon}}\, x\, f''(0) \tag{10}$$

Thus, $f''(0)$ is proportional to surface shear stress. Because of their relation to physical quantities, we obtain the f, f' and $f''(0)$ in our results.

4. Rational Chebyshev Polynomials

The aim of this work is to apply an important type of spectral methods called the RCC method for solving the boundary value problem (7). We present the rational Chebyshev polynomials and some of their important properties [2].

The well-known Chebyshev polynomial $T_l(\xi)$ is the l-th normalized eigenfunction of the singular Sturm-Liouville problem:

$$\sqrt{1-\xi^2}\Big[\sqrt{1-\xi^2}\, T_l{}'(\xi)\Big]' + l^2\, T_l(\xi) = 0\,, \qquad \xi \in (-1, 1)$$

Also, the Chebyshev polynomials satisfy the following recurrence formula:

$$T_0\,(\xi) = 1, \quad T_1\,(\xi) = \xi,$$

$$T_{n+1}(\xi) = 2\xi T_n(\xi) - T_{n-1}(\xi), \quad n \geq 1$$

which are orthogonal in the interval $[-1, 1]$ with respect to the weight function $\omega(\xi) = \frac{1}{\sqrt{1-\xi^2}}$ i.e.,

$$\int_{-1}^{1} T_i(\xi)\, T_j(\xi)\, \omega(\xi)\, d\xi = \frac{c_i \pi}{2} \delta_{ij}$$

where $c_0 = 2$, $c_i = 1$ for $i \geq 1$ and δ_{ij} is the Kronecker function. As mentioned above, it is clear that the well-known Chebyshev polynomials are valid only for $\xi \in [-1, 1]$. For problems with semi-infinite domain, we use a transformation that maps a semi-infinite interval into a finite domain. By this mapping, we obtain new basis sets for the semi-infinite interval [1]. Boyd [4, 6-8] offered algebraic mapping in the following form:

$$\tau = \frac{L(1+\xi)}{1-\xi} \;\leftrightarrow\; \xi = \frac{\tau - L}{\tau + L}$$

where L is constant. The presented algebraic mapping for every fixed L, maps the semi-infinite interval $[0, \infty)$ into $[-1, 1]$. Thus, new basis sets $R_l(\tau)$ are generated for the semi-infinite interval that are the images under the change-of-coordinate of Chebyshev polynomials:

$$R_l(\tau) = T_l\left(\frac{\tau - L}{\tau + L}\right) = \cos(lt), \quad t = 2\cot^{-1}\left(\sqrt{\frac{\tau}{L}}\right), \quad t \in [0, \pi] \tag{11}$$

So the rational Chebyshev polynomials $R_n(\tau)$ can be defined as the following three-term recurrence relations:

$$R_0(\tau) = 1\,, \quad R_1(\tau) = \frac{\tau - L}{\tau + L}\,,$$

$$R_{n+1}(\tau) = 2\left(\frac{\tau - L}{\tau + L}\right) R_n(\tau) - R_{n-1}(\tau)\,, \quad n \geq 1$$

It can be shown that $R_l(\tau)$ is the l-th eigenfunction of the singular Sturm-Liouville problem:

$$(\tau + L)\frac{\sqrt{\tau}}{L}\left[(\tau + L)\sqrt{\tau} R_l'(\tau)\right]' + l^2 R_l(\tau) = 0\,, \quad \tau \in (0, \infty)$$

and rational Chebyshev polynomials are orthogonal with respect to the weight function $\omega(\tau) = \frac{\sqrt{L}}{\sqrt{\tau}(L+\tau)}$ in the interval $[0, \infty)$, with the orthogonality property:

$$\int_0^{\infty} R_i(\tau)\, R_j(\tau)\, \omega(\tau)\, d\tau = \frac{c_i \pi}{2} \delta_{ij} \tag{12}$$

where $c_0 = 2$, $c_i = 1$ for $i \geq 1$.

Here, we introduce basic properties of the rational Chebyshev polynomials [2].

