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ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO THE DIFFUSION EQUATION

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We study asymptotic behavior of solutions to an initial value problem posed for heat equation. For which, we construct an approximate solution to the initial value problem in terms of derivatives of Gaussian by incorporating the moments of initial function. Spatial shifts are introduced into the leading order term as well as penultimate term of the approximation. This paper is continuation to the work of Yanagisawa [14].

Key words : Asymptotic behavior; Cole-Hopf transformation; heat equation.

1. INTRODUCTION

In this article, we study the large time asymptotic to the solutions of the Cauchy problem

$$\phi_t = \phi_{xx}, \quad x \in \mathbb{R}, \ t > 0, \tag{1}$$

$$\phi(x,0) = \phi_0(x), \quad x \in \mathbb{R}, \tag{2}$$

where

$$(1+|x|)^{k+3+\epsilon}\phi_0 \in L^1(\mathbb{R}), \tag{3}$$

for any positive integer k and a small real number ϵ. This work is continuation to the work done by Yanagisawa [14]. We denote the *j-th order moment* of the function ϕ_0, where j is a non negative integer, by

$$M_j = \int_{\mathbb{R}} x^j \phi_0(x) dx.$$

It is well known [4] that Hopf transformation reduces the viscous Burgers equation to heat equation. Hence, for studying the asymptotic behavior of solutions to the Burgers equation, one may first investigate the asymptotic behavior of solutions to the heat equation.

Chern and Liu [1] studied large time asymptotics of solutions to heat equation by constructing a self similar solution, which converges to the true solution with order $O(t^{-1+\frac{1}{2p}})$ in L^p norm when $t \to \infty$. If $\phi_0(x)e^{|x|}$ is bounded on $\mathbb{R}$, the decay rate of the solution $u(x,t)$ of (1)-(2) is given in Rubinstein and Rubinstein [11] by

$$\begin{aligned} \phi(x,t) \quad &\asymp \quad \frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}} \left[M_0 + t^{-1} \left(\frac{xM_1}{2} - \frac{M_2}{4} \right) \right. \\ &\qquad \left. + \frac{1}{32} t^{-2} \left(4x^2 M_2 - 4xM_3 + M_4 \right) + O(t^{-3}) \right], \quad \text{as } t \to \infty, \end{aligned}$$

when x is bounded on $\mathbb{R}$. Witelski and Bernoff [13] considered the heat equation with nonnegative compactly supported integrable initial functions. They introduced three parameters into the approximate solution, these correspond to the mass, center of mass and variance of the given initial data. Miller and Bernoff [9] studied the heat equation with nonnegative piecewise initial data assuming that third order moment of initial data exists. Using the approximation obtained in Witelski and Bernoff [13], Miller and Bernoff [9] derived the rate of convergence to be of order $O(t^{-2+\frac{1}{2p}})$ in L^p norm when $t \to \infty$. Kloosterziel [8] considered the initial data, which is nonnegative and square integrable with respect to weight function $e^{\frac{x^2}{2}}$ on $\mathbb{R}$. They then expanded the solution of the heat equation in terms of its self-similar solutions. This expansion reveals the large time asymptotics quickly. Srinivasa Rao and Engu [12] studied the relations between the approximations given by Kloosterziel [8] and Miller and Bernoff [9]. Assuming $x^{2n}\phi_0 \in L^1(\mathbb{R})$, ϕ_0 is bounded and not changing its sign on $\mathbb{R}$, Kim and Ni [6] constructed an asymptotic solution

$$\phi_n(x,t) = \sum_{i=1}^{n} \frac{\rho_i}{\sqrt{4\pi t}} e^{-\frac{-(x-c_i)^2}{4t}}$$

to (1)-(2), where ρ_i and c_i are constants to be determined based on the moments of ϕ_0, showing that

$$||\phi(.,t) - \phi_n(.,t)||_p = O(t^{-\frac{2n+1}{2}+\frac{1}{2p}}), \;\; t \to \infty.$$

Kim [7] generalized the theory of the truncated moment problem to include complex measures and then obtained the large time behavior of solutions to heat equation by constructing the approximations as linear combination of heat kernels. Using the generalization of truncated moment problem [7], Engu, Sahoo and Manasa [3] studied the N-wave solutions for heat equation. Duoandikoetxea and Zuazua [2] constructed an approximate solution for heat equation in the following form;

$$\psi_{2n}(x,t) = \sum_{j=0}^{2n-1} (-1)^j \frac{\gamma_i}{\sqrt{4\pi t}} \partial_x^j \left(e^{-\frac{x^2}{4t}} \right) \text{ for } t > 0,\ x \in \mathbb{R}. \tag{4}$$

They showed that the approximate solution ψ_{2n} converges to the solution ϕ with an order $O\left(t^{-\frac{2n+1}{2}+\frac{1}{2p}}\right)$ in L^p-norm when $t \to \infty$.

