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p, q-ANALOGUE OF A LINEAR TRANSFORMATION PRESERVING LOG-CONVEXITY

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In this paper, we establish the preserving log-convexity of linear transformation associated with p, q-analogue of Pascal triangle, i.e., if the sequence of nonnegative numbers $\{x_n\}_n$ is log-convex, then $y_n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{pq} x_k$ so is it for $q \neq p \geq 1$.

Key words : Log-convexity; linear transformations; p, q-binomial coefficient.

1. INTRODUCTION

We give some necessary termonologie and notation. A sequence of nonnegative numbers $\{x_k\}_k$ is *log-concave* (or *log-convex*) if $x_{i-1}x_{i+1} \leq x_i^2$ (or $x_{i-1}x_{i+1} \geq x_i^2$) for all $i > 0$, which is equivalent to (relevant results can see [4] and [17])

$$x_{i-1}x_{j+1} \leq x_i x_j \text{ (or } x_{i-1}x_{j+1} \geq x_i x_j\text{) for } j \geq i \geq 1. \tag{1}$$

Given two real polynomials $f(x)$ and $g(x)$, we write $f(x)$-$g(x) \geq_x 0$ if $f(x)$-$g(x)$ has nonnegative coefficients as a polynomial in x. Let $\{f_n(x)\}_n$ be a sequence of polynomials in x, we say $\{f_n(x)\}_n$ is x-log-convex (or x-log-concave if

$$f_{n-1}(x)f_{n+1}(x) - f_n(x)^2 \geq_x 0 \text{ (or } f_{n-1}(x)f_{n+1}(x) - f_n(x)^2 \leq_x 0\text{)}, \tag{2}$$

for all $n \geq 1$. Clearly, if the sequence $\{f_n(x)\}_n$ is x-log-convex, then for each fixed positive number x, the sequence $\{f_n(x)\}_n$ is log-convex. The converse is not true in general. If the opposite inequality in (2) holds, then the sequence $\{f_n(x)\}_n$ is x-log-concave. The concept of the x-log-concavity was first suggested by Stanley [17] and this has been of much interest on this subject. We refer the reader to Sagan [14] for further information about the x-log-concavity.

Perhaps the simplest example of x-log-convex polynomials is the $n!$ and x-factorial $[n]_x!$ where $[n]_x = 1 + x + \cdots + x^{n-1}$. There are lot of examples of x-log-convex polynomials. Liu and Wang [11] etablished the x-log-convexity of the following polynomials:

1. Bell polynomials or generating function of the Stirling numbers of the second kind: $B_n(x) = \sum_{k=0}^{n} S(n,k)x^k$;
2. Eulerian polynomials: $A_n(x) = \sum_{k=0}^{n} A(n,k)x^k$;
3. The x-Schröder number: $r_n(x) = \sum_{k=0}^{n} \binom{n+k}{n-k} C_k x^{n-k}$ where C_k is the Catalan number;
4. The x-central Delannoy number: $D_n(x) = \sum_{k=0}^{n} \binom{n+k}{n-k} b_k x^{n-k}$ where $b_k = \binom{2k}{k}$ central binomial coefficient.

Chen *et al*. [5] proved that the Narayana polynomials of type A $N_n(x) = \sum_{k=0}^{n} \frac{1}{n}\binom{n}{k}\binom{n}{k+1}x^k$ are x-log-convex. Chen *et al*. [6] showed that the Narayana polynomials of type B $N_n(x) = \sum_{k=0}^{n} \binom{n}{k}^2 x^k$ are x-log-convex.

Let $\{x_k\}_k$ be a sequence of nonnegative numbers. Let us consider the following linear transformation of sequence

$$y_n = \sum_{k=0}^{n} a(n,k)x_k, \quad (n \geq 0), \tag{3}$$

where $\{a(n,k)\}_{0 \leq k \leq n}$ is a triangular array of nonnegative numbers.

We say that the linear transformation (3) preserves the log-convexity if the log-convexity of $\{x_n\}$ implies that of $\{y_n\}$.

So far there exist a lot of linear transformations preserving log-convexity. For example, it is well known that the following linear transformations,

$$z_n = \sum_{k=0}^{n} \binom{n}{k} x_k, \tag{4}$$

$$b_n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x_k, \tag{5}$$

preserves the log-convexity, see respectively [8] and [13].

