Indian J. Pure Appl. Math., **49**(3): 527-537, September 2018

DOI: 10.1007/s13226-018-0282-7

NEW IDENTITIES INVOLVING GENERALIZED FIBONACCI AND GENERALIZED LUCAS NUMBERS

W. M. Abd-Elhameed and N. A. Zeyada
Department of Mathematics, Faculty of Science, Cairo University,
Giza, Egypt
e-mails: walee_9@yahoo.com; nzeyada@gmail.com

(*Received* 28 *May* 2017; *after final revision* 13 *September* 2017;
accepted 12 *October* 2017)

This paper presents two new identities involving generalized Fibonacci and generalized Lucas numbers. One of these identities generalize the two well-known identities of Sury and Marques which are recently developed. Some other interesting identities involving the famous numbers of Fibonacci, Lucas, Pell and Pell-Lucas numbers are also deduced as special cases of the two derived identities. Performing some mathematical operations on the introduced identities yield some other new identities involving generalized Fibonacci and generalized Lucas numbers.

Key words : Generalized Fibonacci and Lucas numbers; Lucas and Fibonacci numbers; Pell and Pell-Lucas numbers; recurrence relation.

1. Introduction

The Fibonacci, Lucas, Pell and Pell-Lucas sequences $(F_i)_{i\geq 0}$, $(L_i)_{i\geq 0}$, $(P_i)_{i\geq 0}$ and $(Q_i)_{i\geq 0}$, may be constructed by means of the recurrence relations

$$F_{i+2} = F_{i+1} + F_i, \quad F_0 = 0,\ F_1 = 1,$$

$$L_{i+2} = L_{i+1} + L_i, \quad L_0 = 2,\ L_1 = 1,$$

$$P_{i+2} = 2\,P_{i+1} + P_i, \quad P_0 = 0,\ P_1 = 1,$$

and

$$Q_{i+2} = 2\,Q_{i+1} + Q_i, \quad Q_0 = 2,\ Q_1 = 2.$$

The two sequences of Fibonacci and Lucas numbers have been extensively studied in the beautiful book of Koshy [7], while the Pell and Pell-Lucas sequences have been studied in [5, 8]. The Fibonacci numbers and polynomials serve in many applications in several branches (see for example [1, 13]).

Many generalizations of the above sequences of numbers are introduced by many authors. For example, the k Fibonacci sequence generated by the recurrence relation

$$F^k_{i+2} = k\, F^k_{i+1} + F^k_i,\ F^k_0 = 0,\ F^k_1 = 1,\ k \text{ is a nonzero real constant,}$$

is a generalization for Fibonacci sequence. This sequence is considered in a variety of papers (see, for example [3]). Some other generalizations can be found in [4, 11, 14, 15].

In this paper, we introduce the two generalized Fibonacci and Lucas sequences generated by the following two recurrence relations:

$$U^{a,b}_{i+2} = a\, U^{a,b}_{i+1} + b\, U^{a,b}_i, \quad U^{a,b}_0 = 0,\ U^{a,b}_1 = 1, \tag{1}$$

and

$$V^{a,b}_{i+2} = a\, V^{a,b}_{i+1} + b\, V^{a,b}_i, \quad V^{a,b}_0 = 2,\ V^{a,b}_1 = a. \tag{2}$$

The main advantage of introducing the two generalized sequences $(U^{a,b}_i)$ and $(V^{a,b}_i)$ is that several famous sequences can be deduced as special cases of them. In fact, the celebrated sequences of Fibonacci (F_i), Lucas (L_i), Pell (P_i), Pell-Lucas (Q_i) and Mersenne numbers (M_i) are particular sequences of $(U^{a,b}_i)$ and $(V^{a,b}_i)$. Explicitly, we have

$$F_i = U^{1,1}_i, \quad L_i = V^{1,1}_i, \quad P_i = U^{2,1}_i,$$

$$Q_i = V^{2,1}_i, \quad M_i = U^{3,-2}_i.$$

Some other important sequences of numbers can be deduced as special cases of the two generalized sequences $(U^{a,b}_i)$ and $(V^{a,b}_i)$, see [6].