Generalizing the works of Duoandikoetxea and Zuazua [2] and Miller and Bernoff [9], Yanagisawa [14] introduced an asymptotic approximate solution

$$\phi^k(x,t) = \sum_{j=0}^{k-1} (-1)^j \frac{M_j}{j!} \partial_x^j G_t(x) + (-1)^k \frac{M_k}{k!} \partial_x^k G_{t+(t_k)_+}(x - \gamma_k), \text{ for } t > 0,\ x \in \mathbb{R} \tag{5}$$

to the Cauchy problem (1)-(2), where

$$G_t(x) = \frac{1}{\sqrt{4\pi t}} \text{exp} \left(-\frac{x^2}{4t} \right).$$

They chose space shift γ_k and time shift $(t_k)_+ = \max(t_k, 0)$ by making $(k+1)^{th}$ and $(k+2)^{th}$ moments of $\phi(x,t) - \phi^k(x,t)$ vanish when $t \to 0^+$. As $t \to \infty$, they derived the convergence order to be $O(t^{(\frac{1}{2p}-2-\frac{k}{2})})$ when $(t_k)_+ > 0$ and $O(t^{(\frac{1}{2p}-\frac{3}{2}-\frac{k}{2})})$ when $(t_k)_+ = 0$ with respect to L^p-norm, $1 \leqslant p \leqslant \infty$.

Our main result is stated herewith:

***Theorem* 1.1** — *Let $\phi(x,t)$ be a solution to heat equation (1) subject to (2) satisfying $(1 + |x|)^{k+3+\epsilon} \phi_0 \in L^1(\mathbb{R})$, where k is any positive integer and ϵ is any small positive real number. Then, for $1 \leqslant p \leqslant \infty$, we have*

$$||\phi(.,t) - \phi^k(.,t)||_p = O(t^{-\frac{4+k}{2}+\frac{1}{2p}}), \quad t \to \infty, \tag{6}$$

where ϕ^k is given by

$$\phi^k(x,t) = \begin{cases} \sum_{j=0}^{k-2}(-1)^j \frac{M_j}{j!}\partial_x^j G_t(x) + (-1)^{(k-1)}\frac{M_{k-1}}{(k-1)!}\partial_x^{k-1}G_t\left(x - \frac{M_k}{kM_{k-1}}\right) \\ \qquad \textit{if } M_{k-1} \neq 0, M_{k-1}M_{k+1} = \frac{k+1}{2k}M_k^2,\ \frac{6k^2}{(k+2)(k+1)}M_{k-1}^2 M_{k+2} = M_k^3, \\ \sum_{j=0}^{k-2}(-1)^j \frac{M_j}{j!}\partial_x^j G_t(x) + (-1)^{(k-1)}\frac{M_{k-1}}{(k-1)!}\partial_x^{k-1}G_t(x-b_0) \\ +\frac{(-1)^k}{k!}[M_k - kM_{k-1}b_0]\partial_x^k G_t\left(x - \frac{2M_{k+1}-k(k+1)M_{k-1}b_0^2}{2(k+1)(M_k - kM_{k-1}b_0)}\right) \\ \qquad \textit{if } M_{k-1} \neq 0, \\ \sum_{j=0}^{k-2}(-1)^j \frac{M_j}{j!}\partial_x^j G_t(x) + (-1)^k \frac{M_k}{k!}\partial_x^k G_t\left(x - \frac{M_{k+1}}{(k+1)M_k}\right) \\ \qquad \textit{if } M_{k-1} = 0, M_k \neq 0, M_{k+2}M_k = \frac{k+2}{2(k+1)}[M_{k+1}]^2, \end{cases} \tag{7}$$

where $b_0(\neq \frac{M_k}{kM_{k-1}})$ *is any real valued solution of the quartic equation*

$$\begin{aligned} &[k^2(k+1)(k+2)M_{k-1}^2]b_0^4 - [4k(k+1)(k+2)M_{k-1}M_k]b_0^3 \\ &+[12k(k+2)M_{k-1}M_{k+1}]b_0^2 - [24kM_{k-1}M_{k+2}]b_0 \\ &+12[2M_kM_{k+2} - \frac{(k+2)}{(k+1)}M_{k+1}^2] = 0. \end{aligned} \tag{8}$$

The existence of real valued solution of the quartic equation (8) is discussed in Remark 4.

