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EGOROFF'S THEOREM IN MEASURABLE OPERATOR SPACES ASSOCIATED WITH A VON NEUMANN ALGEBRA[1]

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In this article, some kinds of convergence about the $\tau-$measurable operators affiliated with a von Neumann algebra are defined. Moreover, the relationships among these kinds of convergence are considered.

Key words : Measurable operator; uniformly convergent; almost uniformly convergent; convergent; convergent in measure.

1. Introduction

The theory of von Neumann algebra, which can be thought as "non-commutative measure theory", is an important and well-developed branch of operator algebras. We get into the paper with the preliminary notions of von Neumann algebras. Details can be seen in [1-10].

Let $\mathscr{H}$ be a Hilbert space with an inner product $\langle , \rangle$, denote by $B(\mathscr{H})$ the set of all bounded linear mappings from $\mathscr{H}$ to itself. The topology induced by the norm on $B(\mathscr{H})$ is called the norm topology. The strong (operator) topology on $B(\mathscr{H})$ is that defined by the semi-norms $x \mapsto \|x\xi\|$ as ξ runs through $\mathscr{H}$, thus a net $\{x_\lambda\}_{\lambda\in\Lambda}$ converges strongly to x on $\mathscr{H}$ (denoted by $x_\lambda \xrightarrow{SOT} x$) means $x\xi = \lim\limits_{\lambda} x_\lambda\xi$ for all $\xi \in \mathscr{H}$, that is $\lim\limits_{\lambda} \|x_\lambda\xi - x\xi\| = 0$ for all $\xi \in \mathscr{H}$. The topology generated by the semi-norms $x \mapsto |\langle x\xi, \eta\rangle|$ $(\xi, \eta \in \mathscr{H})$ is called the weak (operator) topology on $B(\mathscr{H})$, then $\{x_\lambda\}_{\lambda\in\Lambda}$ converges weakly to x (denoted by $x_\lambda \xrightarrow{WOT} x$) if and only if $\langle x\xi, \eta\rangle = \lim\limits_{\lambda}\langle x_\lambda\xi, \eta\rangle$ $(\forall \xi, \eta \in \mathscr{H})$.

If $\mathscr{B}$ is a subset of $B(\mathscr{H})$, we define its commutant as $\mathscr{B}' = \{x \in B(\mathscr{H}) : \ xy = yx$ for all $y \in \mathscr{B}\}$, and the double commutant $\mathscr{B}'' = (\mathscr{B}')'$. It is easy to verify $\mathscr{B} \subseteq \mathscr{B}''$ and $\mathscr{B}' = \mathscr{B}'''$. Let $\mathscr{B}$ be a $*$-subalgebra of $B(\mathscr{H})$ with the identity I, then $\mathscr{B}$ is strongly (hence weakly) dense in $\mathscr{B}''$. We call $\mathscr{M}$ a von Neumann algebra if $\mathscr{M}$ is a strongly(weakly) closed $*$-subalgebra of $B(\mathscr{H})$ containing I. Let $\mathscr{M}$ be a $*$-algebra on a Hilbert space $\mathscr{H}$ and $I \in \mathscr{M}$, then $\mathscr{M}$ is a von Neumann algebra if and only if $\mathscr{M} = \mathscr{M}''$.

An element $x \in \mathscr{M}$ is self-adjoint if $x = x^*$. Set $\mathscr{M}_{sa} = \{x \in \mathscr{M} | x = x^*\}$. We define the spectrum of x to be the set $\sigma(x) = \{\lambda \in \mathbb{C} | \lambda I - x$ is not invertible$\}$. An element $x \in \mathscr{M}$ is positive (denoted by $x \succcurlyeq \theta$ where θ is the zero element in $\mathscr{M}$) if $x \in \mathscr{M}_{sa}$ and $\sigma(x) \subset \mathbb{R}^+$, set $\mathscr{M}_+ = \{x \in \mathscr{M} | x \succcurlyeq \theta\}$. Then we make $\mathscr{M}_{sa}$ a poset by defining $x \preccurlyeq y$ to mean $y - x \in \mathscr{M}_+$. If an element $p \in \mathscr{M}$ satisfies $p = p^* = p^2$, p is called a projection, denote by $\mathcal{P}(\mathscr{M})$ the set of projections in $\mathscr{M}$. For $e \in \mathcal{P}(\mathscr{M})$, set $e^\perp = I - e$. The lattice operations on $\mathcal{P}(\mathscr{M})$ is defined by $e \vee f$ and $e \wedge f$, where $e \vee f$ is the projection from $\mathscr{H}$ onto its closed subspace $\overline{\text{span}(e(\mathscr{H}), f(\mathscr{H}))}$ and $e \wedge f$ is the projection from $\mathscr{H}$ onto its closed subspace $e(\mathscr{H}) \cap f(\mathscr{H})$. Then $\mathcal{P}(\mathscr{M})$ forms a complete lattice, namely, in which each subset has both a supremum and an infimum.