One of the most important properties of the generalized Fibonacci numbers $U^{a,b}_i$ and the generalized Lucas numbers $V^{a,b}_i$ is that they have the following Binet forms:

$$U^{a,b}_i = \frac{\left(a + \sqrt{a^2 + 4\, b}\right)^i - \left(a - \sqrt{a^2 + 4\, b}\right)^i}{2^i \sqrt{a^2 + 4\, b}}, \tag{3}$$

and

$$V^{a,b}_i = \frac{\left(a + \sqrt{a^2 + 4\, b}\right)^i + \left(a - \sqrt{a^2 + 4\, b}\right)^i}{2^i}. \tag{4}$$

Our main aim in this paper is to develop new identities involving the two sequences of numbers namely, generalized Fibonacci and generalized Lucas numbers. These identities enable us to deduce several new identities of the celebrated Fibonacci, Lucas, Pell, and Pell Lucas numbers. The organization of the paper is as follows. The next section presents new identities involving generalized

Fibonacci and generalized Lucas numbers. The two identities of Sury and Marques are also deduced as special cases in this section. Section 3 is devoted to introducing some other new identities involving generalized Fibonacci and generalized Lucas numbers based on performing some mathematical operations on the introduced identities developed in Section 2. Finally, some conclusions are reported in Section 4.

2. Identities Generalizing Identities of Sury and Marques

This section is devoted to stating and proving two theorems, in which two new identities involving generalized Fibonacci and generalized Lucas numbers are given. In addition, several identities of Fibonacci, Lucas, Pell and Pell-Lucas numbers are exhibited including the two identities of Sury and Marques. The following lemma is needed in the sequel.

Lemma 1 — For every nonzero real constant b, the following recurrence relation is satisfied:

$$b\,U_i^{a,b} + U_{i+2}^{a,b} = V_{i+1}^{a,b}. \tag{5}$$

Proof : If we make use of the two substitutions

$$A = a + \sqrt{a^2 + 4\,b}, \quad B = a - \sqrt{a^2 + 4\,b},$$

then it is easy to see that

$$A\,B = -4\,b, \quad A + B = 2\,a, \quad A - B = 2\,\sqrt{a^2 + 4\,b},$$

and therefore, the two Binet forms of $U_i^{a,b}$ and $V_i^{a,b}$ given in (3) and (4) can be written in the following alternative forms:

$$U_i^{a,b} = \frac{A^i - B^i}{2^{i-1}\,(A-B)},$$

and

$$V_i^{a,b} = \frac{A^i + B^i}{2^i}.$$

Now

$$b\,U_i^{a,b}(x) + U_{i+2}^{a,b}(x) = b\,\frac{A^i - B^i}{2^{i-1}\,(A-B)} + \frac{A^{i+2} - B^{i+2}}{2^{i+1}(A-B)}. \tag{6}$$

Equation (6) after performing some algebraic manipulations is turned into

$$b\,U_i^{a,b} + U_{i+2}^{a,b} = \frac{A^{i+1} + B^{i+1}}{2^{i+1}} = V_{i+1}^{a,b}. \tag{7}$$

This completes the proof of Lemma 1. □

***Theorem* 1** — *If m is an arbitrary nonnegative integer, and $t \in \mathbb{R} - \{0\}$, then the following identity holds*

$$t^{m+1}\, U_{m+1}^{a,b} = \frac{1}{a} \sum_{i=0}^{m} t^i \left\{(a\,t-2)U_{i+1}^{a,b} + V_i^{a,b}\right\}. \tag{8}$$