This paper is organised as follows. In Section 2, we construct an approximate solution to the Cauchy problem (1)-(2) and then extend these results to viscous Burgers equation. Section 3 presents the comparisions of the proposed asymptotic approximate solution with Yanagisawa [14]'s asymptotic approximate solution (5) as well as with Kim and Ni [6]'s asymptotic approximate solution.

2. An Asymptotic Approximation

In this section, we propose an asymptotic approximate solution $\phi^k(x,t)$ to the solution of the initial value problem (1)-(2) by

$$\phi^k(x,t) = \sum_{j=0}^{k-2}(-1)^j \frac{M_j}{j!}\partial_x^j G_t(x) + a_0\partial_x^{k-1}G_t(x-b_0) + a_1\partial_x^k G_t(x-b_1), \tag{9}$$

where the arbitrary constants a_0, a_1, b_0 and b_1 are to be chosen appropriately. Here $G_t(z)$ represents one-dimensional heat kernel

$$G_t(z) = \frac{1}{\sqrt{4\pi t}}\exp\left(-\frac{z^2}{4t}\right).$$

As an outcome of this, we will obtain higher order convergence of ϕ^k as mentioned in the Theorem 1.1.

We now prove few Lemmas which are needed to prove Theorem 1.1.

Lemma 2.1 — Let ϕ be the solution to the problem (1)-(2) and ϕ^k be an asymptotic approximation given in (9), where k is any positive integer. Then we have the following:

$$\lim_{t\to 0^+}\int_{\mathbb{R}} x^{k-1}[\phi(x,t)-\phi^k(x,t)]dx = M_{k-1}-(-1)^{k-1}(k-1)!a_0, \tag{10}$$

$$\lim_{t\to 0^+}\int_{\mathbb{R}} x^{k}[\phi(x,t)-\phi^k(x,t)]dx = M_{k}-(-1)^{k-1}k!a_0b_0-(-1)^{k}k!a_1, \tag{11}$$

$$\begin{aligned}\lim_{t\to 0^+}\int_{\mathbb{R}} x^{k+1}[\phi(x,t)-\phi^k(x,t)]dx &= M_{k+1}-(-1)^{k-1}\frac{(k+1)!}{2}a_0b_0^2\\ &\quad -(-1)^{k}(k+1)!a_1b_1,\end{aligned} \tag{12}$$

$$\begin{aligned}\lim_{t\to 0^+}\int_{\mathbb{R}} x^{k+2}[\phi(x,t)-\phi^k(x,t)]dx &= M_{k+2}-(-1)^{k-1}\frac{(k+2)!}{3!}a_0b_0^3\\ &\quad -(-1)^{k}\frac{(k+2)!}{2}a_1b_1^2.\end{aligned} \tag{13}$$

PROOF : Using integration by parts, we have

$$\lim_{t\to 0^+}\int_{\mathbb{R}} x^{j}\partial_x^l G_t(x)dx = (-1)^l j!\delta_{lj}, \quad \text{for any non-negative integers } l \text{ and } j, \tag{14}$$

where δ_{lj} represents Kronecker's delta. We first prove (10). Making use of (14), it is easy to see that

$$\begin{aligned}\lim_{t\to 0^+}\int_{\mathbb{R}} x^{k-1}\partial_x^{k-1}G_t(x-b_0)dx &= \lim_{t\to 0^+}\int_{\mathbb{R}}(y+b_0)^{k-1}\partial_y^{k-1}G_t(y)dy\\ &= (-1)^{k-1}(k-1)!\end{aligned} \tag{15}$$

and

$$\begin{aligned}\lim_{t\to 0^+}\int_{\mathbb{R}} x^{k-1}\partial_x^{k}G_t(x-b_1)dx &= \lim_{t\to 0^+}\int_{\mathbb{R}}(y+b_1)^{k-1}\partial_y^{k}G_t(y)dy\\ &= 0.\end{aligned} \tag{16}$$

