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STABILITY OF BASIC SEQUENCES VIA NONLINEAR ε-ISOMETRIES[1]

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In this paper, Let X, Y be two real Banach spaces and $\varepsilon \geq 0$. A mapping $f : X \to Y$ is said to be a standard ε-isometry provided $f(0) = 0$ and

$$|\|f(x) - f(y)\| - \|x - y\|| \leq \varepsilon, \text{ for all } x, y \in X.$$

If $\varepsilon = 0$, then it is simply called a standard isometry. We prove a sufficient and necessary condition for which $\{f(x_n)\}_{n\geq1}$ is a basic sequence of Y equivalent to $\{x_n\}_{n\geq1}$ whenever $\{x_n\}_{n\geq1}$ is a basic sequence in X and $f : X \to Y$ is a nonlinear standard isometry. As a corollary we obtain the stability of basic sequences under the perturbation by nonlinear and non-surjective standard ε-isometries.

Key words : ε-isometry, basic sequence, stability, Banach space.

1. Introduction

Throughout this paper, let X, Y be two real Banach spaces and $\varepsilon \geq 0$ and f will be nonlinear. A mapping $f : X \to Y$ is said to be a standard ε-isometry provided $f(0) = 0$ and

$$|\|f(x) - f(y)\| - \|x - y\|| \leq \varepsilon, \text{ for all } x, y \in X.$$

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If $\varepsilon = 0$, then it is simply called a standard isometry. For a subset A of X and Y we denote by span A the vector space spanned by A.

The Mazur-Ulam theorem [8] says that every surjective standard isometry $f : X \to Y$ is a linear isometry. The study of surjective standard ε-isometry has been considered by many mathematicians (see, for instance, Theorem 1.1 [10] which gives a sharp estimate for the Hyers-Ulam problem [6]). For non-surjective standard isometry, we mention the theorem in [5] which was proved in 1968 by Figiel that for every standard isometry $f : X \to Y$, there is a linear operator $P : L(f) \to X$ with $\|P\| = 1$ so that $Pf = Id$ on X, where $L(f)$ is the closure of span $f(X)$ in Y (see also [2]).

***Theorem* 1.1** — *(Omladič-Šemrl). If $f : X \to Y$ is a surjective standard ε-isometry, then there is a surjective linear isometry $U : X \to Y$ such that*

$$\|f(x) - U(x)\| \leq 2\varepsilon, \ for \ all \ x \in X.$$

In 2015, Cheng showed the following sharp weak stability theorem in [3].

***Theorem* 1.2** — *(Cheng et al.). Let $f : X \to Y$ be a standard $\varepsilon-$ isometry for some $\varepsilon \geq 0$. Then for every $x^* \in X^*$, there exists $\varphi(x^*) \in L(f)^*$ with $\|\varphi(x^*)\| = \|x^*\| = r$ such that*

$$|\langle \varphi(x^*), f(x)\rangle - \langle x^*, x\rangle| \leq 2\varepsilon r, \quad for \ all \ x \in X.$$

In 2014, Dai and Dong [4] study the stability of Banach spaces for convexity and smoothness under the perturbation by a nonlinear ε-isometry, indeed, they prove that if such f exists, then X admits some convexity and smoothness inherited from Y.

Definition 1.3 — We define M to be $P^*(X^*)$, where $P^* : X^* \to L(f)^*$ is a linear isometry and $f : X \to Y$ is a standard isometry by Figiel theorem.