PROOF : We will prove (8) by induction. It is clear that relation (8) is valid for $m = 0$. Now, assume that (8) holds and we have to show that

$$t^{m+2}\, U_{m+2}^{a,b} = \frac{1}{a} \sum_{i=0}^{m+1} t^i \left\{(a\,t-2)U_{i+1}^{a,b} + V_i^{a,b}\right\}. \tag{9}$$

If we make use of the valid relation (8), then the right hand side of (9) can be written as

$$\begin{aligned}\frac{1}{a} \sum_{i=0}^{m+1} t^i \left\{(a\,t-2)U_{i+1}^{a,b} + V_i^{a,b}\right\} &= t^{m+1}\, U_{m+1}^{a,b} + \frac{1}{a}\, t^{m+1}\,(a\,t-2)\, U_{m+2}^{a,b} + \frac{1}{a}\, t^{m+1}\, V_{m+1}^{a,b} \\ &= t^{m+1} \left\{U_{m+1}^{a,b} + \left(t - \frac{2}{a}\right) U_{m+2}^{a,b} + \frac{1}{a}\, V_{m+1}^{a,b}\right\}. \end{aligned} \tag{10}$$

Based on the recurrence relation (5), relation (10) takes the form

$$\begin{aligned}\frac{1}{a} \sum_{i=0}^{m+1} t^i \left\{(a\,t-2)U_{i+1}^{a,b} + V_i^{a,b}\right\} &= t^{m+1} \left\{U_{m+1}^{a,b} + \left(t - \frac{2}{a}\right) U_{m+2}^{a,b} + \frac{1}{a}\left(U_{m+2}^{a,b} + b\, U_m^{a,b}\right)\right\} \\ &= t^{m+1} \left\{t\, U_{m+2}^{a,b} - \frac{1}{a}\left(U_{m+2}^{a,b} - a\, U_{m+1}^{a,b} - b\, U_m^{a,b}\right)\right\}. \end{aligned}$$

Making use of (1), the latter equation can be turned into

$$\frac{1}{a} \sum_{i=0}^{m+1} t^i \left\{(a\,t-2)U_{i+1}^{a,b} + V_i^{a,b}\right\} = t^{m+2}\, U_{m+2}^{a,b}.$$

Theorem 1 is now proved. □

Several identities involving Fibonacci, Lucas, Pell, and Pell-Lucas numbers can be deduced as special cases of Theorem 1. The results are given in the following corollaries.

Corollary 1 — For $a = b = 1$, identity (8) reduces to

$$t^{m+1}\, F_{m+1} = \sum_{i=0}^{m} t^i \left\{(t-2)F_{i+1} + L_i\right\}. \tag{11}$$

The following two identities are obtained as special cases of identity (11).

Corollary 2 — (Sury's identity [9, 12]). For $t = 2$, identity (11) reduces to

$$2^{m+1}\, F_{m+1} = \sum_{i=0}^{m} 2^i\, L_i. \tag{12}$$

Corollary 3 — (Marques' identity [10]). For $t = 3$, identity (11) reduces to

$$3^{m+1}\, F_{m+1} = \sum_{i=0}^{m} 3^i\, L_i + \sum_{i=0}^{m+1} 3^{i-1}\, F_i. \tag{13}$$

Remark 1 : The two identities of Sury and Marques were also deduced through introducing another kind of sequences of generalized numbers [2].

Corollary 4 — If we set $a\,t = 2$ in (8), then we get

$$t^{m+1}\, U_{m+1}^{\frac{2}{t},b} = \frac{1}{2}\, \sum_{i=0}^{m} t^{i+1}\, V_i^{\frac{2}{t},b}. \tag{14}$$

Corollary 5 — If we set $a = 2, b = 1$ in (8), then we get

$$t^{m+1}\, P_{m+1} = \frac{1}{2} \sum_{i=0}^{m} t^i\, \left\{2(t-1)P_{i+1} + Q_i\right\}, \tag{15}$$

and in particular, we have

$$P_{m+1} = \frac{1}{2} \sum_{i=0}^{m} Q_i. \tag{16}$$

Now, we give another identity involving generalized Lucas and Fibonacci numbers.