In [2], the authors established that the ordinary multinomial transformation

$$t_n = \sum_{k=0}^{n} \binom{n}{k}_s x_k, \tag{6}$$

preserve the log-convexity, where the ordinary multinomial number $\binom{n}{k}_s$ is defined as the k^{th} coefficient in the development

$$(1 + x + x^2 + \cdots + x^s)^n = \sum_{k \in \mathbb{Z}} \binom{n}{k}_s x^k, \tag{7}$$

with $\binom{n}{k}_s = 0$ for $k > sn$ or $k < 0$.

The p, q-analogue of the usual binomial coefficient is called the p, q-binomial coefficient and defined for $p \neq q$ by

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \prod_{i=1}^{k} \frac{p^{n-i+1} - q^{n-i+1}}{p^i - q^i}, \quad 0 \leq k \leq n. \tag{8}$$

Using the notation $[k]_{p,q} = \frac{p^k - q^k}{p-q}$ and $[n]_{p,q}! = [n]_{p,q}[n-1]_{p,q} \cdots [1]_{p,q}$ we can rewrite (8) as follows

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \frac{[n]_{p,q}!}{[n-k]_{p,q}![k]_{p,q}!}, \tag{9}$$

where $\begin{bmatrix} n \\ 0 \end{bmatrix}_{p,q} = 1$ and $\begin{bmatrix} 0 \\ k \end{bmatrix}_{p,q} = \delta_k$. It is well known that the coefficient $\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}$ satisfy the following recurrence relation:

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_{p,q} = p^k \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} + q^{n-k+1} \begin{bmatrix} n \\ k-1 \end{bmatrix}_{p,q}. \tag{10}$$

These coefficients, as for usual binomial coefficients, are built as for the Pascal's triangle, known as p, q-Pascal triangle. In Table 1, the first few values of the p, q-binomial coefficients are provided.

n/k	0	1	2	3
0	1			
1	1	1		
2	1	$p+q$	1	
3	1	p^2+pq+q^2	p^2+pq+q^2	1
4	1	$p^3+p^2q+pq^2+q^3$	$\left(p^2+pq+q^2\right)\left(p^2+q^2\right)$	$\cdots$

Table 1: The p, q-Pascal triangle.

The p, q-binomial coefficients reduces to the q-binomial coefficients (elements of q-Pascal triangle) when $p = 1$. For the theory of p, q-binomial coefficients see [7].

The objective of this paper is to show that the log-convexity is preserved under the p, q-binomial transformation using a theorem of Liu and Wang [11, Theorem 4.8], who established the connection between linear transformations preserving the log-convexity and the x-log-convexity. As an application, we give the log-convexity of the p-Galois numbers.

2. MAIN RESULT

Our main result is as follows.

***Theorem* 1** — *Let $q > p \geq 1$ be two real numbers. Then the linear transformation*

$$y_n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{pq} x_k, \quad n = 0, 1, ... \tag{11}$$

preserves log-convexity.

Liu and Wang [11] obtained a sufficient condition on a triangular array which ensures the corresponding transformation is log-convexity preserving.

Given a triangular array $\{a(n, k)\}_{0 \leq k \leq n}$ of nonnegative real numbers, consider the linear transformation

$$y_n = \sum_{k=0}^{n} a(n, k) x_k, \quad n = 0, 1, 2, \ldots \tag{12}$$

For convenience, let $a(n, k) = 0$ unless $0 \leq k \leq n$. For $0 \leq t \leq 2n$, define

$$a_k(n, t) = a(n+1, k)a(n-1, t-k) + a(n+1, t-k)a(n-1, k) - 2a(n, k)a(n, t-k),$$

if $0 \leq k < \left\lfloor \frac{t}{2} \right\rfloor$, and

$$a_k(n, t) = a(n-1, k)a(n+1, k) - a(n, k)^2,$$

if t is even and $k = t/2$. Also, define

$$A_n(x) = \sum_{k=0}^{n} a(n, k) x^k, \quad n = 0, 1, 2, \ldots$$

It is clear that if the linear transformation (12) preserves the log-convexity, then for each positive number x, the sequence $\{A_n(x)\}$ is log-convex.

The sufficient condition of Liu and Wang is stated as follows.