Let $\Omega = [0, \infty)$ and $\omega(\tau) = \frac{\sqrt{L}}{\sqrt{\tau}(L+\tau)}$ be a non-negative, integrable, real valued weight function over the semi-infinite interval Ω. We define a normed space $L^2_\omega(\Omega)$, as follows:

$$L^2_\omega\left(\Omega\right) = \{\upsilon\,|\upsilon \text{ is measurable on } \Omega \text{ and } ||\upsilon||_\omega \le \infty\}$$

where

$$||\upsilon||_\omega = \left(\int_0^\infty |\upsilon\left(\tau\right)|^2 \omega\left(\tau\right) d\tau\right)^{\frac{1}{2}}$$

and $||\,.\,||_\omega$ is the norm induced from the inner product $\langle\,.\,,\,.\,\rangle_\omega$ of the space $L^2_\omega(\Omega)$, i.e.,

$$\langle u, \upsilon\rangle_\omega = \int_0^\infty \upsilon\left(\tau\right) u\left(\tau\right) \omega\left(\tau\right) d\tau$$

Hence, from the orthogonality relation of rational Chebyshev polynomials (12), we get that the rational Chebyshev polynomials $R_l(\tau)$ make a set of complete orthogonal basis for $L^2_\omega(\Omega)$ [18, 25].

For any function $f \in L^2_\omega(\Omega)$, we have the following expansion:

$$f\left(\tau\right) = \sum_{i=0}^{\infty} f_i\, R_i\left(\tau\right) \tag{13}$$

with

$$f_i = \frac{\langle f, R_i\rangle_\omega}{\|R_i\|^2_\omega} = \frac{2}{c_i \pi}\int_0^\infty f\left(\tau\right) R_i\left(\tau\right) \omega\left(\tau\right) d\tau$$

where f_i's are the expansion coefficients associated with the family $\{R_i\}_{i\ge 0}$.

5. Rational Chebyshev Collocation Method

For any positive integer N, we define $\Re_N = \text{span}\,\{R_0,\, R_1, \ldots, R_N\}$ and consider the following spectral approximation:

$$f_N\left(\tau\right) = \sum_{k=0}^{N} f_k R_k\left(\tau\right) \tag{14}$$

The main idea of the collocation method is to obtain the coefficients f_k such that the residual function vanishes in the interior collocation points $\{\tau_j\}_{j=0}^N$. In the presented method for solving the problem (7) with boundary conditions (8), we employ the following $N+1$ rational Chebyshev-Gauss-Radau points as the collocation points:

$$\tau_j = L\frac{1+\xi_j}{1-\xi_j} \quad , \quad j = 0, 1, \ldots, N \tag{15}$$

where ξ_j's are the $N+1$ Chebyshev-Gauss-Radau points:

$$\xi_j = -\cos\left(\frac{2j\pi}{2N+1}\right) \quad , \quad j = 0, 1, \ldots, N$$

Therefore, we have a system of nonlinear equations with $N+1$ equations and $N+1$ unknowns f_k (the expansion coefficients of $f_k(\tau)$), that can be solved numerically by Newton's method.

6. Convergence of RCC Method

To investigate the convergence of rational Chebyshev method, we introduce the orthogonal projection [1].

In general, the $L^2_\omega(\Omega)$-Orthogonal projection is defined as the following:

$$P_N : L^2_\omega(\Omega) \to \Re_N \quad \text{by: } \langle P_N f - f, \phi \rangle_\omega = 0\,, \quad \forall \phi \in \Re_N$$

where $P_N f(\tau) = f_N(\tau)$.

The equation (14) shows that f_N is the orthogonal projection of f upon $\Re_N$ with respect to the weighted inner product $\langle\, .\,,\, .\,\rangle_\omega$.