In this paper, we study the stability of basic sequences under the perturbation by ε-isometries and show the following: (i) If Y is reflexive and $f : X \to Y$ is a standard isometry then $\{f(x_n)\}_{n\geq 1}$ is a basic sequence equivalent to $\{x_n\}_{n\geq 1}$ if and only if M separates points of $\overline{\mathrm{span}}\{f(x_n)\}_{n\geq 1}$ whenever $\{x_n\}_{n\geq 1}$ is a basic sequence in X; (ii) If Y is reflexive, and admitting KKP (we say that Y has the Kadets-Klee property, for short KKP if whenever $\{y_n\}_{n\geq 1} \subset Y$ and $y \in Y$ such that $\{y_n\}_{n\geq 1}$ converges weakly to y and $\lim_{n\to\infty} \|y_n\| = \|y\|$, then $\{y_n\}_{n\geq 1}$ converges to y in norm topology), $f : X \to Y$ is a standard ε-isometry, and assume that M separates points of $\overline{\mathrm{span}}\{U(x_n)\}_{n\geq 1}$ for a standard isometry U (see Theorem 2.10). (In particular, when Y is strictly convex), then there is a sequence $\{\lambda_n\}_{n\geq 1} \subset \mathbb{R}^{\aleph_0}$ which tends to ∞ such that $\{f(\lambda_n x_n)\}_{n\geq 1}$ is a basic sequence equivalent to $\{\lambda_n x_n\}_{n\geq 1}$ whenever $\{x_n\}_{n\geq 1}$ is a basic sequence in X.

2. Main Results

For definitions and known results on basic sequences in Banach spaces we refer to [1]. According to [1], for a basic sequence $\{x_n\}_{n\geq 1}$ of X we also denote by $[x_n]$ the closure of the vector space spanned by $\{x_n\}_{n\geq 1}$.

Next we recall Rosenthal's theorem as Theorem 2.1 and Lindenstrauss-Zippin theorem as Theorem 2.2.

***Theorem* 2.1** — *(Rosenthal's ℓ_1 theorem [11]). Let $\{x_n\}_{n\geq 1}$ be a bounded sequence in an infinite-dimensional Banach space X. Then either:*

(i) $\{x_n\}_{n\geq 1}$ has a subsequence which is weakly Cauchy, or

(ii) $\{x_n\}_{n\geq 1}$ has a subsequence which is basic and equivalent to the canonical basis of ℓ_1.

***Theorem* 2.2** — *(Lindenstrauss-Zippin [7]). A Banach space X has a unique unconditional basis (up to equivalence) if and only if X is isomorphic to one of the following three spaces: c_0, ℓ_1, or ℓ_2.*

We first study the following so called basic sequences preserving problem.

Problem 2.3 : Let $f : X \to Y$ be a standard isometry. Then whether $\{f(x_n)\}_{n\geq 1}$ is a basic sequence of Y equivalent to $\{x_n\}_{n\geq 1}$ whenever $\{x_n\}_{n\geq 1}$ is a basic sequence in X.

The following example gives a positive answer to Problem 2.3 for seminormalized unconditional bases of ℓ_1.

Example 2.4 : Assume that $f : \ell_1 \to Y$ is a standard isometry, and assume that $\{x_n\}_{n\geq 1}$ is a seminormalized unconditional basis in ℓ_1. Then $\{f(x_n)\}_{n\geq 1}$ is a seminormalized unconditional basis in Y equivalent to $\{x_n\}_{n\geq 1}$.

PROOF : Suppose that $\{x_n\}_{n\geq 1}$ is a seminormalized unconditional basis in ℓ_1. We first show for every sequence of scalars $\{a_n\}_{n\geq 1}$ satisfying that $\sum_{n=1}^{\infty} a_n x_n$ is convergent $\sum_{n=1}^{\infty} a_n f(x_n)$ is also convergent. By Theorem 2.2 $\{x_n\}_{n\geq 1}$ is equivalent to the canonical basis $\{e_n\}_{n\geq 1}$ of ℓ_1, $\sum_{n=1}^{\infty} a_n e_n$ is convergent and hence $\sum_{n=1}^{\infty} |a_n|$ converges. Therefore, we deduce that $\sum_{n=1}^{\infty} a_n f(x_n)$ converges absolutely and hence converges. Let $\{x_n^*\}_{n\geq 1}$ be the biorthogonal functionals associated with $\{x_n\}_{n\geq 1}$, then $\{\varphi(x_n^*)\}_{n\geq 1}$ is the biorthogonal functionals associated with $\{f(x_n)\}_{n\geq 1}$. Let us define $T : [x_n] \to \overline{\text{span}}\{f(x_n)\}_{n\geq 1}$ by $T(\sum_{n=1}^{\infty} a_n x_n) = \sum_{n=1}^{\infty} a_n f(x_n)$. T is well defined, injective and linear. we prove that T is continuous by using the Closed Graph theorem. Suppose $\{u_i\} \to u$ in $[x_n]$ and $Tu_i \to v$ in Y. Thus $u_i = \sum_{n=1}^{\infty} x_n^*(u_i)(x_n)$ and $u = \sum_{n=1}^{\infty} x_n^*(u)(x_n)$. It follows from the continuity of x_n^* and $\varphi(x_n^*)$ that $x_n^*(u_i) \to x_n^*(u)$ and $\varphi(x_n^*)(Tu_i) = x_n^*(u_i) \to \varphi(x_n^*)(v)$ for all n. Hence, we deduce that $x_n^*(u) = \varphi(x_n^*)(v)$ for all n. Therefore $Tu = v$ and so T is continuous. It follows from Figiel's Theorem or Theorem 1.2 that T^{-1} is also continuous on $T([x_n])$ with $\|T^{-1}\| = 1$, so T is an isomorphism between $[x_n]$ and $T([x_n])$. Therefore, $\{f(x_n)\}_{n\geq 1}$ is a seminormalized unconditional basis and we complete the proof. $\square$

