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THE PETERSON RECURRENCE FORMULA FOR THE CHROMATIC DISCRIMINANT OF A GRAPH

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The absolute value of the coefficient of q in the chromatic polynomial of a graph G is known as the *chromatic discriminant* of G and is denoted $\alpha(G)$. There is a well known recurrence formula for $\alpha(G)$ that comes from the deletion-contraction rule for the chromatic polynomial. In this paper we prove another recurrence formula for $\alpha(G)$ that comes from the theory of Kac-Moody Lie algebras. We start with a brief survey on many interesting algebraic and combinatorial interpretations of $\alpha(G)$. We use two of these interpretations (in terms of acyclic orientations and spanning trees) to give two bijective proofs for our recurrence formula of $\alpha(G)$.

Key words : Chromatic discriminant; acyclic orientations; spanning trees.

1. INTRODUCTION

Let G be a simple graph and let $\chi(G, q)$ denote its chromatic polynomial. The absolute value of the coefficient of q in $\chi(G, q)$ is known as the *chromatic discriminant* of the graph G [11, 14] and is denoted $\alpha(G)$. It is an important graph invariant with numerous algebraic and combinatorial interpretations. For instance, letting q denote a fixed vertex of the graph G, it is well known that each of the following sets has cardinality $\alpha(G)$:

(1) Acyclic orientations of G with unique sink at q [7],

(2) Maximum G-parking functions relative to q [2],

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(3) Minimal q-critical states [6, Lemmas 14.12.1 and 14.12.2],

(4) Spanning trees of G without broken circuits [3],

(5) Conjugacy classes of Coxeter elements in the Coxeter group associated to G [4, 12, 13],

(6) Multilinear Lyndon heaps on G [9, 10, 16].

In addition, $\alpha(G)$ is also equal to the dimension of the root space corresponding to the sum of all simple roots in the Kac-Moody Lie algebra associated to G [1, 15].

We have the following recurrence formula for $\alpha(G)$ (see for instance [5]) which is an immediate consequence of the well-known *deletion-contraction rule* for the chromatic polynomial:

$$\alpha(G) = \alpha(G\backslash e) + \alpha(G/e), \tag{1.1}$$

where e is any edge of G. Here, $G\backslash e$ denotes G with e deleted and G/e denotes the simple graph obtained from G by identifying the two ends of e (*i.e.*, *contracting* e *to a single vertex*) and removing any multiple edges that result.

Yet another recurrence formula for $\alpha(G)$ was obtained in [15] using its connection to root multiplicities of Kac-Moody Lie algebras. To state this, we introduce some notation: for a graph G, we let $V(G)$ and $E(G)$ denote its vertex and edge sets respectively. We say that the ordered pair (G_1, G_2) is an *ordered partition of* G, if G_1 and G_2 are non-empty subgraphs of G whose vertex sets form a partition of $V(G)$, i.e., they are disjoint and their union is $V(G)$. When we don't care about the ordering of G_1, G_2, we call the set $\{G_1, G_2\}$ an *unordered partition of* G. We say that an edge e *straddles* G_1 and G_2 if one end of e is in G_1 and the other in G_2.

We then have:

Proposition 1 — [15]

$$2\, e(G)\, \alpha(G) = \sum_{\substack{(G_1, G_2) \\ \text{ordered partitions of } G}} \alpha(G_1)\, \alpha(G_2)\, e(G_1, G_2). \tag{1.2}$$

Here $e(G)$ is the total number of edges in G, $e(G_1, G_2)$ is the number of edges that straddle G_1 and G_2, and the sum ranges over ordered partitions of G.

We note that the recurrence formula (1.2) does not seem to follow directly from (1.1). In [15], (1.2) was derived from the *Peterson recurrence formula* [8] for root multiplicities of Kac-Moody Lie algebras. The goal of this paper is to give a purely combinatorial (bijective) proof of (1.2).

To construct a bijective proof, we need sets whose cardinalities are the left and right hand sides of (1.2). We in fact give two bijective proofs, starting with the interpretations of $\alpha(G)$ in terms of acyclic orientations and spanning trees.

2. ACYCLIC ORIENTATIONS WITH UNIQUE FIXED SINK

In this section we give a bijective proof of the recurrence formula (1.2) in terms of acyclic orientations.

We recall that an acyclic orientation of G is an assignment of arrows to its edges such that there are no directed cycles in the resulting directed graph. A sink in an acyclic orientation is a vertex which only has incoming arrows. The set of all acyclic orientations of G is denoted $\mathcal{A}(G)$. For a vertex q of G, the set of all acyclic orientations in which q is the unique sink is denoted $\mathcal{A}(G, q)$. It is well-known that the cardinality of $\mathcal{A}(G, q)$ is independent of q and equals $\alpha(G)$ [7]. The following characterization of $\mathcal{A}(G, q)$ is immediate.