Now, in order to estimate $\|P_N f - f\|_\omega$, we define the normed space:

$$H^r_\omega(\Omega) = \left\{ \upsilon \,|\, \upsilon \text{ is measurable on } \Omega \text{ and } ||\upsilon||_{r,\omega} < \infty \right\}$$

where for the non-negative integer r, the norm is induced by:

$$||\upsilon||_{r,\omega} = \left(\sum_{k=0}^{r} \left\| (\tau+1)^{\frac{r}{2}+k} \frac{d^k}{d\tau} \upsilon \right\|^2_\omega \right)^{\frac{1}{2}}$$

Therefore, we present the following theorem for the convergence.

Theorem — *For any $f \in H^r_\omega(I)$ and $f \geq 0$,*

$$||P_N f - f||_\omega \leq c N^{-r} ||f||_{r,\omega}$$

Proof : see [18].

It is clear from this theorem that the rational Chebyshev approximation is convergent exponentially.

7. Applying RCC Method for Hiemenz Flow

Now, we employ the RCC method for solving the problem (7) with the boundary conditions (8).

We apply $f_N(\tau)$ on the function $f(\tau)$ in equation (7). Note that from the definitions of $R_N(\tau)$, $f_N(\tau)$, we have $R_i'(\infty) = 0$ for $i = 0, 1, \ldots, N$, $f_N'(\infty) = 0$. To satisfy the boundary conditions (8), an extra simple term is added to the equation (14) and the following approximation is considered:

$$\tilde{f}_N(\tau) = \tau + \sum_{k=0}^{N} f_k\, R_k(\tau) \tag{16}$$

where $\tilde{f}_N'(\infty) = 1$. Thus, the boundary condition $f'(\infty) = 1$ is already satisfied. Now, if we replace $f(\tau)$ with approximate solution $\tilde{f}_N(\tau)$ into the equation (7), then we obtain the residual function as the following form:

$$\mathrm{Res}(\tau) = \tilde{f}_N'''(\tau) + \tilde{f}_N(\tau)\,\tilde{f}_N''(\tau) - \left(\tilde{f}_N'(\tau)\right)^2 + 1 \tag{17}$$

As mentioned above, for earning the coefficients f_k, we equalize the equation (17) to zero at rational Chebyshev-Gauss-Radau collocation points (15). So, we have:

$$\begin{cases} \mathrm{Res}_N(\tau_j) = 0\,, \quad j = 1, 2, \ldots, N-1 \\ \tilde{f}_N(0) = 0 \\ \tilde{f}_N'(0) = 0 \end{cases} \tag{18}$$

System (18) contains $N+1$ nonlinear equations, and we solve numerically by Newton's method.

The complexity of the RCC method is the finding of the suitable map parameter L. To overcome this problem, Boyd [7] presented some suggestions for optimizing the map parameter L.

8. Presentation of Results

In the section of results, the rational Chebyshev solution of the equation (7) with the boundary conditions (8) for different number of the collocation points, N, with obtaining suitable L, is presented. As mentioned earlier, the $f''(0)$ is proportional to surface shear stress and is an important point of the function. Hence, we have calculated it. Also, in order to checking the accuracy of the results of RCC method, a fourth-order Runge-Kutta method along with a shooting method has been used for solving the equation (7) and the errors of the RCC method is calculated respect to this fourth-order Runge-Kutta method.

The approximations of the $f''(0)$ computed by the RCC method for several values of N, with obtaining suitable L and their absolute errors have been shown in Table 1. The absolute errors have been calculated with respect to the fourth-order Runge-Kutta solution. From Table 1, it is seen that by increasing the number of the collocation points and obtaining suitable L, the absolute values of the errors decrease which shows the stability and rapid convergence of the RCC method.

Table 1: Numerical results for the $f''(0)$ and their absolute errors for several values of N

N	***L***	$\boldsymbol{f''}(0)$	**Error**
5	0.75	1.2298144887	2.773×10^{-03}
10	1.62	1.2315877500	9.999×10^{-04}
15	2.33	1.2324955242	9.211×10^{-05}
20	2.24	1.2325836168	4.014×10^{-06}
25	2.76	1.2325881629	5.317×10^{-07}
30	3.24	1.2325877115	8.030×10^{-08}
35	3.76	1.2325876085	2.270×10^{-08}
40	4.43	1.2325876434	1.220×10^{-08}
Runge-Kutta		1.2325876312	

The logarithmic graph of the absolute coefficients $|f_i|$ of the rational Chebyshev functions in the approximate solutions for $N = 40$ by choosing suitable $L = 4.43$ has been shown in Fig. 2. The graph shows the stability and convergence of the RCC method.

