In this paper, let $X$, $Y$ be two real Banach spaces and $\varepsilon \geq 0$. A mapping $f : X \to Y$ is said to be a standard $\varepsilon$-isometry provided $f(0) = 0$ and

$$|||f(x) - f(y)|| - ||x - y||| \leq \varepsilon, \text{ for all } x, y \in X.$$ 

If $\varepsilon = 0$, then it is simply called a standard isometry. We prove a sufficient and necessary condition for which $\{f(x_n)\}_{n \geq 1}$ is a basic sequence of $Y$ equivalent to $\{x_n\}_{n \geq 1}$ whenever $\{x_n\}_{n \geq 1}$ is a basic sequence in $X$ and $f : X \to Y$ is a nonlinear standard isometry. As a corollary we obtain the stability of basic sequences under the perturbation by nonlinear and non-surjective standard $\varepsilon$-isometries.

**Key words**: $\varepsilon$-isometry, basic sequence, stability, Banach space.

1. **Introduction**

Throughout this paper, let $X$, $Y$ be two real Banach spaces and $\varepsilon \geq 0$ and $f$ will be nonlinear. A mapping $f : X \to Y$ is said to be a standard $\varepsilon$-isometry provided $f(0) = 0$ and

$$|||f(x) - f(y)|| - ||x - y||| \leq \varepsilon, \text{ for all } x, y \in X.$$ 

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If \( \varepsilon = 0 \), then it is simply called a standard isometry. For a subset \( A \) of \( X \) and \( Y \) we denote by \( \text{span} \, A \) the vector space spanned by \( A \).

The Mazur-Ulam theorem [8] says that every surjective standard isometry \( f : X \to Y \) is a linear isometry. The study of surjective standard \( \varepsilon \)-isometry has been considered by many mathematicians (see, for instance, Theorem 1.1 [10] which gives a sharp estimate for the Hyers-Ulam problem [6]).

For non-surjective standard isometry, we mention the theorem in [5] which was proved in 1968 by Figiel that for every standard isometry \( f : X \to Y \), there is a linear operator \( P : L(f) \to X \) with \( \|P\| = 1 \) so that \( Pf = Id \) on \( X \), where \( L(f) \) is the closure of span \( f(X) \) in \( Y \) (see also [2]).

**Theorem 1.1** — (Omladič-Šemrl). If \( f : X \to Y \) is a surjective standard \( \varepsilon \)-isometry, then there is a surjective linear isometry \( U : X \to Y \) such that

\[
\|f(x) - U(x)\| \leq 2\varepsilon, \quad \text{for all } x \in X.
\]

In 2015, Cheng showed the following sharp weak stability theorem in [3].

**Theorem 1.2** — (Cheng et al.). Let \( f : X \to Y \) be a standard \( \varepsilon \)-isometry for some \( \varepsilon \geq 0 \). Then for every \( x^* \in X^* \), there exists \( \varphi(x^*) \in L(f)^* \) with \( \|\varphi(x^*)\| = \|x^*\| = r \) such that

\[
|\langle \varphi(x^*), f(x) \rangle - \langle x^*, x \rangle| \leq 2\varepsilon r, \quad \text{for all } x \in X.
\]

In 2014, Dai and Dong [4] study the stability of Banach spaces for convexity and smoothness under the perturbation by a nonlinear \( \varepsilon \)-isometry, indeed, they prove that if such \( f \) exists, then \( X \) admits some convexity and smoothness inherited from \( Y \).

**Definition 1.3** — We define \( M \) to be \( P^*(X^*) \), where \( P^* : X^* \to L(f)^* \) is a linear isometry and \( f : X \to Y \) is a standard isometry by Figiel theorem.