Now we will prove a sufficient and necessary condition for which $\{f(x_n)\}_{n\geq 1}$ is a basic sequence equivalent to $\{x_n\}_{n\geq 1}$ in Theorem 2.8.

Lemma 2.5 — Let $\varphi : X^* \to L(f)^*$ be a mapping as defined in Theorem 1.2 with $\|\varphi(x^*)\| = \|x^*\|$ such that for all $x \in X$,

$$|\langle \varphi(x^*), f(x)\rangle - \langle x^*, x\rangle| \leq 2\|x^*\|\varepsilon. \tag{2.1}$$

Then the following statements hold

(i) There exist a closed subspace N of $L(f)^*$ and a linear isometry U from X into N^* such that for each $x^* \in S_{X^*}$ and $x \in X$,

$$\langle \varphi(x^*), U(x)\rangle = \langle x^*, x\rangle; \tag{2.2}$$

(ii) If, in addition, $\varepsilon = 0$, then we may choose for each $x \in X$, $U(x)$ of (i) to be $f(x)$ restricted on N, i.e., for each $x^* \in S_{X^*}$ and $x \in X$, the following equality holds

$$\langle \varphi(x^*), f(x)\rangle = \langle x^*, x\rangle. \tag{2.3}$$

PROOF : (i) The proof can be found in ([4], Proposition 4.5, pp. 1003-1004)

(ii) Let $\varepsilon = 0$ in (2.1). Obviously, $\varphi : X^* \to L(f)^*$ is a linear isometry. Moreover, we have

$$M = N = \varphi(X^*) = \mathrm{span}\{\varphi(x^*) : x^* \in \mathrm{S}_{X^*}\}.$$

It follows that $f : X \to M^*$ is a linear isometry and (2.3) holds. $\square$

Lemma 2.6 — Let $f : X \to Y$ be a standard isometry. Let M and φ be as in (ii) of Lemma 2.5. Assume that $\{x_n\}_{n\geq 1}$ is a basic sequence with basis constant K in X. If we define a new norm $|||\cdot|||$ on $\mathrm{span}\{f(x_n)\}_{n\geq 1}$ for all $y \in \mathrm{span}\{f(x_n)\}_{n\geq 1}$ by

$$|||y||| = \sup_{y^* \in S_M} y^*(y) = \sup_{x^* \in S_{X^*}} \langle \varphi(x^*), y\rangle,$$

then $\{f(x_n)\}_{n\geq 1}$ with respect to the norm $|||\cdot|||$ is a basis of $\overline{\mathrm{span}}\{f(x_n)\}_{n\geq 1}$ isometrically equivalent to $\{x_n\}_{n\geq 1}$.

We assume for the rest of the paper that Y is a reflexive Banach space.

Lemmas 2.7 — Let f, M and φ be as in (ii) of Lemma 2.5. If we denote $\mathrm{span}\{f(x_n)\}_{n\geq 1}$ by E and denote the dual of E by E^*, then M separates points of E. If, in addition, M separates points of $\overline{E}^{\|\cdot\|}$ then E^* with respect to its original dual norm $\|\cdot\|^*$ is linear isomorphic to M.