The exponential index of convergence, r, has been shown in Fig. 3 that r is calculated as following:

$$r \equiv \lim_{i\to\infty} \frac{\log|\log(|f_i|)|}{\log(i)}$$

from the Fig. 3, it is seen that $r < 1$ and according to Boyd [4], we conclude that the spectral approximation (14) has subgeometric convergence.

The comparison between the numerical solution given by Howarth [21] and approximation solution of the problem (7) with RCC method have been shown in Fig. 4. This figure displays that the velocity profile $f'(\eta)$ obtained by the RCC method agree with the boundary conditions (8). Also, between the results obtained by the RCC method and the Howarth values for all values of η, a very good adaption is seen.

The comparison of the $f''(0)$ calculated by the present work with the values obtained by Wang [32], Howarth [21] and calculated by the fourth-order Runge-Kutta method has been given in Table 2; which shows the introduced solution is highly accurate.

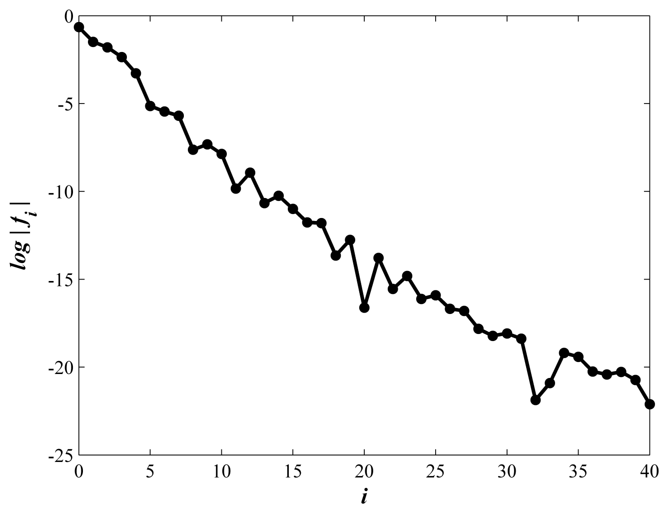

Figure 2 : Logarithmic graph of absolute coefficients $|f_i|$ of rational Chebyshev functions in approximate solution for $N = 40$ and $L = 4.43$

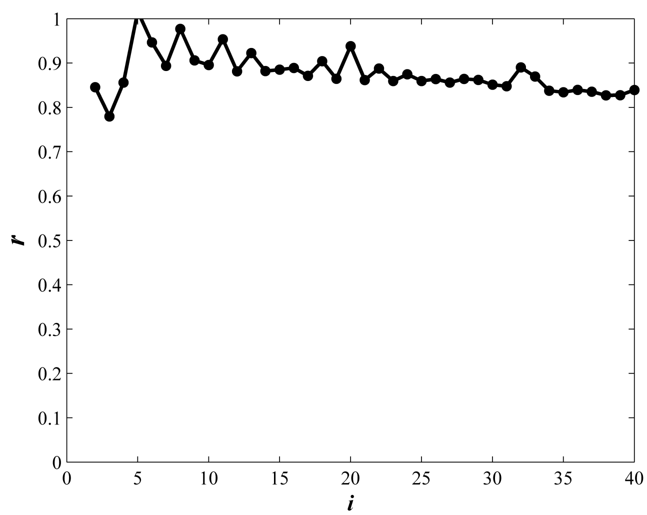

Figure 3 : The exponential index of convergence r versus i

In Table 3, the approximations of the $f''(0)$ calculated by the RCC method together with their absolute errors for large numbers of the collocation points, $N = 30, 35, 40$ and different choices of L from 1 to 8, have been shown. It is seen that for $N = 30$ which N is a sufficiently large number, for all values of L in this interval, the results are the same as that of Runge-Kutta to 5 digits decimal. Also for $N = 35$ to 6 digits decimal and for $N = 40$ to 7 digits decimal. So we can conclude that if N is chosen large enough, almost every L in this interval achieves good results and not need to calculate optimized L, which is complicated and time consuming process.