It then follows from (14)-(16) that

$$\begin{aligned}&\lim_{t\to 0^+}\int_{\mathbb{R}} x^{k-1}[\phi(x,t)-\phi^k(x,t)]dx\\ &= M_{k-1}-a_0\lim_{t\to 0^+}\int_{\mathbb{R}} x^{k-1}\partial_x^{k-1}G_t(x-b_0)dx - a_1\lim_{t\to 0^+}\int_{\mathbb{R}} x^{k-1}\partial_x^{k}G_t(x-b_1)dx\\ &= M_{k-1}-(-1)^{k-1}(k-1)!a_0,\end{aligned}$$

which proves (10). We now proceed to prove (11). In view of (14), we have

$$\begin{aligned}
\lim_{t\to 0^+}\int_{\mathbb{R}} x^k \partial_x^{k-1} G_t(x-b_0)dx &= \lim_{t\to 0^+}\int_{\mathbb{R}} (y+b_0)^k \partial_y^{k-1} G_t(y)dy \\
&= (-1)^{k-1}(k-1)!\binom{k}{1} b_0 \\
&= (-1)^{k-1}k! b_0 \qquad (17)
\end{aligned}$$

and

$$\begin{aligned}
\lim_{t\to 0^+}\int_{\mathbb{R}} x^k \partial_x^{k} G_t(x-b_1)dx &= \lim_{t\to 0^+}\int_{\mathbb{R}} (y+b_1)^k \partial_y^{k} G_t(y)dy \\
&= (-1)^k k!. \qquad (18)
\end{aligned}$$

Now (11) can be derived from (14), (17) and (18) as follows:

$$\begin{aligned}
&\lim_{t\to 0^+}\int_{\mathbb{R}} x^k[\phi(x,t)-\phi^k(x,t)]dx \\
&= M_k - a_0 \lim_{t\to 0^+}\int_{\mathbb{R}} x^k \partial_x^{k-1} G_t(x-b_0)dx - a_1 \lim_{t\to 0^+}\int_{\mathbb{R}} x^k \partial_x^{k} G_t(x-b_1)dx \\
&= M_k - (-1)^{k-1}k! a_0 b_0 - (-1)^k k! a_1.
\end{aligned}$$

Next, using (14), we see that

$$\begin{aligned}
\lim_{t\to 0^+}\int_{\mathbb{R}} x^{k+1} \partial_x^{k-1} G_t(x-b_0)dx &= \lim_{t\to 0^+}\int_{\mathbb{R}} (y+b_0)^{k+1} \partial_y^{k-1} G_t(y)dy \\
&= (-1)^{k-1}(k-1)!\binom{k+1}{2} b_0^2 \\
&= (-1)^{k-1}\frac{(k+1)!}{2} b_0^2 \qquad (19)
\end{aligned}$$

and

$$\begin{aligned}
\lim_{t\to 0^+}\int_{\mathbb{R}} x^{k+1} \partial_x^{k} G_t(x-b_1)dx &= \lim_{t\to 0^+}\int_{\mathbb{R}} (y+b_1)^{k+1} \partial_y^{k} G_t(y)dy \\
&= (-1)^k k!\binom{k+1}{1} b_1 \\
&= (-1)^k (k+1)! b_1. \qquad (20)
\end{aligned}$$

Therefore, from (14), (19)-(20), we get

$$\begin{aligned}
&\lim_{t\to 0^+}\int_{\mathbb{R}} x^{k+1}[\phi(x,t)-\phi^k(x,t)]dx \\
&= M_{k+1} - a_0 \lim_{t\to 0^+}\int_{\mathbb{R}} x^{k+1} \partial_x^{k-1} G_t(x-b_0)dx - a_1 \lim_{t\to 0^+}\int_{\mathbb{R}} x^{k+1} \partial_x^{k} G_t(x-b_1)dx \\
&= M_{k+1} - (-1)^{k-1}\frac{(k+1)!}{2} a_0 b_0^2 - (-1)^k (k+1)! a_1 b_1,
\end{aligned}$$

which verifies the equality (12). Finally to prove (13), we use (14) to find that

$$\begin{aligned}\lim_{t\to 0^+}\int_{\mathbb{R}} x^{k+2}\partial_x^{k-1}G_t(x-b_0)dx &= \lim_{t\to 0^+}\int_{\mathbb{R}}(y+b_0)^{k+2}\partial_y^{k-1}G_t(y)dy \\ &= (-1)^{k-1}(k-1)!\binom{k+2}{3}b_0^3 \\ &= (-1)^{k-1}\frac{(k+2)!}{3!}b_0^3 \qquad (21)\end{aligned}$$

and

$$\begin{aligned}\lim_{t\to 0^+}\int_{\mathbb{R}} x^{k+2}\partial_x^{k}G_t(x-b_1)dx &= \lim_{t\to 0^+}\int_{\mathbb{R}}(y+b_1)^{k+2}\partial_y^{k}G_t(y)dy \\ &= (-1)^{k}k!\binom{k+2}{2}b_1^2 \\ &= (-1)^{k}\frac{(k+2)!}{2}b_1^2. \qquad (22)\end{aligned}$$