An element $u \in B(\mathscr{H})$ is called a partial isometry if u is isometric on $(\ker u)^\perp$. For $x \in B(\mathscr{H})$, let $|x| = (x^*x)^{\frac{1}{2}}$, then there is a unique partial isometry u such that $x = u|x|$. This is called the polar decomposition for x. If x is an element of a von Neumann algebra $\mathscr{M}$, then $u, |x| \in \mathscr{M}$. For $x \in B(\mathscr{H})_+$, there is a unique spectral measure $\{e_\lambda(x)\}$ relative to $(\sigma(x), \mathscr{H})$ such that

$$x = \int_{\sigma(x)} \lambda de_\lambda(x),$$

which is called the resolution of the identity for x. If $x \in \mathscr{M}_+$, then $e_\lambda(x) \in \mathscr{M}$.

Let $\mathscr{M}$ be a von Neumann algebra. A trace on $\mathscr{M}$ is a mapping $\tau : \mathscr{M}_+ \to [0, \infty]$ satisfying:

1. for $x, y \in \mathscr{M}_+, \ \lambda \in \mathbb{R}_+, \ \tau(x + \lambda y) = \tau(x) + \lambda\tau(y)$;
2. for $x \in \mathscr{M}, \ \tau(x^*x) = \tau(xx^*)$.

A trace τ is normal if for any bounded monotonic increasing net $\{x_\lambda\}$ in $\mathscr{M}_+$, $\sup\limits_\lambda \tau(x_\lambda) = \tau(\sup\limits_\lambda x_\lambda)$. Furthermore, we say that τ is finite if $\tau(I) < \infty$, and semi-finite if for any $x \in \mathscr{M}_+$, there is a $y \in \mathscr{M}_+$ such that $y \preccurlyeq x$ and $\tau(y) < \infty$. Also, if for $x \in \mathscr{M}_+$, $\tau(x) = 0 \Rightarrow x = \theta$, then τ is faithful.

***Theorem* 1.1** — *[11]. Let $\mathscr{M}$ be a von Neumann algebra, τ is a faithful normal semifinite trace on $\mathscr{M}$.*

1. For $\{e_\alpha\}_{\alpha\in\Lambda} \subseteq \mathcal{P}(\mathscr{M})$,
$$\tau(\bigvee_{\alpha\in\Lambda} e_\alpha) \leqslant \sum_{\alpha\in\Lambda} \tau(e_\alpha).$$
In addition, if $\{e_\alpha\}_{\alpha\in\Lambda}$ is pairwise orthogonal, then
$$\tau(\bigvee_{\alpha\in\Lambda} e_\alpha) = \sum_{\alpha\in\Lambda} \tau(e_\alpha).$$

2. Let $\{e_n\} \subseteq \mathcal{P}(\mathscr{M})$ be a monotonic decreasing sequence, if $\tau(e_{n_0}) < \infty$ for some n_0, set $e = \bigwedge_{n=1}^{\infty} e_n$, then
$$\tau(e) = \lim_{n\to\infty} \tau(e_n).$$

Operators described above are all bounded linear operators on Hilbert space, however, there are a lot of unbounded operators, such as derivation operator. Next, we present some prior knowledge about the unbounded operators. Details can be seen in [5, 11].

Let $\mathscr{H}$ be a Hilbert space. An operator in $\mathscr{H}$ means a linear mapping $x : D(x) \to \mathscr{H}$ whose domain $D(x)$ is a subspace of $\mathscr{H}$. If $D(x)$ is dense in $\mathscr{H}$, x is called a densely defined operator in $\mathscr{H}$. We call $G(x) = \{(\xi, x\xi) : \xi \in D(x)\}$ the graph of x, x is called a closed operator if $G(x)$ is a closed subspace of $\mathscr{H} \oplus \mathscr{H}$. If $\overline{G(x)}$ is the graph of some operator in $\mathscr{H}$, this operator is called the closure of x, denoted by $[x]$ and x is called pre-closed. We say $x \subset y$ if $D(x) \subset D(y)$ and $x\xi = y\xi$ for any $\xi \in D(x)$. Let x and y be operators in $\mathscr{H}$, we define $D(x+y) = D(x) \bigcap D(y),\ D(xy) = \{\xi \in D(y) : y\xi \in D(x)\}$.