Table 3: Numerical results for $f''(0)$ and their absolute errors for several values of L and N.

	$\boldsymbol{N} = 30$		$\boldsymbol{N} = 35$		$\boldsymbol{N} = 40$	
L	$\boldsymbol{f''}(0)$	**Error**	$\boldsymbol{f''}(0)$	**Error**	$\boldsymbol{f''}(0)$	**Error**
1.0	1.23254296	4.47×10^{-05}	1.23258300	4.63×10^{-06}	1.23258980	2.17×10^{-06}
1.5	1.23258242	5.21×10^{-06}	1.23258675	8.83×10^{-07}	1.23258809	4.60×10^{-07}
2.0	1.23259151	3.88×10^{-06}	1.23258700	6.31×10^{-07}	1.23258774	1.09×10^{-07}
2.5	1.23258579	1.84×10^{-06}	1.23258794	3.11×10^{-07}	1.23258761	2.37×10^{-08}
3.0	1.23258897	1.34×10^{-06}	1.23258751	1.17×10^{-07}	1.23258767	3.82×10^{-08}
3.5	1.23258668	9.50×10^{-07}	1.23258778	1.46×10^{-07}	1.23258765	2.05×10^{-08}
4.0	1.23258794	3.10×10^{-07}	1.23258754	9.02×10^{-08}	1.23258766	3.28×10^{-08}
4.5	1.23258823	6.02×10^{-07}	1.23258772	9.43×10^{-08}	1.23258764	1.43×10^{-08}
5.0	1.23258704	5.95×10^{-07}	1.23258769	5.64×10^{-08}	1.23258767	3.75×10^{-08}
5.5	1.23258724	3.86×10^{-07}	1.23258756	6.69×10^{-08}	1.23258765	2.19×10^{-08}
6.0	1.23258827	6.35×10^{-07}	1.23258766	3.44×10^{-08}	1.23258765	1.66×10^{-08}
6.5	1.23258832	6.89×10^{-07}	1.23258775	1.24×10^{-07}	1.23258767	3.51×10^{-08}
7.0	1.23258739	2.45×10^{-07}	1.23258766	3.22×10^{-08}	1.23258768	3.30×10^{-08}
7.5	1.23258667	9.58×10^{-07}	1.23258754	8.57×10^{-08}	1.23258764	1.46×10^{-08}
8.0	1.23258697	6.58×10^{-07}	1.23258758	5.40×10^{-08}	1.23258764	1.41×10^{-08}

Table 4: Approximation of $f(\eta)$ for present method, [2], [3] and Runge-Kutta method

η	**Howarth** [21]	**Ref** [3]	**Runge-Kutta**	**RCC** ($\boldsymbol{N} = 40$)
0.0	0	0	0	0
0.2	0.0233	0.233355	0.0233222492	0.0233222570
0.6	0.1867	0.186715	0.1867009886	0.1867009935
1.0	0.4592	0.459236	0.4592270144	0.4592270171
1.4	0.7966	0.796657	0.7966517822	0.7966517836
1.8	1.1688	1.168855	1.1688554750	1.1688554755
2.0	1.3619	1.361968	1.3619741617	1.3619741619
2.4	1.7552	1.755238	1.7552538771	1.7552538766
2.8	2.1529	2.152965	2.1529965081	2.1529965067
3.0	2.3525	2.352516	2.3525566765	2.3525566747

rational Chebyshev polynomials as the basis functions. Note that these basis functions have some advantages: easy to compute, rapid convergence and completeness. This method reduces the solution of a nonlinear ordinary differential equation to the solution of a system of algebraic equations. The comparison between the numerical solution given by Howarth [21], Wang [32], Abbasbandy *et al.* [3], the fourth-order Runge-Kutta solution and approximated by the current work, shows that RCC method provides more accurate and numerically stable solutions than those obtained by other methods and demonstrates the validity of the present method for boundary value problems.