Using (14), (21)-(22), it can be seen that

$$\begin{aligned}&\lim_{t\to 0^+}\int_{\mathbb{R}} x^{k+2}[\phi(x,t)-\phi^k(x,t)]dx \\ &= M_{k+2} - a_0\lim_{t\to 0^+}\int_{\mathbb{R}} x^{k+2}\partial_x^{k-1}G_t(x-b_0)dx - a_1\lim_{t\to 0^+}\int_{\mathbb{R}} x^{k+2}\partial_x^{k}G_t(x-b_1)dx \\ &= M_{k+2} - (-1)^{k-1}\frac{(k+2)!}{3!}a_0b_0^3 - (-1)^k\frac{(k+2)!}{2}a_1b_1^2.\end{aligned}$$

This completes the proof. □

Remark 1 : It is easy to see that the moments of initial function, upto the order $k-2$, were incorporated in the asymptotic approximate solution (9) as if

$$\lim_{t\to 0^+}\int_{\mathbb{R}} x^j[\phi(x,t)-\phi^k(x,t)]dx = 0, \quad j = 0,1,2\ldots,k-2,$$

provided $k > 2$.

By virtue of above Lemma 2.1, we list out some key observations:

1. The choice of

$$a_0 := (-1)^{k-1}\frac{M_{k-1}}{(k-1)!}, \qquad (23)$$

coefficient in penultimate term of the asymptotic approximate solution (9), makes the $(k-1)^{th}$ order moment of ϕ to agrees with that of ϕ^k as $t \to 0^+$.

2. If we choose
$$a_1 := \frac{(-1)^k}{k!}[M_k - kM_{k-1}b_0], \tag{24}$$
in (9) and a_0 as in (23), then k^{th} order initial moment of ϕ agrees with that of ϕ^k for any arbitrary constant b_0.

3. Choosing
$$b_1 := \frac{2M_{k+1} - k(k+1)M_{k-1}b_0^2}{2(k+1)(M_k - kM_{k-1}b_0)}. \tag{25}$$
in (9) and a_0, a_1 as in (23)-(24), we notice that $(k+1)^{th}$ order initial moment of ϕ agrees with that of ϕ^k for any arbitrary value of b_0.

4. Finally, if we assume that $(k+2)^{th}$ order initial moment of ϕ agrees with that of ϕ^k, then from (13) we have
$$M_{k+2} - (-1)^{k-1}\frac{(k+2)!}{3!}a_0b_0^3 - (-1)^k\frac{(k+2)!}{2}a_1b_1^2 = 0. \tag{26}$$

   Substituting the expressions (23)-(25) of a_0, a_1 and b_1 in the equation (26), we get an equation in b_0;
$$\begin{aligned}&[k^2(k+1)(k+2)M_{k-1}^2]b_0^4 - [4k(k+1)(k+2)M_{k-1}M_k]b_0^3\\ &+[12k(k+2)M_{k-1}M_{k+1}]b_0^2 - [24kM_{k-1}M_{k+2}]b_0\\ &+12[2M_kM_{k+2} - \frac{(k+2)}{(k+1)}M_{k+1}^2] = 0.\end{aligned}$$

   Hence, by assigning any solution of the above equation to b_0, we notice that $(k+2)^{th}$ order initial moment of ϕ also agrees with that of ϕ^k.

5. The conditions $M_{k-1} = 0$ and
$$\lim_{t\to 0^+}\int_{\mathbb{R}} x^j[\phi(x,t) - \phi^k(x,t)]dx = 0, \quad 0 \leqslant j < k+2,$$
reduce the proposed approximation (9) to (54).

Motivated by the work of Yanagisawa [14], we now introduce a function $F(\phi_0)$ by

$$
\begin{aligned}
&F(\phi_0)(x) = \\
&\int_{-\infty}^{x}\left(\int_{-\infty}^{x_1}\cdots\int_{-\infty}^{x_{k+1}}\left(\int_{-\infty}^{x_{k+2}}\phi_0(x_{k+3})dx_{k+3}\right)dx_{k+2}\cdots\right)dx_1 \\
&-\sum_{j=0}^{k-2}(-1)^j\frac{M_j}{j!}\int_0^x\left(\int_0^{x_1}\cdots\int_0^{x_{k-j}}\left(\int_0^{x_{k+1-j}}H_0(x_{k+2-j})dx_{k+2-j}\right)dx_{k+1-j}\cdots\right)dx_1 \\
&-a_0\int_{b_0}^{x}\left(\int_{b_0}^{x_1}\left(\int_{-\infty}^{x_2}H_{b_0}(x_3)dx_3\right)dx_2\right)dx_1 - a_1\int_{b_1}^{x}\left(\int_{-\infty}^{x_1}H_{b_1}(x_2)dx_2\right)dx_1, \qquad (27)
\end{aligned}
$$

where $H_a(x)$ is the Heaviside's unit step function

$$H_a(x) = \begin{cases} 0 \text{ if } & x \leqslant a, \\ 1 \text{ if } & x > a. \end{cases}$$

The following Lemma talks about the integrability of the function $F(\phi_0)$ on $\mathbb{R}$.