If x is a densely defined operator, we can define its adjoint operator x^*. Set
$$D(x^*) = \{\eta \in \mathscr{H} |\ \langle x\xi,\ \eta\rangle (\xi \in D(x)) \text{ is a continuous linear functional on } D(x)\},$$
then we get an operator $x^* :\ D(x^*) \to \mathscr{H}$ satisfying
$$\langle x\xi, \eta\rangle = \langle \xi, x^*\eta\rangle,\ (\forall \xi \in D(x)).$$
x^* is a closed operator. If $x^* = x,\ x$ is called self-adjoint; x is positive if x is self-adjoint and $\langle x\xi, \xi\rangle \geqslant 0$ $(\forall \xi \in D(x))$. Let x be a positive operator, there is a unique spectral decomposition for x :
$$x = \int_{\sigma(x)} \lambda de_\lambda(x), \text{ where } e_\lambda(x) \text{ denote the spectral measure corresponding to } (0, \lambda).$$
Usually, we use $e_\lambda^\perp(x)$ denote the spectral measure corresponding to (λ, ∞).

Let $\mathscr{M}$ be a von Neumann algebra in $B(\mathscr{H})$. Recall that a closed densely defined operator x in $\mathscr{H}$ is affiliated with $\mathscr{M}$, denote it by $x\eta\mathscr{M}$, if $xu = ux$ for any unitary $u \in \mathscr{M}'$. Obviously, if $x \in \mathscr{M}$, then $x\eta\mathscr{M}$. Let $x = u|x|$ be the polar decomposition for x, then $x\eta\mathscr{M} \Leftrightarrow x^*\eta\mathscr{M} \Leftrightarrow |x|$ and u are affiliated with $\mathscr{M}$.

Definition 1.2 — [11]. An affiliated operator x is said to be τ-measurable if for any $\delta > 0$, there is a projection $e \in \mathcal{P}(\mathscr{M})$ such that

$$e(\mathscr{H}) \subseteq D(x) \text{ and } \tau(e^{\perp}) < \delta.$$

Let $L_0(\mathscr{M}, \tau)$ denote the set of all τ-measurable operators.

***Theorem* 1.3** — *[11].Let x be an operator affiliated with $\mathscr{M}$, then*

1. *for any $\lambda \geqslant 0$, $\tau(e_\lambda^{\perp}(|x|)) = \tau(e_\lambda^{\perp}(|x^*|))$;*
2. *$x \in L_0(\mathscr{M}) \Leftrightarrow \tau(e_\lambda^{\perp}(|x|)) < \infty$ for some $\lambda \geqslant 0 \Leftrightarrow x^* \in L_0(\mathscr{M})$.*

In the following, by $\mathscr{M}$ we denote a von Neumann algebra on $\mathscr{H}$ with a normal semi-finite faithful trace τ, and by $L_0(\mathscr{M})$ we denote the set of all τ-measurable operators.

2. Main Results

The Egoroff's theorem is a fundamental and important theorem in real analysis, which states that on a set with finite measure if a sequence of almost everywhere(a.e.) finite measurable functions converges a.e., then it is almost uniformly convergent. In this paper, we will consider convergent, uniformly convergent, almost uniformly convergent and convergence in measure in $L_0(\mathscr{M})$ and prove that the theorem holds in $L_0(\mathscr{M})$ as well.

In this section, firstly, by analogy with the convergence about the measurable function, we can make the following definition:

Definition 2.1 — Let $\{x_k\} \subseteq L_0(\mathscr{M}),\ x \in L_0(\mathscr{M})$, set $\mathscr{D} = \big(\bigcap\limits_{k=1}^{\infty} D(x_k)\big) \cap D(x)$.

1. If for any $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $k > N$,

$$\|x_k\xi - x\xi\| < \varepsilon\|\xi\| \ (\forall \xi \in \mathscr{D}),$$

   then the sequence x_k is called uniformly convergent to x on $\mathscr{D}$, denote it by $x_k \rightrightarrows x$.