In this paper, we study the stability of basic sequences under the perturbation by \( \varepsilon \)-isometries and show the following: (i) If \( Y \) is reflexive and \( f : X \to Y \) is a standard isometry then \( \{f(x_n)\}_{n \geq 1} \) is a basic sequence equivalent to \( \{x_n\}_{n \geq 1} \) if and only if \( M \) separates points of \( \overline{\text{span}} \{f(x_n)\}_{n \geq 1} \) whenever \( \{x_n\}_{n \geq 1} \) is a basic sequence in \( X \); (ii) If \( Y \) is reflexive, and admitting KKP (we say that \( Y \) has the Kadets-Klee property, for short KKP if whenever \( \{y_n\}_{n \geq 1} \subset Y \) and \( y \in Y \) such that \( \{y_n\}_{n \geq 1} \) converges weakly to \( y \) and \( \lim_{n \to \infty} \|y_n\| = \|y\| \), then \( \{y_n\}_{n \geq 1} \) converges to \( y \) in norm topology), \( f : X \to Y \) is a standard \( \varepsilon \)-isometry, and assume that \( M \) separates points of \( \overline{\text{span}} \{U(x_n)\}_{n \geq 1} \) for a standard isometry \( U \) (see Theorem 2.10). (In particular, when \( Y \) is strictly convex), then there is a sequence \( \{\lambda_n\}_{n \geq 1} \subset \mathbb{R}^{N_0} \) which tends to \( \infty \) such that \( \{f(\lambda_n x_n)\}_{n \geq 1} \) is a basic sequence equivalent to \( \{\lambda_n x_n\}_{n \geq 1} \) whenever \( \{x_n\}_{n \geq 1} \) is a basic sequence in \( X \).
2. MAIN RESULTS

For definitions and known results on basic sequences in Banach spaces we refer to [1]. According to [1], for a basic sequence \( \{x_n\}_{n \geq 1} \) of \( X \) we also denote by \([x_n]\) the closure of the vector space spanned by \( \{x_n\}_{n \geq 1} \).

Next we recall Rosenthal’s theorem as Theorem 2.1 and Lindenstrauss-Zippin theorem as Theorem 2.2.

**Theorem 2.1** — (Rosenthal’s \( \ell_1 \) theorem [11]). Let \( \{x_n\}_{n \geq 1} \) be a bounded sequence in an infinite-dimensional Banach space \( X \). Then either:

(i) \( \{x_n\}_{n \geq 1} \) has a subsequence which is weakly Cauchy, or

(ii) \( \{x_n\}_{n \geq 1} \) has a subsequence which is basic and equivalent to the canonical basis of \( \ell_1 \).

**Theorem 2.2** — (Lindenstrauss-Zippin [7]). A Banach space \( X \) has a unique unconditional basis (up to equivalence) if and only if \( X \) is isomorphic to one of the following three spaces: \( c_0 \), \( \ell_1 \), or \( \ell_2 \).

We first study the following so called basic sequences preserving problem.

**Problem 2.3** : Let \( f : X \to Y \) be a standard isometry. Then whether \( \{f(x_n)\}_{n \geq 1} \) is a basic sequence of \( Y \) equivalent to \( \{x_n\}_{n \geq 1} \) whenever \( \{x_n\}_{n \geq 1} \) is a basic sequence in \( X \).

The following example gives a positive answer to Problem 2.3 for seminormalized unconditional bases of \( \ell_1 \).

**Example 2.4** : Assume that \( f : \ell_1 \to Y \) is a standard isometry, and assume that \( \{x_n\}_{n \geq 1} \) is a seminormalized unconditional basis in \( \ell_1 \). Then \( \{f(x_n)\}_{n \geq 1} \) is a seminormalized unconditional basis in \( Y \) equivalent to \( \{x_n\}_{n \geq 1} \).