***Theorem* 2** — *Let m be a nonnegative integer and $t \in \mathbb{R} - \{0\}$. The following identity holds*:

$$t^{m+1}\, V_{m+1}^{a,b} = 2 + \sum_{i=0}^{m} t^i\, \left\{\left[(a^2 + 2\,b)\,t - a\right] U_i^{a,b} + b(a\,t-2)\, U_{i-1}^{a,b}\right\}. \tag{17}$$

PROOF : Theorem 2 can be proved using a similar procedure to that followed in the proof of Theorem 1. □

Several important identities can be deduced as special cases of identity (17). They are given in the following corollaries.

Corollary 6 — For $a = b = 1$, identity (17) reduces to

$$t^{m+1}\, L_{m+1} = 2 + \sum_{i=0}^{m} t^i\, \left\{(3\,t-1)F_i + (t-2)\, F_{i-1}\right\}, \tag{18}$$

and in particular, the following two identities hold:

$$2^{m+1}\,L_{m+1}=2+5\,\sum_{i=0}^{m}2^{i}\,F_{i}, \tag{19}$$

and

$$3^{m+1}\,L_{m+1}=2+\sum_{i=0}^{m}3^{i}\left\{8\,F_{i}+F_{i-1}\right\}. \tag{20}$$

Remark 2 : The two identities (19) and (20) are the counterparts of the two identities of Sury in (12) and Marques in (13) for Lucas numbers.

Corollary 7 — If we set $a=2, b=1$ in (17), then we get

$$t^{m+1}\,Q_{m+1}=2+\sum_{i=0}^{m}t^{i}\,\left\{(3\,t-1)P_{i}+(t-1)\,P_{i-1}\right\}, \tag{21}$$

and in particular, the following two identities hold:

$$Q_{m+1}=2+4\,\sum_{i=0}^{m}P_{i}, \tag{22}$$

and

$$(\tfrac{1}{3})^{m+1}Q_{m+1}=2-\frac{4}{3}\sum_{i=0}^{m}(\tfrac{1}{3})^{i}\,P_{i-1}. \tag{23}$$

Corollary 8 — If we set $a\,t=2$ in (17), then we get

$$t^{m+1}\,V_{m+1}^{\frac{2}{t},b}=2+2\,\sum_{i=0}^{m}t^{i}\,\left\{\mathrm{t}\left(b+\frac{2}{\mathrm{t}^{2}}\right)-\frac{1}{t}\right\}\,V_{i}^{\frac{2}{t},b}. \tag{24}$$

Corollary 9 — If we set $b=\frac{a(1-a\,t)}{2\,t}$ in (17), then we get

$$t^{m+1}\,V_{m+1}^{a,\frac{a(1-a\,t)}{2\,t}}=2+\frac{a}{2}\sum_{i=0}^{m}t^{i-1}\,\left\{(1-a\,t)(a\,t-2)\,U_{i-1}^{a,\frac{a(1-a\,t)}{2\,t}}\right\}, \tag{25}$$

and in particular, we have

$$\left(\tfrac{1}{3}\right)^{m+1}\,L_{m+1}=2-\frac{5}{9}\,\sum_{i=0}^{m}\left(\tfrac{1}{3}\right)^{i-1}\,F_{i-1}. \tag{26}$$

3. Some New Identities Introduced from (8) and (17)

This section is devoted to introducing some other new identities involving generalized Fibonacci and generalized Lucas numbers. These identities can be introduced by performing some mathematical operations on the two main identities (8) and (17).