2. If for any $\delta > 0$, there is an $e_\delta \in \mathcal{P}(\mathscr{M})$ such that $\tau(e_\delta^{\perp}) < \delta$ and $x_k \rightrightarrows x$ on $e_\delta(\mathscr{H}) \cap \mathscr{D}$, we call $\{x_k\}$ almost uniformly convergent to x, and denote it by $x_k \xrightarrow{a.u.} x$.

3. If for any $\xi \in \mathscr{D}$, any $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that for all $k > N$, $\|x_k\xi - x\xi\| < \varepsilon\|\xi\|$, we call $\{x_k\}$ is convergent to x and denote it by $x_k \to x$.

4. If for any $\varepsilon > 0$, $\lim\limits_{k\to\infty} \tau(e_\varepsilon^\perp(|x_k - x|)) = 0$, $\{x_k\}$ is called convergent to x in measure, denoted by $x_k \xrightarrow{d} x$.[11]

The definitions above strongly rely on the denseness of $\mathscr{D}$. The following proposition guarantees that the above definitions are well-defined.

Proposition 2.2 — Let $\{x_n\} \subseteq L_0(\mathscr{M})$ and $\mathscr{D}_0 = \bigcap\limits_{n=1}^{\infty} D(x_n)$, then $\mathscr{D}_0$ is dense in $\mathscr{H}$.

PROOF : Since $x_n \in L_0(\mathscr{M})$, for any $k \in \mathbb{N}$, there is an $e_{n,k} \in \mathcal{P}(\mathscr{M})$ such that

$$e_{n,k}(\mathscr{H}) \subseteq D(x_n) \text{ and } \tau(e_{n,k}^\perp) \leqslant \frac{1}{2^{nk}}.$$

Let $f_n = \bigwedge\limits_{k\geqslant n} (\bigwedge\limits_{m=1}^{\infty} e_{m,k})$, then $\{f_n\}$ is a sequence of monotonic increasing projections and

$$\begin{aligned}\tau(f_n^\perp) = \tau\left(\bigvee_{k\geqslant n}\left(\bigvee_{m=1}^{\infty} e_{m,k}^\perp\right)\right) &\leqslant \sum_{k\geqslant n}\sum_{m=1}^{\infty}\tau(e_{m,k}^\perp) \\ &\leqslant \sum_{k\geqslant n}\sum_{m=1}^{\infty}\frac{1}{2^{mk}} = \sum_{k\geqslant n}\frac{\frac{1}{2^k}}{1-\frac{1}{2^k}} = \sum_{k\geqslant n}\frac{1}{2^k-1} \\ &\leqslant \sum_{k\geqslant n}\frac{1}{2^{k-1}} \leqslant \frac{1}{2^n}.\end{aligned}$$

Since τ is faithful, $f_n^\perp \xrightarrow{SOT} \theta$, and $f_n \xrightarrow{SOT} I$. Therefore $\bigcup\limits_{n=1}^{\infty} f_n(H)$ is dense in $\mathscr{H}$. Since $f_n(\mathscr{H}) \subseteq \bigcap\limits_{m=1}^{\infty} e_{m,n}(\mathscr{H}) \subseteq \bigcap\limits_{m=1}^{\infty} D(x_m) = \mathscr{D}_0$, $\mathscr{D}_0$ is dense in $\mathscr{H}$. □

Remark 2.3 : The above convergences of the measurable operators in $L_0(\mathscr{M})$ is the generalization of the convergences of the operators in $\mathscr{M}$. In fact, if $\{x_k\} \subseteq \mathscr{M}$, $x \in \mathscr{M}$, then $x_k \Rightarrow x$ means $x_k \xrightarrow{\|\cdot\|} x$ and $x_k \to x$ means $x_k \xrightarrow{SOT} x$.

We will give the relationships of these kinds of convergence in the following. Let $\mathscr{D} = \big(\bigcap\limits_{k=1}^{\infty} D(x_k)\big) \cap D(x)$.

***Theorem* 2.4** — *Let* $\{x_k\} \subseteq L_0(\mathscr{M})$, $x \in L_0(\mathscr{M})$.

1. If $x_k \Rightarrow x$, *then* $x_k \xrightarrow{a.u.} x$.

2. If $x_k \xrightarrow{a.u.} x$, *then* $x_k \to x$.

PROOF : It suffices to show 2.