**Proof** : Suppose that \( \{x_n\}_{n \geq 1} \) is a seminormalized unconditional basis in \( \ell_1 \). We first show for every sequence of scalars \( \{a_n\}_{n \geq 1} \) satisfying that \( \sum_{n=1}^{\infty} a_n x_n \) is convergent \( \sum_{n=1}^{\infty} f(x_n) \) is also convergent. By Theorem 2.2 \( \{x_n\}_{n \geq 1} \) is equivalent to the canonical basis \( \{e_n\}_{n \geq 1} \) of \( \ell_1 \), \( \sum_{n=1}^{\infty} a_n e_n \) is convergent and hence \( \sum_{n=1}^{\infty} |a_n| \) converges. Therefore, we deduce that \( \sum_{n=1}^{\infty} a_n f(x_n) \) converges absolutely and hence converges. Let \( \{x^*_n\}_{n \geq 1} \) be the biorthogonal functionals associated with \( \{x_n\}_{n \geq 1} \), then \( \{\varphi(x^*_n)\}_{n \geq 1} \) is the biorthogonal functionals associated with \( \{f(x_n)\}_{n \geq 1} \). Let us define \( T : \{x_n\} \to \text{span}\{f(x_n)\}_{n \geq 1} \) by \( T(\sum_{n=1}^{\infty} a_n x_n) = \sum_{n=1}^{\infty} a_n f(x_n) \). \( T \) is well defined, injective and linear. we prove that \( T \) is continuous by using the Closed Graph theorem. Suppose \( \{u_i\} \to u \) in \([x_n]\) and \( T u_i \to v \) in \( Y \). Thus \( u_i = \sum_{n=1}^{\infty} x^*_n(u_i)(x_n) \) and \( u = \sum_{n=1}^{\infty} x^*_n(u)(x_n) \). It follows from the continuity of \( x^*_n \) and \( \varphi(x^*_n) \) that \( x^*_n(u_i) \to x^*_n(u) \) and \( \varphi(x^*_n)(Tu_i) = x^*_n(u_i) \to \varphi(x^*_n)(v) \) for
all \( n \). Hence, we deduce that \( x_n^*(u) = \varphi(x_n^*)(v) \) for all \( n \). Therefore \( Tu = v \) and so \( T \) is continuous. It follows from Figiel’s Theorem or Theorem 1.2 that \( T^{-1} \) is also continuous on \( T([x_n]) \) with \( \|T^{-1}\| = 1 \), so \( T \) is an isomorphism between \( [x_n] \) and \( T([x_n]) \). Therefore, \( \{f(x_n)\}_{n \geq 1} \) is a semi-normalized unconditional basis and we complete the proof.

Now we will prove a sufficient and necessary condition for which \( \{f(x_n)\}_{n \geq 1} \) is a basic sequence equivalent to \( \{x_n\}_{n \geq 1} \) in Theorem 2.8.

**Lemma 2.5** — Let \( \varphi : X^* \to L(f)^* \) be a mapping as defined in Theorem 1.2 with \( \|\varphi(x^*)\| = \|x^*\| \) such that for all \( x \in X \),

\[
|\langle \varphi(x^*), f(x) \rangle - \langle x^*, x \rangle| \leq 2\|x^*\|\varepsilon. \tag{2.1}
\]

Then the following statements hold

(i) There exist a closed subspace \( N \) of \( L(f)^* \) and a linear isometry \( U \) from \( X \) into \( N^* \) such that for each \( x^* \in S_{X^*} \) and \( x \in X \),

\[
\langle \varphi(x^*), U(x) \rangle = \langle x^*, x \rangle; \tag{2.2}
\]

(ii) If, in addition, \( \varepsilon = 0 \), then we may choose for each \( x \in X \), \( U(x) \) of (i) to be \( f(x) \) restricted on \( N \), i.e., for each \( x^* \in S_{X^*} \) and \( x \in X \), the following equality holds

\[
\langle \varphi(x^*), f(x) \rangle = \langle x^*, x \rangle. \tag{2.3}
\]

**Proof:** (i) The proof can be found in ([4], Proposition 4.5, pp. 1003-1004)

(ii) Let \( \varepsilon = 0 \) in (2.1). Obviously, \( \varphi : X^* \to L(f)^* \) is a linear isometry. Moreover, we have

\[
M = N = \varphi(X^*) = \text{span}\{\varphi(x^*) : x^* \in S_{X^*}\}.
\]

It follows that \( f : X \to M^* \) is a linear isometry and (2.3) holds. \( \Box \)