***Theorem* 2** — *[11, Theorem 4.8]. Assume that the polynomials*

$$A_n(x) = \sum_{k=0}^{n} a(n, k) x^k$$

form a x-log-convex sequence. For any given n and t, if there exists an integer $r = r(n,t)$ such that $a_k(n,t) \geq 0$ for $k \leq r$ and $a_k(n,t) \leq 0$ for $k > r$, then the linear transformation with respect to the triangular array $\{a(n,k)\}_{0\leq k\leq n}$ is log-convexity preserving.

Let $A_n(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{pq} x^k$. Firstly, we show that the polynomial $A_n(x)$ is x-log-convex, so we give the following lemma.

Lemma 1 — Let $j, l \geq 0$ and $n \geq 1$ be integers and let $q > p \geq 1$ be two real numbers. Then we have the following inequality:

$$q^l \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{pq} x^k \sum_{h=0}^{n+1} \begin{bmatrix} n+1 \\ h \end{bmatrix}_{pq} \left(\frac{p^j x}{q^l}\right)^h \geq_x \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{pq} \left(\frac{p^j x}{q^l}\right)^k \sum_{h=0}^{n+1} \begin{bmatrix} n+1 \\ h \end{bmatrix}_{pq} x^h. \tag{13}$$

PROOF : We proceed by induction over n. Taking $n = 1$, the left of the inequality (13) is

$$q^l + (p^{j+1} + p^j q + q^l)x + \frac{p^{2j} + p^{j+1}q^l + p^j q^{l+1}}{q^l} x^2 + \frac{p^{2j}}{q^l} x^3, \tag{14}$$

and the right of the inequality (13) is

$$1 + \frac{p^j + pq^l + q^{l+1}}{q^l} x + \frac{p^{j+1} + p^j q + q^l}{q^l} x^2 + \frac{p^j}{q^l} x^3. \tag{15}$$

It is obvious that (14)$\geq_x$(15).

Now, we suppose that the inequality (13) holds for $n-1$. Then using the recurrence relation, we get

$$\begin{aligned}
& q^l \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{pq} x^k \sum_{h=0}^{n+1} \begin{bmatrix} n+1 \\ h \end{bmatrix}_{pq} \left(\frac{p^j x}{q^l}\right)^h - \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{pq} \left(\frac{p^j x}{q^l}\right)^k \sum_{h=0}^{n+1} \begin{bmatrix} n+1 \\ h \end{bmatrix}_{pq} x^h \\
= \; & q^l \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{pq} (px)^k \sum_{h=0}^{n} \begin{bmatrix} n \\ h \end{bmatrix}_{pq} \left(\frac{p^{j+1} x}{q^l}\right)^h + q^{n+l+1} \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{pq} (px)^k \\
& \sum_{h=0}^{n} \begin{bmatrix} n \\ h \end{bmatrix}_{pq} \left(\frac{p^j x}{q^{l+1}}\right)^{h+1} + q^{n+l} \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{pq} (\frac{x}{q})^{k+1} \sum_{h=0}^{n} \begin{bmatrix} n \\ h \end{bmatrix}_{pq} \left(\frac{p^{j+1} x}{q^l}\right)^h \\
& + q^{2n+l+1} \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{pq} (\frac{x}{q})^{k+1} \sum_{h=0}^{n} \begin{bmatrix} n \\ h \end{bmatrix}_{pq} \left(\frac{p^j x}{q^{l+1}}\right)^{h+1} \\
& - \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{pq} \left(\frac{p^{j+1} x}{q^l}\right)^k \sum_{h=0}^{n} \begin{bmatrix} n \\ h \end{bmatrix}_{pq} (px)^h - q^{n+1} \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{pq} \left(\frac{p^{j+1} x}{q^l}\right)^k \\
& \sum_{h=0}^{n} \begin{bmatrix} n \\ h \end{bmatrix}_{pq} (\frac{x}{q})^{h+1} - q^n \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{pq} \left(\frac{p^j x}{q^{l+1}}\right)^{k+1} \sum_{h=0}^{n} \begin{bmatrix} n \\ h \end{bmatrix}_{pq} (px)^h \\
& - q^{2n+1} \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{pq} \left(\frac{p^j x}{q^{l+1}}\right)^{k+1} \sum_{h=0}^{n} \begin{bmatrix} n \\ h \end{bmatrix}_{pq} (\frac{x}{q})^{h+1}.
\end{aligned}$$