PROOF : Since M separates points of $\overline{E}^{\|\cdot\|}$, for all $y^* \in M$ there is $x^* \in X^*$ (assume without loss of generality that $X = [x_n]$) such that $y^* = \varphi(x^*)$ and we can define a bounded linear operator T from M into E^* for all $e \in \overline{E}^{\|\cdot\|}$ by

$$T(y^*)(e) = T(\varphi(x^*))(e) = \varphi(x^*)(e).$$

Since that $\varphi(x^*)(f(x_n)) = 0$ for all natural number n implies $x^* = 0$, T is injective. On the other hand, the Hahn-Banach theorem entails that $\overline{T(M)}^{w*} = E^*$, where w^*-topology on E^* denotes the topology induced by $\overline{E}^{\|\cdot\|}$. Since Y is reflexive, it suffices to show that $\overline{T(M)}^{w*} = \overline{T(M)}^{w} = \overline{T(M)}^{\|\cdot\|} \subseteq T(M)$.

Assume that $(y_n^*) \subset T(M)$ is convergent to $y_0^* \in \overline{T(M)}^{\|\cdot\|}$ in the norm topology. Then for all $n \in N$ there exist $x_n^* \in X^*$ such that

$$\lim_n \varphi(x_n^*)(e) = y_0^*(e), \;\; \text{for all } e \in \overline{E}^{\|\cdot\|}.$$

Thus

$$\lim_n \varphi(x_n^*)(f(x_m)) = \lim_n x_n^*(x_m)$$

exists for all $m \in N$.

We deduce from the boundedness of $\{x_n^*\}_{n\geq 1}$ that there exists $x^* \in X^*$ and a subsequence $\{x_{n_i}^*\}_{i\geq 1} \subset \{x_n^*\}_{n\geq 1}$ such that

$$\lim_i x_{n_i}^*(x) = x^*(x), \;\; \text{for all } x \in X.$$

Therefore,

$$\lim_i \varphi(x_{n_i}^*)(f(x_m)) = \lim_i x_{n_i}^*(x_m) = x^*(x_m) = \varphi(x^*)(f(x_m)).$$

Hence,

$$\lim_i \varphi(x_{n_i}^*)(e) = \varphi(x^*)(e),$$

exists for all $e \in \overline{E}^{\|\cdot\|}$. Then $y_0^* = T(\varphi(x^*)) \in T(M)$. It follows from the Inverse Mapping Theorem that T is a linear isomorphism from M to E^*. □

***Theorem* 2.8** — *Let $f : X \to Y$ be a standard isometry. If $\{x_n\}_{n\geq 1}$ is a basic sequence in X, then $\{f(x_n)\}_{n\geq 1}$ is a basic sequence in Y if and only if M separates points of $\overline{E}^{\|\cdot\|}$.*

PROOF : If $\{f(x_n)\}_{n\geq 1}$ is a basic sequence in Y, for every nonzero element $e \in \overline{E}^{\|\cdot\|}$ there is a unique sequence $\{a_n\}_{n\geq 1}$ of scalars such that

$$e = \sum_{n=1}^{\infty} a_n f(x_n).$$

We deduce for every $x^* \in X^*$ that

$$\varphi(x^*)(e) = \sum_{n=1}^{\infty} a_n \langle \varphi(x^*), f(x_n) \rangle = \langle x^*, \sum_{n=1}^{\infty} a_n x_n \rangle.$$

Hence M separates points of $\overline{E}^{\|\cdot\|}$. Using Lemma 2.6 and Lemma 2.7 we can easily complete the proof of (if) part. Indeed, for every sequence of scalars $\{a_n\}_{n\geq 1}$ and all the integers k, l such that $k \leq l$,

$$\|\sum_{n=1}^{k} a_n f(x_n)\| = \sup_{y^* \in B_{E^*}} \sum_{n=1}^{k} a_n y^*(f(x_n)) \tag{2.4}$$