Since $x_k \xrightarrow{a.u.} x$, for any $n \in \mathbb{N}$, there is an $e_n \in \mathcal{P}(\mathscr{M})$ such that $\tau(e_n^{\perp}) < \frac{1}{n}$ and $x_k \Rightarrow x$ on $e_n(\mathscr{H}) \bigcap \mathscr{D}$. That is, for any $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that for all $k > N$,

$$\|x_k\xi - x\xi\| < \varepsilon\|\xi\| \ (\forall \xi \in e_n(\mathscr{H}) \cap \mathscr{D}).$$

Let $e = \bigvee_{n=1}^{\infty} e_n$, then $\tau(e^{\perp}) = 0$ and $e = I$. Therefore, for any $\xi \in \mathscr{D}$, there is an $n \in \mathbb{N}$ such that $\xi \in \mathscr{D} \cap e_n(\mathscr{H})$. Then for any $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that for all $k > N$, $\|x_k\xi - x\xi\| < \varepsilon\|\xi\|$, i.e. $x_k \to x$. □

In order to prove the relationship of convergence and almost uniformly convergence, we show the following lemma first.

Lemma 2.5 — Suppose $\{x_k\} \subseteq L_0(\mathscr{M}),\ x \in L_0(\mathscr{M})$ and $\xi \in \mathscr{D}$ such that $\|x_k\xi - x\xi\| \to 0$. Let e be the projection from $\mathscr{H}$ onto the closed subspace generated by these ξ, and $e_{lk} = e_{\frac{1}{l}}(|x_k - x|)$, then $e = \bigwedge_{l=1}^{\infty} \bigvee_{j=1}^{\infty} \bigwedge_{k=j}^{\infty} e_{lk}$, $e^{\perp} = \bigvee_{l=1}^{\infty} \bigwedge_{j=1}^{\infty} \bigvee_{k=j}^{\infty} e_{lk}^{\perp}$.

PROOF : For $\xi \in e_{lk}$,

$$\|x_k\xi - x\xi\|^2 = \||x_k - x|\xi\|^2 = \int_{(0,\frac{1}{l})} \lambda^2 d\langle e_\lambda(|x_k - x|)\xi, \xi\rangle \leqslant \frac{\|\xi\|}{l^2}.$$

On the other hand, if $\xi \in e_{lk}^{\perp}$,

$$\|x_k\xi - x\xi\|^2 = \int_{(\frac{1}{l},\infty)} \lambda^2 d\langle e_\lambda(|x_k - x|)\xi, \xi\rangle \geqslant \frac{\|\xi\|}{l^2}.$$

Therefore,

$$\begin{aligned} &\xi \in \mathscr{D},\ \|x_k\xi - x\xi\| \to 0 \\ \Leftrightarrow &\forall l \in \mathbb{N},\ \exists N \in \mathbb{N},\ \forall k > N : \|x_k\xi - x\xi\| \leqslant \frac{\|\xi\|}{l} \\ \Leftrightarrow &\forall l \in \mathbb{N},\ \exists N \in \mathbb{N},\ \forall k > N : \xi \in e_{lk}(\mathscr{H}). \end{aligned}$$

The second equation is immediate from the fact that $(\bigvee_{\alpha\in\Lambda} e_\alpha)^{\perp} = \bigwedge_{\alpha\in\Lambda} e_\alpha^{\perp}$. □

Lemma 2.6 — Let $\{x_k\} \subseteq L_0(\mathscr{M}),\ x \in L_0(\mathscr{M})$. If $x_k \to x$ and $\tau(I) < \infty$, then for any $l \in \mathbb{N},\ \lim_{j\to\infty} \tau(\bigvee_{k=j}^{\infty} e_{lk}^{\perp}) = 0$.

PROOF : If $x_k \to x$, then $e^{\perp} = \bigvee_{l=1}^{\infty} \bigwedge_{j=1}^{\infty} \bigvee_{k=j}^{\infty} e_{lk}^{\perp} = \theta$. Therefore,

$$\bigwedge_{j=1}^{\infty} \bigvee_{k=j}^{\infty} e_{lk}^{\perp} = \theta \ (\forall l \in \mathbb{N}).$$

Set $e_j = \bigvee\limits_{k=j}^{\infty} e_{lk}^{\perp},\ \{e_j\}$ is monotonic decreasing and $\tau(e_1) \leqslant \tau(I) < \infty$, so

$$\lim_{j\to\infty} \tau(e_j) = \tau(\bigwedge_{j=1}^{\infty} e_j) = 0.$$

that is,

$$\lim_{j\to\infty} \tau(\bigvee_{k=j}^{\infty} e_{lk}^{\perp}) = 0.\Box$$

***Theorem* 2.7** — *Let* $\{x_k\} \subseteq L_0(\mathscr{M}),\ x \in L_0(\mathscr{M}),\ \tau(I) < \infty$. *If* $x_k \to x$, *then* $x_k \xrightarrow{a.u.} x$.