Table 5: Comparison between RCC solution and Runge-Kutta solution for $f(\eta)$, $f'(\eta)$ and $f''(\eta)$ with $N = 40$ and $L = 4.43$.

	f		f'		f''	
η	**Runge-Kutta**	**RCC (*N*=40)**	**Runge-Kutta**	**RCC (*N*=40)**	**Runge-Kutta**	**RCC (*N*=40)**
0.0	0	0	0	0	1.23258763	1.23258764
0.2	0.02332225	0.02332226	0.22661243	0.22661242	1.03445417	1.03445419
0.4	0.08805658	0.08805659	0.41445612	0.41445611	0.84632541	0.84632542
0.6	0.18670099	0.18670099	0.56628053	0.56628052	0.67517140	0.67517142
0.8	0.31242302	0.31242302	0.68593746	0.68593745	0.52513133	0.52513134
1.0	0.45922701	0.45922702	0.77786528	0.77786527	0.39801295	0.39801295
1.4	0.79665178	0.79665178	0.89680865	0.89680865	0.21100317	0.21100317
1.8	1.16885547	1.16885547	0.95683379	0.95683379	0.09996382	0.09996382
2.0	1.36197416	1.36197416	0.97321674	0.97321674	0.06582538	0.06582538
2.4	1.75525388	1.75525388	0.99054940	0.99054940	0.02602026	0.02602026
2.8	2.15299651	2.15299651	0.99704567	0.99704567	0.00904886	0.00904886
3.0	2.35255668	2.35255667	0.99842416	0.99842416	0.00507796	0.00507796
3.2	2.55232541	2.55232540	0.99918606	0.99918606	0.00275489	0.00275489
3.6	2.95214967	2.95214967	0.99980325	0.99980325	0.00073126	0.00073126
4.0	3.35210930	3.35210930	0.99995843	0.99995843	0.00016867	0.00016867

References

1. S. Abbasbandy, H. R. Ghehsareh and I. Hashim, An approximate solution of the MHD flow over a non-linear stretching sheet by rational Chebyshev collocation method, *U.P.B. Sci. Bull., Ser. A*, **74** (2012), 47-58.
2. S. Abbasbandy, T. Hayat, H. R. Ghehsareh and A. Alsaedi, MHD Falkner-Skan flow of Maxwell fluid by rational Chebyshev collocation method, *Appl. Math. Mech.*, **34** (2013), 921-930.
3. S. Abbasbandy, K. Parand, S. Kazem and A. R. Sanaei Kia, A numerical approach on Hiemenz flow problem using radial basis functions, *Eur. Int. J. Industrial Math.*, **5**(1) (2013), 65-73.
4. J. P. Boyd, *Chebyshev and Fourier spectral methods, 2nd Ed.*, Springer, Berlin, (2000).
5. J. P. Boyd, Orthogonal rational functions on a semi-infinite interval, *J. Comput. Phys*., **70**(1) (1987), 63-88.
6. J. P. Boyd, Spectral methods using rational basis functions on an infinite interval, *J. Comput. Phys*., **69** (1987), 112-142.
7. J. P. Boyd, The optimization of convergence for Chebyshev polynomial methods in an unbounded domain, *J. Comput. Phys*., **45** (1982), 43-79.
8. J. P. Boyd, C. Rangan and P. H. Bucksbaum, Pseudospectral methods on a semi-infinite interval with application to the hydrogen atom: A comparison of the mapped Fourier-sine method with Laguerre series and rational Chebyshev expansions, *J. Comput. Phys*., **188** (2003), 56-74.
9. C. Canuto, M. Y. Hussaini, A. Quarteroni and T. A. Zang, *Spectral methods in fluids dynamics*, Springer, Berlin, (1990).
10. E. H. W. Cheng, M. N. Ozisik and J. C. William, Nonsteady three-dimensional stagnation point flow, *J. Appl. Mech*., **38** (1971), 282-287.
11. A. Davey, Boundary layer flow at a saddle point of attachment, *J. Fluid Mech*., **10** (1961), 593-610.
12. D. Funaro, Computational aspects of pseudospectral Laguerre approximations, *Appl. Numer. Math*., **6** (1990), 447-457.
13. D. Funaro and O. Kavian, Approximation of some diffusion evolution equations in unbounded domains by Hermite functions, *Math. Comput*., **57** (1991), 597-619.
14. H. R. Ghehsareh, B. Soltanalizadeh and S. A. Abbasbandy, A matrix formulation to the wave equation with non-local boundary condition, *Int. J. Comput. Math*., **88** (2011), 1681-1696.
15. S. Goldstein, *Modern developments in fluid dynamics*, Oxford University Press, (1938).
16. B. Y. Guo, Error estimation of Hermite spectral method for nonlinear partial differential equations, *Math. Compt*., **68**(227) (1999), 1067-1078.
17. B. Y. Guo and J. Shen, Laguerre-Galerkin method for nonlinear partial differential equations on a semi-infinite interval, *Numer. Math*., **86** (2000), 635-654.