$$= \sup_{y^* \in TT^{-1}B_{E^*}} \sum_{n=1}^{k} a_n y^*(f(x_n)) \tag{2.5}$$

$$\leq \sup_{y^* \in \|T^{-1}\| TB_M} \sum_{n=1}^{k} a_n y^*(f(x_n)) \tag{2.6}$$

$$\leq \|T\| \cdot \|T^{-1}\| \cdot \sup_{x^* \in B_{[x_n]^*}} \sum_{n=1}^{k} a_n \varphi(x^*) f(x_n) \tag{2.7}$$

$$\leq \|T\| \cdot \|T^{-1}\| \sup_{x^* \in B_{[x_n]^*}} \sum_{n=1}^{k} a_n x^*(x_n) \tag{2.8}$$

$$\leq K\|T\| \cdot \|T^{-1}\| \cdot \|\sum_{n=1}^{l} a_n x_n\| \tag{2.9}$$

$$\leq K\|T\| \cdot \|T^{-1}\| \cdot \|\sum_{n=1}^{l} a_n f(x_n)\| \tag{2.10}$$

and we complete the proof. □

As a result of Theorem 2.8, we obtain the stability of basic sequences under the perturbation by non-surjective standard ε-isometries.

Lemma 2.9 — Let X be separable with dense set $\{x_m\}$ and admitting KKP. If $\varepsilon \geq 0$, $f : X \to Y$ is a ε-isometry, then there is an isometry $U : X \to Y$ such that

$$U(x_m) \equiv \lim_k \frac{f(n_k x_m)}{n_k} \text{ exists for all } m \in N.$$

PROOF : Given $x \in X$, by the Rosenthal's ℓ_1 theorem (i.e., Theorem 2.1), there exists a weakly Cauchy subsequence $\{\frac{f(n_k x)}{n_k}\}_{k=1}^{\infty}$ of $\{\frac{f(nx)}{n}\}_{n\geq 1}$. Since Y is reflexive, $\{\frac{f(n_k x)}{n_k}\}_{k=1}^{\infty}$ is a w-convergent sequence in Y. Let $\{x_m\}_{m=1}^{\infty}$ be a norm-dense sequence of X. By a standard argument of principal diagonal, we can without loss of generality assume that $\{n_k\}_{k=1}^{\infty}$ satisfies that

$$U(x_m) \equiv w - \lim_k \frac{f(n_k x_m)}{n_k} \text{ exists for all } m \in N.$$

By Theorem 1.2, for each $x^* \in S_{X^*}$, there is a functional $\phi \in S_{Y^*}$ such that

$$|\langle \phi, f(x)\rangle - \langle x^*, x\rangle| \leq 2\varepsilon, \text{ for all } x \in X.$$

Hence

$$|\langle \phi, \frac{f(n_k x_m)}{n_k}\rangle - \langle x^*, x_m\rangle| \leq \frac{2\varepsilon}{n_k}, \text{ for all } m,\ k \in N.$$

Let k tend to ∞, we deduce that for all $m \in N$,

$$\langle \phi, U(x_m)\rangle = \langle x^*, x_m\rangle. \tag{2.11}$$

Given $m,\ n \in N$, by the Hahn-Banach theorem and (2.11) we can choose a norm-attaining functional $x^* \in S_{X^*}$ such that

$$\begin{aligned} \|x_m - x_n\| &= x^*(x_m - x_n) \\ &= \langle \phi, U(x_m)\rangle - \langle \phi, U(x_n)\rangle \\ &\leq \|U(x_m) - U(x_n)\|. \end{aligned} \tag{2.12}$$

On the other hand, by the w-lower semicontinuous argument of a norm in Y, we deduce that for every $m, n \in N$

$$\begin{aligned} \|U(x_m) - U(x_n)\| &= \|w - \lim_k \frac{f(n_k x_m) - f(n_k x_n)}{n_k}\| \\ &\leq \liminf_k \frac{\|n_k x_m - n_k x_n\| + \varepsilon}{n_k} \\ &= \|x_m - x_n\|. \end{aligned} \tag{2.13}$$