**Lemma 2.6** — Let \( f : X \to Y \) be a standard isometry. Let \( M \) and \( \varphi \) be as in (ii) of Lemma 2.5. Assume that \( \{x_n\}_{n \geq 1} \) is a basic sequence with basis constant \( K \) in \( X \). If we define a new norm \( \|\cdot\| \) on \( \text{span}\{f(x_n)\}_{n \geq 1} \) for all \( y \in \text{span}\{f(x_n)\}_{n \geq 1} \) by

\[
\|y\| = \sup_{y^* \in S_M} y^*(y) = \sup_{x^* \in S_{X^*}} \langle \varphi(x^*), y \rangle,
\]

then \( \{f(x_n)\}_{n \geq 1} \) with respect to the norm \( \|\cdot\| \) is a basis of \( \text{span}\{f(x_n)\}_{n \geq 1} \) isometrically equivalent to \( \{x_n\}_{n \geq 1} \).
We assume for the rest of the paper that $Y$ is a reflexive Banach space.

**Lemma 2.7** — Let $f$, $M$ and $\varphi$ be as in (ii) of Lemma 2.5. If we denote $\text{span}\{f(x_n)\}_{n \geq 1}$ by $E$ and denote the dual of $E$ by $E^*$, then $M$ separates points of $E$. If, in addition, $M$ separates points of $E$ with respect to its original dual norm $\| \cdot \|^*$ then $E^*$ is linear isomorphic to $M$.

**Proof:** Since $M$ separates points of $E$, for all $y^* \in M$ there is $x^* \in X^*$ (assume without loss of generality that $X = [x_n]$) such that $y^* = \varphi(x^*)$ and we can define a bounded linear operator $T$ from $M$ into $E^*$ for all $e \in E$ by

$$T(y^*)(e) = T(\varphi(x^*))(e) = \varphi(x^*)(e).$$

Since that $\varphi(x^*)(f(x_n)) = 0$ for all natural number $n$ implies $x^* = 0$, $T$ is injective. On the other hand, the Hahn-Banach theorem entails that $T(M)^{w^*} = E^*$, where $w^*$-topology on $E^*$ denotes the topology induced by $E^{||}$. Since $Y$ is reflexive, it suffices to show that $T(M)^{w^*} = T(M)^w = T(M)^{||} \subseteq T(M)$.

Assume that $(y^*_n) \subset T(M)$ is convergent to $y^*_0 \in T(M)^{||}$ in the norm topology. Then for all $n \in N$ there exist $x^*_n \in X^*$ such that

$$\lim_n \varphi(x^*_n)(e) = y^*_0(e), \quad \text{for all } e \in E^{||}.$$ 

Thus

$$\lim_n \varphi(x^*_n)(f(x_m)) = \lim_n x^*_n(x_m)$$

exists for all $m \in N$.

We deduce from the boundedness of $\{x^*_n\}_{n \geq 1}$ that there exists $x^* \in X^*$ and a subsequence $\{x^*_n\}_{i \geq 1} \subset \{x^*_n\}_{n \geq 1}$ such that

$$\lim_i x^*_n(x) = x^*(x), \quad \text{for all } x \in X.$$ 

Therefore,

$$\lim_i \varphi(x^*_n)(f(x_m)) = \lim_i x^*_n(x_m) = x^*(x_m) = \varphi(x^*)(f(x_m)).$$

Hence,

$$\lim_i \varphi(x^*_n)(e) = \varphi(x^*)(e),$$

exists for all $e \in E^{||}$. Then $y^*_0 = T(\varphi(x^*)) \in T(M)$. It follows from the Inverse Mapping Theorem that $T$ is a linear isomorphism from $M$ to $E^*$.

□
**Theorem 2.8** — Let \( f : X \to Y \) be a standard isometry. If \( \{x_n\}_{n\geq 1} \) is a basic sequence in \( X \), then \( \{f(x_n)\}_{n\geq 1} \) is a basic sequence in \( Y \) if and only if \( M \) separates points of \( \overline{E} \).