***Theorem* 3** — *If m is an arbitrary nonnegative integer, and $t \in \mathbb{R} - \{0\}$, then the following identity holds*

$$(m+1)\, t^{m+1}\, U_{m+1}^{a,b} = \frac{1}{a} \sum_{i=0}^{m} t^{i} \left\{ \left((i+1)\, a\, t - 2\, i \right) U_{i+1}^{a,b} + i\, V_i^{a,b} \right\}. \tag{27}$$

PROOF : If both sides of (8) are differentiated with respect to t, then we get

$$(m+1)\, t^{m}\, U_{m+1}^{a,b} = \frac{1}{a} \sum_{i=0}^{m} \left\{ \left((i+1)\, a\, t^{i} - 2\, i\, t^{i-1} \right) U_{i+1}^{a,b} + i\, t^{i-1}\, V_i^{a,b} \right\}, \tag{28}$$

which can be written in the form

$$(m+1)\, t^{m+1}\, U_{m+1}^{a,b} = \frac{1}{a} \sum_{i=0}^{m} t^{i} \left\{ \left((i+1)\, a\, t - 2\, i \right) U_{i+1}^{a,b} + i\, V_i^{a,b} \right\}. \square \tag{29}$$

Corollary 10 — If we set $a\,t = 2$ in (27), then we get

$$(m+1) \left(\frac{2}{a} \right)^{m+1} U_{m+1}^{a,b} = \frac{1}{a} \sum_{i=0}^{m} \left(\frac{2}{a} \right)^{i} \left\{ 2\, U_{i+1}^{a,b} + i\, V_i^{a,b} \right\}, \tag{30}$$

and in particular, we have the following two identities

$$2^{m+1}\,(m+1)\, F_{m+1} = \sum_{i=0}^{m} 2^{i}\,(2\, F_{i+1} + i\, L_i), \tag{31}$$

and

$$(m+1)\, P_{m+1} = \frac{1}{2} \sum_{i=0}^{m} (2\, P_{i+1} + i\, Q_i). \tag{32}$$

Now, identity (31) together with Sury's identity (12) yields a new identity. The following corollary exhibits this identity.

Corollary 11 — For every nonnegative integer m, the following identity holds

$$m\, 2^{m+1} F_{m+1} = \sum_{i=0}^{m} 2^{i} \left(2\, F_{i+1} + (i-1)\, L_i \right). \tag{33}$$

PROOF : If we subtract $2^{m+1}\,F_{m+1}$ form (31) and make use of Sury's identity (12), then identity (33) can be followed. □

The following theorem generalizes the result of Theorem 3.

***Theorem* 4** — *If m and q are arbitrary nonnegative integers, and $t\in\mathbb{R}-\{0\}$, then the following identity holds*

$$(m-q+2)_q\,t^{m+1}\,U^{a,b}_{m+1}=\frac{1}{a}\sum_{i=0}^{m}t^i\,\Big\{(i-q+2)_{q-1}\,((i+1)\,a\,t-2\,(i-q+1))\,U^{a,b}_{i+1}+(i-q+1)_q\,V^{a,b}_i\Big\}\,, \tag{34}$$

where $(z)_q$ denotes the Pochhammer symbol defined as: $(z)_q=\dfrac{\Gamma(z+q)}{\Gamma(z)}$.

PROOF : Relation (34) can be followed by differentiating both sides of relation (8) q times. □

Remark 3 : The result of Theorem 4 generalizes the two results of Theorems 1 and 3, by setting respectively, $q=0$ and $q=1$, noting the two simple properties

$$(z)_0=1,\quad (z)_{-1}=\frac{1}{z-1}.$$

Corollary 12 — If we set $a\,t=2$ in (34), then we get

$$(m-q+2)_q\left(\frac{2}{a}\right)^{m+1}U^{a,b}_{m+1}=\frac{1}{a}\sum_{i=0}^{m}\left(\frac{2}{a}\right)^{i}\Big\{2\,q\,(i-q+2)_{q-1}\,U^{a,b}_{i+1}+(i-q+1)_q\,V^{a,b}_i\Big\}\,, \tag{35}$$

and in particular, we have the following two identities

$$(m-q+2)_q\,2^{m+1}\,F_{m+1}=\sum_{i=0}^{m}2^i\,\{2\,q\,(i-q+2)_{q-1}\,F_{i+1}+(i-q+1)_q\,L_i\}\,, \tag{36}$$

and

$$(m-q+2)_q\,P_{m+1}=\frac{1}{2}\sum_{i=0}^{m}\{2\,q\,(i-q+2)_{q-1}\,P_{i+1}+(i-q+1)_q\,Q_i\}\,. \tag{37}$$