Therefore, combining (2.12) and (2.13) $U:\{x_m\}_{m=1}^{\infty}\to Y$ is an isometry. Obviously, U has a unique extension $\overline{U}:X\to Y$ such that $\overline{U}$ is also an isometry and for all $x\in X$,

$$\langle \phi, \overline{U}(x)\rangle = \langle x^*, x\rangle. \tag{2.14}$$

Since $\lim_k \|\frac{f(n_k x_m)}{n_k}\| = \|x_m\|$ for all $m\in N$ and Y has KKP, we deduce that the limit

$$U(x_m) \equiv w-\lim_k \frac{f(n_k x_m)}{n_k} \equiv \lim_k \frac{f(n_k x_m)}{n_k} \text{ exists for all } m\in N.\Box$$

***Theorem* 2.10** — *Let $\{x_n\}_{n\geq 1}$ be a basic sequence of X and Y be admitting KKP. If $\varepsilon>0$, $f:X\to Y$ is a ε-isometry, and assume that M separates points of $\overline{E}=\overline{\text{span}}\{U(x_n)\}_{n\geq 1}$ with respect to U defined in Lemma 2.9 (In particular, when Y is strictly convex) then there is a sequence $\{\lambda_n\}_{n\geq 1}\subset\mathbb{R}^{\aleph_0}$ which tends to ∞ such that $\{f(\lambda_n x_n)\}_{n\geq 1}$ is a basic sequence equivalent to $\{\lambda_n x_n\}_{n\geq 1}$.*

PROOF : Assume without loss of generality that $\overline{\text{span}}\{x_n\}_{n\geq 1}=X$. By Lemma 2.9, we can choose norm dense sequence containing $\{x_n\}_{n\geq 1}$ such that

$$U(x_n)\equiv \lim_k \frac{f(n_k x_n)}{n_k} \text{ exists for all } n\in N.$$

We deduce from Theorem 2.8 that $\{U(x_n)\}_{n\geq 1}$ is a basic sequence equivalent to $\{x_n\}_{n\geq 1}$ (U is even linear when Y is strictly convex [9]). On the other hand, for each $n\in N$ we can choose large enough λ_n such that

$$2K\sum_{n=1}^{\infty}\frac{\|U(x_n)-\frac{f(\lambda_n x_n)}{\lambda_n}\|}{\|U(x_n)\|}=\theta<1.$$

Hence by using principle of small perturbations, $\{\frac{f(\lambda_n x_n)}{\lambda_n}\}_{n\geq 1}$ is also a basic sequence equivalent to $\{x_n\}_{n\geq 1}$, and we complete the proof. $\Box$

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REFERENCES

1. F. Albiac and N. J. Kalton, *Topics in Banach space theory*, Graduate Texts in Mathematics 233, Springer, New York (2006).
2. Y. Benyamini and J. Lindenstrauss, Geometric nonlinear functional analysis I, *Amer. Math. Soc. Colloquium Publications*, Providence, RI, **48** (2000).
3. L. Cheng, Q. Cheng, K. Tu and J. Zhang, A universal theorem for stability of ε-isometries of Banach spaces, *J. Funct. Anal.*, **269** (2015), 199-214.
4. D. Dai and Y. Dong, Stability of Banach spaces via nonlinear ε-isometry, *J. Math. Anal. Appl.*, **414** (2014), 996-1005.
5. T. Figiel, On non-linear isometric embeddings of normed linear spaces, *Bull. Acad. Polon. Sci. Math. Astro. Phys.*, **16** (1968), 185-188.
6. D. Hyers and S. Ulam, On approximate isometries. *Bull. Amer. Math. Soc.*, **51** (1945), 288-292.
7. J. Lindenstrauss and M. Zippin, Banach spaces with a unique unconditional basis, *J. Funct. Anal.*, **3** (1969), 115-125.
8. S. Mazur and S. Ulam, Sur les transformations isométriques déspaces vectoriels normés. *C.R. Acad. Sci. Paris.*, **194** (1932), 946-948.
9. M. S. Moslehian and G. Sadeghi, A Mazur-Ulam theorem in non-Archimedean normed spaces, *Nonlinear Analysis: TMA.*, **69** (2008), 3405-3408.
10. M. Omladič, P. Šemrl, On non linear perturbations of isometries, *Math. Ann.*, **303** (1995), 617-628.
11. H. P. Rosenthal, A characterization of Banach spaces containing ℓ_1, *Proc. Natl. Acad. Sci. U.S.A.*, **71** (1974), 2411-2413.