**Proof** : If \( \{f(x_n)\}_{n\geq 1} \) is a basic sequence in \( Y \), for every nonzero element \( e \in \overline{E} \) there is a unique sequence \( \{a_n\}_{n\geq 1} \) of scalars such that

\[
e = \sum_{n=1}^{\infty} a_n f(x_n).
\]

We deduce for every \( x^* \in X^* \) that

\[
\varphi(x^*)(e) = \sum_{n=1}^{\infty} a_n \langle \varphi(x^*), f(x_n) \rangle = \langle x^*, \sum_{n=1}^{\infty} a_n x_n \rangle.
\]

Hence \( M \) separates points of \( \overline{E} \). Using Lemma 2.6 and Lemma 2.7 we can easily complete the proof of (if) part. Indeed, for every sequence of scalars \( \{a_n\}_{n\geq 1} \) and all the integers \( k, l \) such that \( k \leq l \),

\[
\| \sum_{n=1}^{k} a_n f(x_n) \| = \sup_{y^* \in B_{E^*}} \sum_{n=1}^{k} a_n y^*(f(x_n)) \leq \| T \| \cdot \| T^{-1} \| \cdot \sum_{n=1}^{k} a_n \varphi(x^*) f(x_n)
\] (2.4)

\[
= \sup_{y^* \in T T^{-1} B_{E^*}} \sum_{n=1}^{k} a_n y^*(f(x_n)) \leq \| T \| \cdot \| T^{-1} \| \cdot \sup_{x^* \in B_{[x_n]^*}} \sum_{n=1}^{k} a_n \varphi(x^*) f(x_n)
\] (2.5)

\[
\leq K \| T \| \cdot \| T^{-1} \| \cdot \sum_{n=1}^{l} a_n x^*(x_n)
\] (2.6)

\[
\leq K \| T \| \cdot \| T^{-1} \| \cdot \sum_{n=1}^{l} a_n x_n \| (2.7)
\]

\[
\leq K \| T \| \cdot \| T^{-1} \| \cdot \sum_{n=1}^{l} a_n f(x_n) \| (2.8)
\]
and we complete the proof. □

As a result of Theorem 2.8, we obtain the stability of basic sequences under the perturbation by non-surjective standard \( \varepsilon \)-isometries.

**Lemma 2.9** — Let \( X \) be separable with dense set \( \{x_m\} \) and admitting KKP. If \( \varepsilon \geq 0 \), \( f : X \to Y \) is a \( \varepsilon \)-isometry, then there is an isometry \( U : X \to Y \) such that

\[
U(x_m) \equiv \lim_{k} \frac{f(n_kx_m)}{n_k}
\]
exists for all \( m \in N \).

**Proof:** Given \( x \in X \), by the Rosenthal’s \( \ell_1 \) theorem (i.e., Theorem 2.1), there exists a weakly Cauchy subsequence \( \{\frac{f(n_kx)}{n_k}\}_{k=1}^{\infty} \) of \( \{\frac{f(nx)}{n}\}_{n \geq 1} \). Since \( Y \) is reflexive, \( \{\frac{f(n_kx)}{n_k}\}_{k=1}^{\infty} \) is a \( w \)-convergent sequence in \( Y \). Let \( \{x_m\}_{m=1}^{\infty} \) be a norm-dense sequence of \( X \). By a standard argument of principal diagonal, we can without loss of generality assume that \( \{n_k\}_{k=1}^{\infty} \) satisfies that

\[
U(x_m) \equiv w - \lim_{k} \frac{f(n_kx_m)}{n_k}
\]
exists for all \( m \in N \).

By Theorem 1.2, for each \( x^* \in S_{X^*} \), there is a functional \( \phi \in S_{Y^*} \) such that

\[
|\langle \phi, f(x) \rangle - \langle x^*, x \rangle| \leq 2\varepsilon,
\]
for all \( x \in X \).