PROOF : Since $x_k \to x,\ \lim\limits_{j\to\infty} \tau(\bigvee\limits_{k=j}^{\infty} e_{lk}^{\perp}) = 0$. For $\frac{\delta}{2^l} > 0$, there is a $J_{l,\delta} \in \mathbb{N}$ such that

$$\tau(\bigvee_{k=J_{l,\delta}}^{\infty} e_{lk}^{\perp}) < \frac{\delta}{2^l}.$$

Set $e_\delta = \bigvee\limits_{l=1}^{\infty} (\bigvee\limits_{k=J_{l,\delta}}^{\infty} e_{lk}^{\perp})$. Then

$$\tau(e_\delta) = \tau\left(\bigvee_{l=1}^{\infty}\left(\bigvee_{k=J_{l,\delta}}^{\infty} e_{lk}^{\perp}\right)\right) \leqslant \sum_{l=1}^{\infty} \tau\left(\bigvee_{k=J_{l,\delta}}^{\infty} e_{lk}^{\perp}\right) \leqslant \sum_{l=1}^{\infty} \frac{\delta}{2^l} = \delta.$$

For any $\xi \in e_\delta^{\perp}(\mathscr{H}) \cap \mathscr{D} \subseteq e_{lk}(\mathscr{H}) \cap \mathscr{D}\ (\forall l \in \mathbb{N}, k > J_{l,\delta})$,

$$\|x_k\xi - x\xi\|^2 \leqslant \frac{\|\xi\|^2}{l^2}.$$

Therefore, for any $\varepsilon > 0$, there is an $l_0 \in \mathbb{N}$ such that $\frac{1}{l_0} < \varepsilon$, when $k > J_{l_0,\delta}$,

$$\|x_k\xi - x\xi\| < \varepsilon\|\xi\|\ (\forall \xi \in e_\delta^{\perp}(\mathscr{H}) \cap \mathscr{D}).$$

That is $x_k \xrightarrow{a.u.} x$. $\Box$

Remark 2.8 : From Theorem 2.7 one can see that under the condition of $\tau(I) < \infty$, a convergent sequence in $L_0(\mathscr{M})$ is almost uniformly convergent. Notice that the demand $\tau(I) < \infty$ is necessary. Indeed, if $\mathscr{H}$ is a separable infinite dimensional Hilbert space, then $\mathscr{M} = B(\mathscr{H})$ is a noncommutative von Neumann algebra. Set $\tau(e) = \dim(e(\mathscr{H}))$, then $\tau(I) = \infty$. Let $\{\xi_n\}_{n=1}^{\infty}$ be an orthonormal basis of $\mathscr{H},\ f_n =$ Proj$[\overline{\text{span}\{\xi_1, \xi_2, \cdots, \xi_n\}}]$. Then $f_n \to I$, but not almost uniformly convergent to I.

The fact that $f_n \to I$ is immediate from the Parseval's equation. But for $\delta = 1$, let $e_\delta = I$, then $\tau(e_\delta^\perp) = 0 < \delta$. On $e_\delta(\mathscr{H})$, for $\varepsilon = \frac{1}{2}$, $\forall N \in \mathbb{N}$, let $\xi_0 \in e_N^\perp(\mathscr{H})$, then

$$\|f_N\xi_0 - \xi_0\| = \|\xi_0\| > \frac{1}{2}\|\xi_0\|.$$

It means that f_n is not almost uniformly convergent to I.

Moreover, in the theorem above, $\{x_k\}$ can only almost uniformly convergent to x, but not uniformly convergent to x. Indeed, let $\mathscr{M} = L^\infty[0,1]$, namely, the set of all essentially bounded complex valued measurable functions on $[0,1]$. $L^\infty[0,1]$ is a Banach space with the essential supremum norm. Set $\mathscr{H} = L^2[0,1] = \{f : [0,1] \to \mathbb{C} | \int_{[0,1]} |f(t)|^2 dt < \infty\}$, then $\mathscr{H}$ is a Hilbert space with the inner product

$$< f, g >= \int_{[0,1]} f(t)\overline{g(t)}dt.$$

For $\varphi \in L^\infty[0,1]$, define $M_\varphi : \mathscr{H} \to \mathscr{H}$ by

$$M_\varphi(f) = \varphi f \ (f \in L^2[0,1]),$$

then M_φ is bounded and moreover $\|M_\varphi\| = \|\varphi\|_\infty$. Since $L^\infty[0,1]$ acting on $L^2[0,1]$ is strongly dense, $\mathscr{M} = L^\infty[0,1] \subseteq B(\mathscr{H})$ is a von Neumann algebra. Set $\varphi_n \in L^\infty[0,1]$ be the characteristic function of $[0, \frac{1}{n}]$, then $\varphi_n \to \theta$ with $\|\varphi_n\|_\infty = 1$, so $\{\varphi_n\}$ is not uniformly convergent to θ.