It follows that

$$
\begin{aligned}
& q^l \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{pq} x^k \sum_{h=0}^{n+1} \begin{bmatrix} n+1 \\ h \end{bmatrix}_{pq} \left(\frac{p^j x}{q^l}\right)^h - \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{pq} \left(\frac{p^j x}{q^l}\right)^k \sum_{h=0}^{n+1} \begin{bmatrix} n+1 \\ h \end{bmatrix}_{pq} x^h \\
= \ & \left\{ q^l \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{pq} (px)^k \sum_{h=0}^{n} \begin{bmatrix} n \\ h \end{bmatrix}_{pq} \left(\frac{p^{j+1} x}{q^l}\right)^h \right. \\
- \ & \left. \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{pq} \left(\frac{p^{j+1} x}{q^l}\right)^k \sum_{h=0}^{n} \begin{bmatrix} n \\ h \end{bmatrix}_{pq} (px)^h \right\} \\
& + x q^n \left\{ q^{l-1} \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{pq} (\frac{x}{q})^k \sum_{h=0}^{n} \begin{bmatrix} n \\ h \end{bmatrix}_{pq} \left(\frac{p^{j+1} x}{q^l}\right)^h \right. \\
& \left. - \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{pq} \left(\frac{p^{j+1} x}{q^l}\right)^k \sum_{h=0}^{n} \begin{bmatrix} n \\ h \end{bmatrix}_{pq} (\frac{x}{q})^h \right\} \\
& + x q^{n-l-1} p^j \left\{ q^{l+1} \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{pq} (px)^k \sum_{h=0}^{n} \begin{bmatrix} n \\ h \end{bmatrix}_{pq} \left(\frac{p^j x}{q^{l+1}}\right)^h \right. \\
& \left. - \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{pq} \left(\frac{p^j x}{q^{l+1}}\right)^k \sum_{h=0}^{n} \begin{bmatrix} n \\ h \end{bmatrix}_{pq} (px)^h \right\} \\
& \qquad + x^2 q^{2n-l-1} p^j \left\{ q^l \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{pq} (\frac{x}{q})^k \sum_{h=0}^{n} \begin{bmatrix} n \\ h \end{bmatrix}_{pq} \left(\frac{p^j x}{q^{l+1}}\right)^h \right. \\
& \qquad \left. - \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{pq} \left(\frac{p^j x}{q^{l+1}}\right)^k \sum_{h=0}^{n} \begin{bmatrix} n \\ h \end{bmatrix}_{pq} (\frac{x}{q})^h \right\},
\end{aligned}
$$

by the induction assumption every term of the above sum is a polynomial on x with nonnegative coefficients. Thus we obtain

$$
q^l \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{pq} x^k \sum_{h=0}^{n+1} \begin{bmatrix} n+1 \\ h \end{bmatrix}_{pq} \left(\frac{p^j x}{q^l}\right)^h \geq_x \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{pq} \left(\frac{p^j x}{q^l}\right)^k \sum_{h=0}^{n+1} \begin{bmatrix} n+1 \\ h \end{bmatrix}_{pq} x^h,
$$

as desired.

PROOF OF THEOREM 1 : From the recurrence relation (10), we have

$$
A_{n+1}(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{pq} (px)^k + q^{n+1} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{pq} (\frac{x}{q})^{k+1},
$$

and

$$
A_n(x) = \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{pq} (px)^k + q^n \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{pq} (\frac{x}{q})^{k+1},
$$

then

$$\begin{aligned} & A_{n+1}(x)\ A_{n-1}(x)\ - \{A_n(x)\}^2 \\ = \ & \left\{ \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{pq} (x)^k \sum_{h=0}^{n} \begin{bmatrix} n \\ h \end{bmatrix}_{pq} (px)^h - \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{pq} (px)^k \sum_{h=0}^{n} \begin{bmatrix} n \\ h \end{bmatrix}_{pq} (x)^h \right\} \\ & + q^n \left\{ \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{pq} (x)^k \sum_{h=0}^{n} \begin{bmatrix} n \\ h \end{bmatrix}_{pq} (\frac{x}{q})^{h+1} - \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{pq} (\frac{x}{q})^{k+1} \sum_{h=0}^{n} \begin{bmatrix} n \\ h \end{bmatrix}_{pq} (x)^h \right\}. \end{aligned}$$

From Lemma 1, we obtain that $A_{n+1}(x)\ A_{n-1}(x)\ - \{A_n(x)\}^2 \geq_x 0$ holds for each n. Thus the polynomial $A_n(x)$ is x-log-convex.