Lemma 2.2 — Assume that $\phi(x,t)$ is a solution to heat equation (1) subject to (2) satisfying $(1+|x|)^{k+3+\epsilon}\phi_0 \in L^1(\mathbb{R})$. Further, let

$$\lim_{t\to 0^+}\int_{\mathbb{R}} x^j[\phi(x,t)-\phi^k(x,t)]dx = 0, \quad 0 \leqslant j \leqslant k+2, \qquad (28)$$

where $\phi^k(x,t)$ is given in (9). Then $F(\phi_0) \in L^1(\mathbb{R})$.

PROOF : As $(1+|x|)^{k+3+\epsilon}\phi_0 \in L^1(\mathbb{R})$, using Lemma A. 1 [see pp.115, [14]], we have

$$
\begin{aligned}
B_1(x) : &= \int_{-\infty}^{x}\left(\int_{-\infty}^{x_1}\cdots\int_{-\infty}^{x_{k+1}}\left(\int_{-\infty}^{x_{k+2}}\phi_0(x_{k+3})dx_{k+3}\right)dx_{k+2}\cdots\right)dx_1 \\
&= \sum_{j=0}^{k+2}\frac{(-1)^j}{j!}\overline{M}_j(x)\frac{x^{k+2-j}}{(k+2-j)!}, \qquad (29)
\end{aligned}
$$

where $\overline{M}_j(x) = \int_{-\infty}^{x} y^j\phi_0(y)dy$.

$$
\begin{aligned}
B_2(x) : &= -\sum_{j=0}^{k-2}(-1)^j\frac{M_j}{j!} \\
&\times\int_0^x\left(\int_0^{x_1}\cdots\int_0^{x_{k-j}}\left(\int_0^{x_{k+1-j}}H_0(x_{k+2-j})dx_{k+2-j}\right)dx_{k+1-j}\cdots\right)dx_1 \\
&= -\sum_{j=0}^{k-2}\frac{(-1)^j}{j!}M_j\frac{x_+^{k+2-j}}{(k+2-j)!}, \qquad (30)
\end{aligned}
$$

where $x_+ = \max\{x, 0\}$. Consider

$$\begin{aligned} B_3(x) : &= -a_0 \int_{b_0}^{x} \left(\int_{b_0}^{x_1} \left(\int_{-\infty}^{x_2} H_{b_0}(x_3) dx_3 \right) dx_2 \right) dx_1 \\ &= \begin{cases} -a_0 \left[\frac{x^3}{3!} - \frac{b_0}{2} x^2 + \frac{b_0^2}{2} x - \frac{b_0^3}{3!} \right], & \text{if } x > b_0 \\ 0, & \text{if } x \leqslant b_0. \end{cases} \end{aligned} \tag{31}$$

$$\begin{aligned} B_4(x) : &= -a_1 \int_{b_1}^{x} \left(\int_{-\infty}^{x_1} H_{b_1}(x_2) dx_2 \right) dx_1 \\ &= \begin{cases} -a_1 \left[\frac{x^2}{2} - b_1 x + \frac{b_1^2}{2} \right], & \text{if } x > b_1 \\ 0, & \text{if } x \leqslant b_1. \end{cases} \end{aligned} \tag{32}$$

We first consider the case $x \to \infty$ for $F(\phi_0)(x)$. Summing up the equations (29)-(32), we get

$$\begin{aligned} F(\phi_0)(x) &= B_1(x) + B_2(x) + B_3(x) + B_4(x) \\ &= \sum_{j=0}^{k+2} \frac{(-1)^j}{(k+2-j)!j!} \{\overline{M}_j(x) - M_j\} x^{k+2-j} + \left(\frac{(-1)^{k-1}}{(k-1)!3!} M_{k-1} - \frac{a_0}{3!} \right) x^3 \\ &\quad + \left(\frac{(-1)^k}{k!2} M_k - \frac{a_1}{2} + \frac{a_0 b_0}{2} \right) x^2 + \left(\frac{(-1)^{k+1}}{(k+1)!} M_{k+1} + a_1 b_1 - \frac{a_0 b_0^2}{2} \right) x \\ &\quad + \left(\frac{(-1)^{k+2}}{(k+2)!} M_{k+2} + \frac{a_0 b_0^3}{3!} - \frac{a_1 b_1^2}{2} \right). \end{aligned} \tag{33}$$