***Theorem* 2.9** — *Let* $\{x_k\} \subseteq L_0(\mathscr{M}),\ x \in L_0(\mathscr{M})$. *If* $x_k \xrightarrow{a.u.} x$, *then* $x_k \xrightarrow{d} x$.

PROOF : For any $\delta > 0$, if $x_k \xrightarrow{a.u.} x$, $\forall \varepsilon > 0,\ \exists\, e_\delta \in \mathcal{P}(\mathscr{M})$ such that $\tau(e_\delta^\perp) < \delta$ and $\exists\, N \in \mathbb{N}$ such that for all $k > N$,

$$\|x_k\xi - x\xi\| < \varepsilon\|\xi\| \ (\forall \xi \in e_\delta(\mathscr{H}) \cap \mathscr{D}).$$

When $k > N$, $e_\varepsilon^\perp(|x_k - x|) \leqslant e_\delta^\perp$ and $\tau(e_\varepsilon^\perp(|x_k - x|)) \leqslant \delta$.

Thus

$$0 \leqslant \varliminf_{k\to\infty} \tau(e_\varepsilon^\perp(|x_k - x|)) \leqslant \varlimsup_{k\to\infty} \tau(e_\varepsilon^\perp(|x_k - x|)) \leqslant \delta.$$

Since δ is arbitrary,

$$\lim_{k\to\infty} \tau(e_\varepsilon^\perp(|x_k - x|)) = 0.$$

i.e. $x_k \xrightarrow{d} x$. □

The next corollary is an immediate consequence of Theorem 2.7 and Theorem 2.9.

Corollary 2.10 — Let $\{x_k\} \subseteq L_0(\mathscr{M}),\ x \in L_0(\mathscr{M}), \tau(I) < \infty$. If $x_k \to x$, then $x_k \xrightarrow{d} x$.

Remark 2.11 : In the corollary above, the condition $\tau(I) < \infty$ is essential. Indeed, as in remark 2.8, the sequence f_n convergent to I, but not convergent in measure. In fact, for any $n \in \mathbb{N}$, f_n is a projection, so $\sigma(f_n) \subseteq \{0,1\}$. In addition, $f_n\xi_{n+1} = \theta,\ (I - f_n)\xi_n = \theta$, so f_n and $f_n^\perp$ are not invertible. Thus $\sigma(f_n) = \{0,1\}$, similarly, $\sigma(f_n^\perp) = \{0,1\}$. Then,

$$f_n^\perp = \int_{\{0,1\}} \lambda de_\lambda = 0e_0 + 1e_1,\ where\ e_0 = f_n, e_1 = f_n^\perp.$$

Therefore,

$$e_\varepsilon(|f_n - I|) = e_\varepsilon(f_n^\perp) = \begin{cases} f_n,\ \varepsilon < 1; \\ f_n^\perp,\ \varepsilon \geqslant 1. \end{cases}$$

Then for $\varepsilon = 1,\ e_\varepsilon^\perp(|f_n - I|) = f_n,\ \tau(e_\varepsilon^\perp(|f_n - I|)) = \tau(f_n) \to \tau(I) = \infty$. Therefore, f_n is not convergent in measure.

***Theorem* 2.12** — *Let* $\{x_k\} \subseteq L_0(\mathscr{M}),\ x \in L_0(\mathscr{M})$. *If* $x_k \xrightarrow{d} x$, *then there is a subsequence* $\{x_{k_i}\}$ *such that* $x_{k_i} \to x$.

PROOF : Since $x_k \xrightarrow{d} x$, $\forall\varepsilon > 0,\ \forall\delta > 0$, there is a $k = k(\varepsilon,\delta) \in \mathbb{N}$ such that

$$0 \leqslant \tau(e_\varepsilon^\perp(|x_k - x|)) < \delta.$$

Let $\varepsilon = \frac{1}{i}\ \delta = \frac{1}{2^i}$, there is a k_i such that

$$0 \leqslant \tau(e_{\frac{1}{i}}^\perp(|x_{k_i} - x|)) < \frac{1}{2^i}.$$

We may suppose that $k_1 < k_2 < \cdots$, then we get the subsequence $\{x_{k_i}\}_{i=1}^\infty$.