Hence

\[
|\langle \phi, \frac{f(n_kx_m)}{n_k} \rangle - \langle x^*, x_m \rangle| \leq \frac{2\varepsilon}{n_k},
\]
for all \( m, k \in N \).

Let \( k \) tend to \( \infty \), we deduce that for all \( m \in N \),

\[
\langle \phi, U(x_m) \rangle = \langle x^*, x_m \rangle.
\] (2.11)

Given \( m, n \in N \), by the Hahn-Banach theorem and (2.11) we can choose a norm-attaining functional \( x^* \in S_{X^*} \) such that

\[
\|x_m - x_n\| = x^*(x_m - x_n)
\]
\[
= \langle \phi, U(x_m) \rangle - \langle \phi, U(x_n) \rangle
\]
\[
\leq \|U(x_m) - U(x_n)\|.
\] (2.12)

On the other hand, by the \( w \)-lower semicontinuous argument of a norm in \( Y \), we deduce that for every \( m, n \in N \)

\[
\|U(x_m) - U(x_n)\| = \|w - \lim_{k} \frac{f(n_kx_m) - f(n_kx_n)}{n_k}\|
\]
\[
\leq \lim_{k} \inf \left| \frac{n_kx_m - n_kx_n}{n_k} + \varepsilon \right|
\]
\[
= \|x_m - x_n\|.
\] (2.13)
Therefore, combining (2.12) and (2.13) $U : \{x_m\}_{m=1}^{\infty} \to Y$ is an isometry. Obviously, $U$ has a unique extension $\tilde{U} : X \to Y$ such that $\tilde{U}$ is also an isometry and for all $x \in X$,

$$\langle \phi, \tilde{U}(x) \rangle = \langle x^*, x \rangle.$$  \hfill (2.14)

Since $\lim_k \|f(n_kx_m)\| = \|x_m\|$ for all $m \in N$ and $Y$ has KKP, we deduce that the limit

$$U(x_m) \equiv w - \lim_k \frac{f(n_kx_m)}{n_k} \equiv \lim_k \frac{f(n_kx_m)}{n_k}$$

exists for all $m \in N$. □

**Theorem 2.10** — Let $\{x_n\}_{n \geq 1}$ be a basic sequence of $X$ and $Y$ be admitting KKP. If $\varepsilon > 0$, $f : X \to Y$ is a $\varepsilon$-isometry, and assume that $M$ separates points of $E = \text{span}\{U(x_n)\}_{n \geq 1}$ with respect to $U$ defined in Lemma 2.9 (In particular, when $Y$ is strictly convex) then there is a sequence $\{\lambda_n\}_{n \geq 1} \subset \mathbb{R}^{\aleph_0}$ which tends to $\infty$ such that $\{f(\lambda_nx_n)\}_{n \geq 1}$ is a basic sequence equivalent to $\{\lambda_nx_n\}_{n \geq 1}$.

**Proof**: Assume without loss of generality that $\text{span}\{x_n\}_{n \geq 1} = X$. By Lemma 2.9, we can choose norm dense sequence containing $\{x_n\}_{n \geq 1}$ such that

$$U(x_n) \equiv w - \lim_k \frac{f(n_kx_n)}{n_k} \equiv \lim_k \frac{f(n_kx_n)}{n_k}$$

exists for all $n \in N$.

We deduce from Theorem 2.8 that $\{U(x_n)\}_{n \geq 1}$ is a basic sequence equivalent to $\{x_n\}_{n \geq 1}$ ($U$ is even linear when $Y$ is strictly convex [9]). On the other hand, for each $n \in N$ we can choose large enough $\lambda_n$ such that

$$2K \sum_{n=1}^{\infty} \frac{\|U(x_n) - \frac{f(\lambda_nx_n)}{\lambda_n}\|}{\|U(x_n)\|} = \theta < 1.$$  \hfill (2.15)

Hence by using principle of small perturbations, $\{\frac{f(\lambda_nx_n)}{\lambda_n}\}_{n \geq 1}$ is also a basic sequence equivalent to $\{x_n\}_{n \geq 1}$, and we complete the proof. □

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