In view of the equality (28), equations (10)-(13) can be written as

$$\begin{aligned} &\frac{(-1)^{k-1}}{(k-1)!3!} M_{k-1} - \frac{a_0}{3!} = 0, \\ &\frac{(-1)^k}{k!2!} M_k - \frac{a_1}{2} + \frac{a_0 b_0}{2} = 0, \\ &\frac{(-1)^{k+1}}{(k+1)!} M_{k+1} + a_1 b_1 - \frac{a_0 b_0^2}{2} = 0, \\ &\frac{(-1)^{k+2}}{(k+2)!} M_{k+2} + \frac{a_0 b_0^3}{3!} - \frac{a_1 b_1^2}{2} = 0. \end{aligned}$$

Thus, we are left with

$$F(\phi_0)(x) = \sum_{j=0}^{k+2} \frac{(-1)^j}{(k+2-j)!j!} \{\overline{M}_j(x) - M_j\} x^{k+2-j}.$$

For $0 < x \leq s < \infty$, we have

$$x^{k+2-j} \leq s^{k+2-j} \text{ if } 0 \leq j \leq k+2. \tag{34}$$

This leads to

$$x^{k+2-j}s^j|\phi_0(s)| \leq s^{k+2}|\phi_0(s)| \text{ if } 0 \leq j \leq k+2.$$

Integrating the above inequality with respect to s from x to ∞ and then dividing by $(k+2-j)!j!$, we arrive at

$$\frac{x^{k+2-j}}{(k+2-j)!j!}\int_x^\infty s^j|\phi_0(s)|ds \leq \frac{1}{(k+2-j)!j!}\int_x^\infty s^{k+2}|\phi_0(s)|ds. \tag{35}$$

Thus,

$$\begin{aligned}
\left|\sum_{j=0}^{k+2}\frac{(-1)^j}{(k+2-j)!j!}\left(\int_x^\infty s^j\phi_0(s)ds\right)x^{k+2-j}\right| &\leqslant \sum_{j=0}^{k+2}\frac{x^{k+2-j}}{(k+2-j)!j!}\int_x^\infty s^j|\phi_0(s)|ds \\
&\leqslant \sum_{j=0}^{k+2}\frac{1}{(k+2-j)!j!}\int_x^\infty s^{k+2}|\phi_0(s)|ds \\
&= o(|x|^{-1-\epsilon}) \quad \text{as} \quad x\to\infty,
\end{aligned}$$

in view of $(1+|x|)^{k+3+\epsilon}\phi_0 \in L^1(\mathbb{R})$. Hence,

$$F(\phi_0)(x) = o(|x|^{-1-\epsilon}) \quad \text{as} \quad x \to \infty. \tag{36}$$

Therefore, there exists a sufficiently large real number $L(>0)$ such that

$$|x^{1+\epsilon}F(\phi_0)(x)| < 1, \ \ \forall x > L.$$

In other words, $|F(\phi_0)(x)| < x^{-1-\epsilon} \ \ \forall x > L$. Thus, we get

$$\int_L^\infty |F(\phi_0)(x)|dx < \int_L^\infty x^{-1-\epsilon}dx = \frac{1}{\epsilon L^\epsilon} < \infty. \tag{37}$$

We now consider the case $x \to -\infty$ for $F(\phi_0)(x)$. If $x < 0$, we have

$$\begin{aligned}
&\left|\sum_{j=0}^{k+2}\frac{(-1)^j}{(k+2-j)!j!}\left(\int_{-\infty}^x s^j\phi_0(s)ds\right)x^{k+2-j}\right| \\
\leqslant &\sum_{j=0}^{k+2}\frac{|x|^{k+2-j}}{(k+2-j)!j!}\int_{-\infty}^x |s|^j|\phi_0(s)|ds \\
\leqslant &\sum_{j=0}^{k+2}\frac{1}{(k+2-j)!j!}\int_{-\infty}^x |s|^{k+2}|\phi_0(s)|ds \\
= \ &o(|x|^{-1-\epsilon}) \quad \text{as} \quad x \to -\infty.
\end{aligned} \tag{38}$$

On the other hand, when x is smaller than minimum of b_0, b_1 and 0, it is to be noted from (30)-(32) that

$$B_2(x) = B_3(x) = B_4(x) = 0$$

and hence

$$F(\phi_0)(x) = \sum_{j=0}^{k+2} \frac{(-1)^j}{(k+2-j)!j!} \overline{M}_j(x) x^{k+2-j}. \tag{39}$$

Hence, in view of (38), we get

$$F(\phi_0)(x) = o(|x|^{-1-\epsilon}) \quad \text{as} \quad x \to -\infty. \tag{40}$$

Repeating as in the $x \to \infty$ case, one obtains that

$$\int_{-\infty}^{M} |F(\phi_0)(x)| dx < \infty \tag{41}$$

for some large number $|M|$ with $M < 0$.