Let $e_n = \bigwedge\limits_{i=n}^\infty e_{\frac{1}{i}}(|x_{k_i} - x|)$. For a fixed $n \in \mathbb{N},\ \forall\xi \in e_n(\mathscr{H}) \cap \mathscr{D}$, for all $i \geqslant n,\ \xi \in e_{\frac{1}{i}}(|x_{k_i} - x|)(\mathscr{H}) \cap \mathscr{D}$, therefore

$$\|x_{k_i}\xi - x\xi\| \leqslant \frac{\|\xi\|}{i}.$$

That is $x_{k_i} \to x$ on $e_n(\mathscr{H}) \cap \mathscr{D}$. Let $e = \bigvee\limits_{n=1}^\infty e_n$, then $x_{k_i} \to x$ on $e(\mathscr{H}) \cap \mathscr{D}$.

Since

$$e^\perp = \bigwedge_{n=1}^\infty e_n^\perp = \bigwedge_{n=1}^\infty (\bigvee_{i=n}^\infty e_{\frac{1}{i}}^\perp(|x_{k_i} - x|)),$$

$\{e_n^\perp\}$ is monotonic decreasing and $\tau(e_1^\perp) < \infty$,

$$\tau(e^\perp) = \lim_{n\to\infty} \tau(e_n^\perp) = \lim_{n\to\infty} \tau\left(\bigvee_{i=n}^\infty e_{\frac{1}{i}}^\perp(|x_{k_i} - x|)\right)$$
$$\leqslant \lim_{n\to\infty} \sum_{i=n}^\infty \tau(e_{\frac{1}{i}}^\perp(|x_{k_i} - x|)) \leqslant \lim_{n\to\infty} \sum_{i=n}^\infty \frac{1}{2^i} = 0.$$

So $e^{\perp}=\theta$ and $e=I$. Therefore $x_{k_i}\to x$ on $\mathscr{D}$. $\square$

Remark 2.13 : If $x_k\xrightarrow{d}x$, one can find a subsequence $\{x_{k_i}\}\subseteq\{x_k\}$ such that $x_{k_i}\to x$, but $\{x_k\}$ is not necessarily convergent. For example, let $\mathscr{M}=B(L^2[0,1])$, a noncommutative von Neumann algebra on $L^2[0,1]$. Let $A_n^i=[\frac{i-1}{n},\frac{i}{n}]$ $(i=1,2,\cdots,n,\ n\geq 1)$, one can rearrange $\{A_n^i\}$ by a new order $A_1^1,A_2^1,A_2^2,A_3^1,A_3^2,A_3^3,\cdots$ and denote this new sequence by $\{E_k\}$. Set $x_k(t)=\chi_{E_k}(t)$, then $x_k\xrightarrow{d}\theta$, but $\{x_k\}$ is not convergent to θ.

***Theorem* 2.14** — *Let $\{x_k\}\subseteq L_0(\mathscr{M}),\ x\in L_0(\mathscr{M}),\ \tau(I)<\infty$. If for any subsequence $\{x_{k_i}\}$ of $\{x_k\}$, there is $\{x_{k_{i_j}}\}\subseteq\{x_{k_i}\}$ such that $x_{k_{i_j}}\to x$, then $x_k\xrightarrow{d}x$.*

PROOF : If $\{x_k\}$ is not convergent to x in measure, there exists $\varepsilon>0,\ \delta>0$, for any $N\in\mathbb{N}$, $\exists\, k_N>N$ such that $\tau(e_\varepsilon^{\perp}(|x_{k_N}-x|))\geqslant\delta$. i.e., $\exists\,\varepsilon>0,\ \delta>0$ and $k_1<k_2<\cdots$ such that for any i

$$\tau(e_\varepsilon^{\perp}(|x_{k_i}-x|))\geqslant\delta.$$

For $\{x_{k_i}\}\subseteq\{x_k\}$, there is a subsequence $\{x_{k_{i_j}}\}$ such that $x_{k_{i_j}}\to x$. Using Corollary 2.10, $x_{k_{i_j}}\xrightarrow{d}x$, which is a contradiction. Thus $x_k\xrightarrow{d}x$. $\square$

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