Lemma 1 — Fix a vertex q of G and let $\lambda \in \mathcal{A}(G)$. Then $\lambda \in \mathcal{A}(G, q)$ if and only if for every $p \in V(G)$, there is a directed path in λ from p to q.

This motivates the following:

Definition 1 — Given a vertex q and an acyclic orientation λ of G, let $V(\lambda, q)$ denote the set of all vertices p for which there is a directed path in λ from p to q. We call this the set of q-reachable vertices in λ.

We record the following simple observation:

Lemma 2 — Let x be a vertex of G.

(a) If $x \notin V(\lambda, q)$, then $x \notin V(\lambda, p)$ for all $p \in V(\lambda, q)$.

(b) In particular, an edge joining p and x with $p \in V(\lambda, q)$ and $x \notin V(\lambda, q)$ is oriented from p to x in λ.

Our next goal is to construct sets A and B whose cardinalities are respectively equal to the left and right hand sides of (1.2) and to exhibit a bijection between them. To this end, we first consider the set $\vec{E}$ of oriented edges of G; an element of $\vec{E}$ is an edge of G with an arrow marked on it (in one of two possible ways). Thus $\vec{E}$ has cardinality $2\, e(G)$. If $\vec{e}$ is an element of $\vec{E}$ corresponding to an edge joining vertices p and q with the arrow pointing from p to q, we call p the *tail* of $\vec{e}$ and q its *head*.

We now define A to be the set consisting of pairs $(\vec{e}, \lambda) \in \vec{E} \times \mathcal{A}(G)$ such that the head of $\vec{e}$ is the unique sink of λ. For a fixed $\vec{e}$, there are $\alpha(G)$ choices for λ since λ ranges over $\mathcal{A}(G, q)$ where q is the head of $\vec{e}$. It is now clear that A has cardinality exactly $2\, e(G)\, \alpha(G)$.

To define B, we first take an ordered partition (G_1, G_2) of G. Let $E(G_1, G_2)$ denote the set of edges straddling G_1 and G_2. Let $B(G_1, G_2)$ denote the set of triples $(e, \lambda_1, \lambda_2)$ where $e \in E(G_1, G_2)$, say e joins p_1 and p_2 with p_i a vertex of G_i, $i = 1, 2$, and λ_i is an acyclic orientation of G_i with unique sink at p_i, $i = 1, 2$. Arguing as before, one concludes that $B(G_1, G_2)$ has cardinality $\alpha(G_1)\,\alpha(G_2)\,e(G_1, G_2)$. We now let B denote the disjoint union of the $B(G_1, G_2)$ over all ordered partitions (G_1, G_2) of G. It clearly has cardinality equal to the right hand side of (1.2).

We now define a map $\varphi : A \to B$ which will turn out to be the bijection we seek. Given $(\vec{e}, \lambda) \in A$, let p and q denote the tail and head of $\vec{e}$ respectively. Note that $\lambda \in \mathcal{A}(G, q)$. Let $V_1 = V(\lambda, p)$ denote the set of p-reachable vertices in λ (definition 1) and let $V_2 = V(G)\backslash V_1$. Observe that $p \in V_1$ and $q \in V_2$. For $i = 1, 2$, let G_i denote the subgraphs of G induced by V_i, and let λ_i denote the restriction of λ to G_i.

We claim that λ_1 has a unique sink at p and λ_2 has a unique sink at q. The first assertion follows simply from Lemma 1. For the second assertion, observe that if $x \in V_2 \subset V(G)$, then there is a directed path from x to q in λ. Since $x \notin V(\lambda, p)$, Lemma 2(a) implies that no vertex of this directed path can lie in V_1. In other words this directed path is entirely within G_2, and we are again done by Lemma 1.

Let e denote the undirected edge joining p and q. We have thus shown that the triple $(e, \lambda_1, \lambda_2)$ is in $B(G_1, G_2) \subset B$. We define $\varphi(\vec{e}, \lambda) = (e, \lambda_1, \lambda_2)$.

To see that φ is a bijection, we describe its inverse map. Let (G_1, G_2) be an ordered partition of G; given a triple $(e, \lambda_1, \lambda_2) \in B(G_1, G_2)$, we construct an acyclic orientation λ of G as follows: on G_1 and G_2, we define λ to coincide with λ_1 and λ_2 respectively. It only remains to define an orientation for the straddling edges (this includes e); we orient all of them pointing from G_1 towards G_2, i.e., such that their tails are in G_1 and their heads in G_2. We let $\vec{e}$ denote the edge e with the above orientation.