Remark 4 : Relations (34), (35), (36) and (37) generalize respectively, relations (27), (30), (31) and (32).

***Theorem* 5** — *Let m and q be nonnegative integers and $t\in\mathbb{R}-\{0\}$. The following identity holds*:

$$(m-q+2)_q\,t^{m+1}\,V^{a,b}_{m+1}=\sum_{i=0}^{m}t^i\,(i-q+2)_{q-1}\Big\{\big((a^2+2\,b)(i+1)\,t-a(i-q+1)\big)\,U^{a,b}_i+(a\,b\,(i+1)\,t-2\,b(i-q+1))\,U^{a,b}_{i-1}\Big\}\,. \tag{38}$$

PROOF : Relation (38) can be followed by differentiating both sides of (17) q times, and making use of the two relations

$$(i-q+1)_q = (i-q+1)(i-q+2)_{q-1},$$

and

$$(i-q+2)_q = (i-q+2)_{q-1}\,(i+1).\square$$

Corollary 13 — If we set $a = b = 1$ in (38), then the following identity is obtained

$$\begin{aligned}(m-q+2)_q\, t^{m+1}\, L_{m+1} = &\sum_{i=0}^{m} t^i\,(i-q+2)_{q-1}\,\{(3(i+1)t-(i-q+1))\,F_i\\ &+((i+1)\,t-2\,(i-q+1))\;F_{i-1}\}\,.\end{aligned} \tag{39}$$

Corollary 14 — If we set $a = 2, b = 1$ in (38), then the following identity is obtained

$$\begin{aligned}(m-q+2)_q\, t^{m+1}\, Q_{m+1} = &\sum_{i=0}^{m} t^i\,(i-q+2)_{q-1}\,\{(6(i+1)t-2(i-q+1))\,P_i\\ &+(2(i+1)\,t-2\,(i-q+1))\;P_{i-1}\}\,.\end{aligned} \tag{40}$$

Other two relations can be obtained by integrating the two main identities (8) and (17) q times. The following theorem exhibits the resulting identity from integrating identity (8) q times.

***Theorem* 6** — *Let m and q be nonnegative integers and $t \in \mathbb{R}-\{0\}$. The following identity holds*:

$$\frac{1}{(m+2)_q}\, t^{m+1}\, U^{a,b}_{m+1} = \frac{1}{a}\sum_{i=0}^{m} \frac{t^i}{(i+1)_q}\left\{\left(\frac{a\,t\,(i+1)}{i+q+1}-2\right)\,U^{a,b}_{i+1}+V^{a,b}_i\right\}. \tag{41}$$

PROOF : If we integrate relation (8) q times, then the following relation is obtained.

$$\frac{1}{(m+2)_q}\, t^{m+1}\, U^{a,b}_{m+1} = \frac{1}{a}\sum_{i=0}^{m} t^i\,\left\{\left(\frac{a\,t}{(i+2)_q}-\frac{2}{(i+1)_q}\right)\,U^{a,b}_{i+1}+\frac{1}{(i+1)_q}\;V^{a,b}_i\right\}. \tag{42}$$

Noting the simple identity

$$(i+2)_q = (i+1)_q\,\frac{i+q+1}{i+1}, \tag{43}$$

relation (42) can be easily turned into (41). $\square$

Corollary 15 — If we set $a = b = 1$ in (41), then the following identity is obtained

$$\frac{1}{(m+2)_q}\, t^{m+1}\, F_{m+1} = \sum_{i=0}^{m} \frac{t^i}{(i+1)_q}\left\{\left(\frac{t\,(i+1)}{i+q+1}-2\right)\,F_{i+1}+L_i\right\}. \tag{44}$$