18. B. Y. Guo, J. Shen and Z. Q. Wang, Chebyshev rational spectral and pseudospectral methods on a semi-infinite interval, *Int. J. Numer. Meth. Engrg.*, **53** (2002), 65-84.
19. K. Hiemenz, Die Grenzschicht an einem in den gleichförmigen Flüssigkeitsstrom eingetauchten geraden Kreiszylinder, *Dingl. Polytech. J.*, **326** (1911), 321-410.
20. F. Homann, Der Einfluss grosser Zähigkeit bei der Strömung um den Zylinder und um die Kugel, *ZAMM Z. Angew. Math. Mech.*, **16** (1936), 153-164.
21. L. Howarth, On the calculation of steady flow in the boundary layer near the surface of a cylinder in a stream, *Aeron. Res. Comm., Rep. Mem.*, **1632** (1934), 1-12.
22. L. Howarth, The boundary layer in three-dimensional flow. Part-II: The flow near a stagnation point, *Phil. Mag., Ser. VII*, **42**(335) (1951), 1433-1440.
23. Y. Maday, B. Pernaud-Thomas and H. Vandeven, Reappraisal of Laguerre type spectral methods, *La Rech. Aerospatiale*, **6** (1985), 13-35.
24. K. Parand, Z. Delafkar and F. Bahari fard, Rational Chebyshev Tau method for solving natural convection of Darcian fluid about a vertical full cone embedded in porous media whit a prescribed wall temperature, *World Academy of Science, Engineering and Technology*, **5**(8) (2011), 1186-1191.
25. K. Parand and M. Shahini, Rational Chebyshev pseudospectral approach for solving Thomas-Fermi equation, *Phys. Lett. A*, **373** (2009), 210-213.
26. B. Sahoo and F. Labropulu, Steady Homann flow and heat transfer of an electrically conducting second grade fluid, *Comput. Math. Appl.*, **63** (2012), 1244-1255.
27. H. Schlichting, *MHD Boundary layer theory, translate by J. Kestin*, McGraw-Hill, (1968).
28. J. Shen, Stable and efficient spectral methods in unbounded domains using Laguerre functions, *SIAM J. Numer. Anal.*, **38**(4) (2000), 1113-1133.
29. J. Shen and T. Tang, *High order numerical methods and algorithms*, Chinese Science Press, Beijing, (2005).
30. J. Shen, T. Tang and L. L. Wang, *Spectral Methods, Algorithms, Analysis and Applications*, Springer, Berlin, (2010).
31. H. Siyyam, Laguerre Tau methods for solving higher order ordinary differential equations, *J. Comput. Anal. Appl.*, **3**(2) (2001), 173-182.
32. C. Y. Wang, Similarity stagnation point solutions of the Navier-Stokes equations- review and extension, *Eur. J. Mech. B/Fluids*, **27** (2008), 678-683.