By the definition, we have

$$\begin{aligned} a_k(n,t) &= \begin{bmatrix} n-1 \\ k \end{bmatrix}_{pq} \begin{bmatrix} n+1 \\ t-k \end{bmatrix}_{pq} + \begin{bmatrix} n-1 \\ t-k \end{bmatrix}_{pq} \begin{bmatrix} n+1 \\ k \end{bmatrix}_{pq} - 2 \begin{bmatrix} n \\ k \end{bmatrix}_{pq} \begin{bmatrix} n \\ t-k \end{bmatrix}_{pq} \\ &= \frac{[n-1]_{pq}!\,[n+1]_{pq}!}{[k]_{pq}!\,[t-k]_{pq}!\,[n-1]_{pq}!\,[n+1-t+k]_{pq}!\,[n+1-k]_{pq}!} A_k(p,q), \end{aligned}$$

when $k < t/2$, and

$$\begin{aligned} a_k(n,t) &= \begin{bmatrix} n-1 \\ k \end{bmatrix}_{pq} \begin{bmatrix} n+1 \\ k \end{bmatrix}_{pq} - \begin{bmatrix} n \\ k \end{bmatrix}_{pq}^2 \\ &= \frac{[n-1]_{pq}!\,[n]_{pq}!}{[k]_{pq}!^2\,[n+1-k]_{pq}!^2} A_k(p,q), \end{aligned}$$

when t even and $k = t/2$, where

$$\begin{aligned} A_k(p,q) &= [n+1]_{pq} \left\{ [n-t+k]_{pq}\,[n+1-t+k]_{pq} + [n-k]_{pq}\,[n+1-k]_{pq} \right\} \\ & \quad -2\,[n]_{pq}\,[n+1-k]_{pq}\,[n+1-t+k]_{pq}\,. \end{aligned}$$

Clearly, $a_k(n,t)$ has the same sign as that of $A_k(p,q)$ for each k. The derivative of $A_k(p,q)$ with respect to k is given as follows:

$$A'_k(p,q) = \begin{cases} 2(2n+1)(2k-t) & \text{when } p=q=1, \\ \frac{\ln q}{(q-1)^3} \left(q^{n-t+k} - q^{n-k}\right) \{ \left(q^{n+1}-1\right) [2q \left(q^{n-t+k} + q^{n-k}\right) & \\ \quad - q - 1] + 2q\,(q^n - 1) \}, & \text{when } p=1 \text{ and } q>1. \end{cases}$$

By simple computation, we check that $A'_k(p,q) \leq 0$. Thus $A_k(p,q)$ changes sign at most once (from nonnegative to nonpositive), and so does $a_k(n,t)$. Hence, by Theorem 2 the linear transformation $y_n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{pq} x_k$ preserves the log-convexity. This completes our proof.

By setting $p = q = 1$ in Theorem 1, we obtain that the linear transformation

$$z_n = \sum_{k=0}^{n} \binom{n}{k} x_k,$$

preserve the log-convexity.

And if $p = 1$ in Theorem 1, we can deduce that the linear transformation

$$z_n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x_k,$$

preserve the log-convexity.

Finaly, we give the p-Galois number [15] as an application of Theorem 1 which is defined by

$$G_{p,q}(n) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{pq}, \quad n = 0, 1, ... \tag{16}$$

with initial values $G_{p,q}(0) = 1$, $G_{p,q}(1) = 2$, and satisfies the recurrence

$$G_{p,q}(n+1) = 2 \sum_{k=0}^{n} p^k \begin{bmatrix} n \\ k \end{bmatrix}_{pq} + (q^k - p^k) G_{p,q}(n-1), \quad n \geq 1. \tag{17}$$

When $p = 1$, the $G_{p,q}(n)$ reduce to the Galois numbers $G_q(n)$ defined by the second order recurrence [9].

$$G_q(n+1) = 2G_q(n) + (q^k - 1) G_q(n-1), \quad n \geq 1. \tag{18}$$

Thus, we obtain the next result.

Corollary 1 — Let $q > p \geq 1$ be two real numbers. Then the sequence of p-Galois numbers

$$G_{p,q}(n) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{pq}, \quad n = 0, 1, ... \tag{19}$$

is log-convex.

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