Corollary 16 — If we set $a = 2, b = 1$ in (41), then the following identity is obtained

$$\frac{1}{(m+2)_q}\, t^{m+1}\, P_{m+1} = \frac{1}{2} \sum_{i=0}^{m} \frac{t^i}{(i+1)_q} \left\{ \left(\frac{2\, t\, (i+1)}{i+q+1} - 2 \right) P_{i+1} + Q_i \right\}. \tag{45}$$

4. CONCLUSIONS

In this paper, we have derived two new formulae involving the two kinds of generalized numbers, namely, generalized Fibonacci and generalized Lucas numbers. Some special formulae involving the celebrated numbers of Fibonacci, Lucas, Pell and Pell-Lucas are deduced. Moreover, The two well-known identities of Sury and Marques are direct consequences of the derived formulae. Other new identities involving the generalized Fibonacci and generalized Lucas numbers are obtained via differentiation and integration operations.

ACKNOWLEDGEMENT

The authors would like to thank the anonymous referee for his valuable and constructive comments, which helped substantially to improve the manuscript in its present form.

REFERENCES

1. W. M. Abd-Elhameed, Y. H. Youssri, N. El-Sissi and M. Sadek, New hypergeometric connection formulae between Fibonacci and Chebyshev polynomials, *Ramanujan J.*, **42** (2017), 347-361.
2. W. M. Abd-Elhameed and N. A. Zeyada, A generalization of generalized Fibonacci and generalized Pell numbers, *Internat. J. Math. Ed. Sci. Tech.*, **48**(1) (2016), 102-107.
3. S. Falcon and A. Plaza, On the Fibonacci $k-$numbers, *Chaos Soliton Fract.*, **32**(5) (2007), 1615-1624.
4. H. H. Gulec, N. Taskara and K. Uslu, A new approach to generalized Fibonacci and Lucas numbers with binomial coefficients, *Appl. Math. Comput.*, **220** (2013), 482-486.
5. A. F. Horadam and J. M. Mahon, Pell and Pell-Lucas polynomials, *Fibonacci Quart.*, **23**(1) (1985), 7-20.
6. Dan Kalman and Robert Mena, The Fibonacci numbers: exposed, *Mathematics Magazine*, (2003), 167-181.
7. T. Koshy, *Fibonacci and Lucas numbers with applications*, **51** (2011), John Wiley & Sons.
8. T. Koshy, Pell and Pell-Lucas polynomials, In *Pell and Pell-Lucas Numbers with Applications*, (2014), 255-281, Springer.
9. H. Kwong, An alternate proof of Sury's Fibonacci-Lucas relation, *Amer. Math. Monthly*, **121**(6) (2014), 514.
10. D. Marques, A new Fibonacci-Lucas relation, *Amer. Math. Monthly*, **122**(7) (2015), 683.

11. R. Melham, Sums involving Fibonacci and Pell numbers, *Port. Math.*, **56**(3) (1999), 309-318.

12. B. Sury, A polynomial parent to a Fibonacci-Lucas relation, *Amer. Math. Monthly*, **121**(3) (2014), 236.

13. D. T. Tollu, Y. Yazlik and N. Taskara, On the solutions of two special types of Riccati difference equation via Fibonacci numbers, *Adv. Difference Equ.*, **174**, 2013.

14. A. Włoch, Some identities for the generalized Fibonacci numbers and the generalized Lucas numbers, *Appl. Math. Comput.*, **219**(10) (2013), 5564-5568.

15. Yasin Y. Yazlik and N. Taskara, A note on generalized-Horadam sequence, *Comput. Math. Appl.*, **63**(1) (2012), 36-41.