We claim $(\vec{e}, \lambda) \in A$. First observe that λ is in fact acyclic; since λ extends λ_i for $i = 1, 2$, any directed cycle of λ must necessarily involve vertices from both G_1 and G_2. But this is impossible since all straddling edges point the same way, from G_1 towards G_2. Let p, q denote the tail and head of $\vec{e}$. It remains to show that λ has a unique sink at q, or equivalently, by Lemma 1, that there is a directed path in λ from any vertex x to q. This is clear if x is a vertex of G_2. If x is in G_1, we have a directed path in λ_1 from x to p. Now the edge $\vec{e}$ is directed from p to q; concatenating this directed path with $\vec{e}$ produces a directed path from x to q in λ as required. We define the map $\psi : B \to A$ by $\psi(e, \lambda_1, \lambda_2) = (\vec{e}, \lambda)$.

Observe that for the λ defined above, the set of p-reachable vertices is exactly $V(G_1)$. This is because edges straddling G_1 and G_2 point away from G_1, so no vertex of G_2 is p-reachable. This implies that $\varphi \circ \psi$ is the identity map on B. Further, it readily follows from Lemma 2(b) that $\psi \circ \varphi$ is the identity map on A. This establishes that φ is a bijection. $\square$

3. Spanning Trees Without Broken Circuits

In this section we give another bijective proof of the recurrence formula (1.2), this time using the fact that $\alpha(G)$ counts the number of spanning trees of G without broken circuits.

Definition 2 — Let σ be a total ordering on the set $E(G)$ of edges of G. Given a circuit in G, it has a unique maximum edge with respect to σ; the set of edges obtained by deleting this edge from the circuit is called a *broken circuit* relative to σ. The set of all broken circuits relative to σ is denoted $B_G(\sigma)$.

Let $S_G(\sigma)$ be the set of all spanning trees of G that contain no broken circuits relative to σ. It is well-known that the cardinality of $S_G(\sigma)$ is independent of the choice of σ, and equals $\alpha(G)$ [3].

Given a total ordering σ on $E(G)$, let $\max(\sigma)$ denote the maximum edge in $E(G)$. The following lemma is immediate.

Lemma 3 — Any spanning tree in $S_G(\sigma)$ contains the edge $\max(\sigma)$.

In the sequel, we will fix for each edge e, a total order σ_e on $E(G)$ for which $\max(\sigma_e) = e$. We will write $B_G(e)$ and $S_G(e)$ for the sets $B_G(\sigma_e)$ and $S_G(\sigma_e)$, respectively.

We now proceed to prove (1.2) in the following equivalent form:

$$e(G)\,\alpha(G) = \sum_{\substack{\{G_1,G_2\} \\ \text{unordered partitions of } G}} \alpha(G_1)\,\alpha(G_2)\,e(G_1,G_2). \tag{3.1}$$

We first define the set A to consist of pairs (e,T) where e is an edge and $T \in S_G(e)$; from the above discussion, A has cardinality $e(G)\,\alpha(G)$.

Next, we define the set B. Given an unordered partition $\{G_1,G_2\}$ of G, define $B(\{G_1,G_2\})$ to be the set of pairs $(e,\{T_1,T_2\})$ where e is an edge that straddles G_1 and G_2 and $T_i \in S_{G_i}(e)$ for $i = 1,2$. Here, $S_{G_i}(e)$ is the set of spanning trees of G_i which contain no broken circuits relative to the total order σ_e restricted to the edges of G_i. We let B denote the disjoint union of $B(\{G_1,G_2\})$ as $\{G_1,G_2\}$ ranges over unordered partitions of G. Clearly B has cardinality equal to the right hand side of (3.1).

We define maps $\varphi : A \to B$ and $\psi : B \to A$ as follows:

Given $(e, T) \in A$, e occurs in T in view of Lemma 3. Deleting e from the spanning tree T will result in a pair of trees T_1, T_2 with vertex sets V_1 and V_2. Let G_i denote the subgraph induced by V_i, $i = 1, 2$; clearly $\{G_1, G_2\}$ is an unordered partition of G and e straddles the G_i. Observe that since the total order on $E(G_i)$ is defined as the restriction of the total order σ_e on $E(G)$, T_i will contain no broken circuits of G_i for $i = 1, 2$, i.e., $T_i \in S_{G_i}(e)$. We set $\varphi(e, T) = (e, \{T_1, T_2\})$.

For the inverse map ψ, let $(e, \{T_1, T_2\}) \in B$. Define T to be the spanning tree of G obtained by adding the edge e to the union of T_1 and T_2. To prove that T contains no broken circuits relative to σ_e, observe that any broken circuit of T cannot lie entirely within T_1 or T_2, and must hence contain the edge e. But e is the maximum edge relative to σ_e, so this cannot be a broken circuit by definition. Thus $(e, T) \in A$, and we define $\psi(e, \{T_1, T_2\}) = (e, T)$.

It is straightforward to check that φ and ψ are indeed inverse maps. $\square$

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