Eventually, (37) and (41) will conclude that $F(\phi_0) \in L^1(\mathbb{R})$.

Having $F(\phi_0)$ as L^1-function under the hypotheses of Lemma 2.2, we indicate its convolution with heat kernel $G_t(x)$ by

$$G_t * F(\phi_0)(x) = \int_{\mathbb{R}} G_t(x-y) F(\phi_0)(y) dy. \tag{42}$$

The following lemma tells that the difference term $\phi - \phi^k$ can be represented as the $(k+3)^{th}$ order spatial derivative of the convolution $G_t * F(\phi_0)$.

Lemma 2.3 — Assume that $\phi(x,t)$ is a solution to heat equation (1) subject to (2) with $(1 + |x|)^{k+3+\epsilon}\phi_0 \in L^1(\mathbb{R})$, where k is any positive integer and ϵ is a small positive real number. Further, let

$$\lim_{t \to 0^+} \int_{\mathbb{R}} x^j [\phi(x,t) - \phi^k(x,t)] dx = 0, \quad 0 \leqslant j \leqslant k+2, \tag{43}$$

where ϕ^k is an asymptotic approximation given in (9). Then,

$$\partial_x^{k+3}(G_t * F(\phi_0)(x)) = \phi(x,t) - \phi^k(x,t) \;\; for \;\; t > 0, \; x \in \mathbb{R}. \tag{44}$$

PROOF : Consider

$$\begin{aligned} &\partial_x^{k+3}(G_t * B_1(x)) = \\ &\int_{\mathbb{R}} G_t(x-y) \partial_y^{k+3} \left[\int_{-\infty}^{y} \left(\int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_{k+1}} \left(\int_{-\infty}^{x_{k+2}} \phi_0(x_{k+3}) dx_{k+3} \right) dx_{k+2} \cdots \right) dx_1 \right] dy \\ &= \phi(x,t). \end{aligned} \tag{45}$$

Similarly, we have

$$\begin{aligned}\partial_x^{k+3}(G_t * B_2(x)) &= \partial_x^{j+1}(G_t * \partial_x^{k+2-j} B_2(x)) \\ &= -\sum_{j=0}^{k-2}(-1)^j \frac{M_j}{j!}\partial_x^{j+1}\int_0^\infty G_t(x-y)dy \\ &= -\sum_{j=0}^{k-2}(-1)^j \frac{M_j}{j!}\partial_x^{j} G_t(x). \end{aligned} \tag{46}$$

For $B_3(x)$, we compute that

$$\begin{aligned}&\partial_x^{k+3}\int_{\mathbb{R}} G_t(x-y)B_3(y)dy \\ &= -a_0\partial_x^k \int_{\mathbb{R}} G_t(x-y)\partial_y^3\left[\int_{b_0}^y\left(\int_{b_0}^{x_1}\left(\int_{-\infty}^{x_2} H_{b_0}(x_3)dx_3\right)dx_2\right)dx_1\right]dy \\ &= -a_0\partial_x^k \int_{\mathbb{R}} G_t(x-y)H_{b_0}(y)dy \\ &= -a_0\partial_x^k \int_{-\infty}^{x} G_t(y-b_0)dy \\ &= -a_0\partial_x^{k-1} G_t(x-b_0). \end{aligned} \tag{47}$$

Similarly for $B_4(x)$, we obtain

$$\partial_x^{k+3}\int_{\mathbb{R}} G_t(x-y)B_4(y)dy = -a_1\partial_x^k G_t(x-b_1). \tag{48}$$

Therefore, summing up the equations (45)-(48) leads to equation (44) as

$$F(\phi_0)(x) = B_1(x) + B_2(x) + B_3(x) + B_4(x).\square$$

PROOF OF THEOREM 1.1 : Using the Rodrigues formula for Hermite polynomials, it is seen that

$$\partial_x^{k+3}G_t(x) = \frac{(-1)^{k+3}}{\sqrt{\pi}(4t)^{\frac{k+4}{2}}}H_{k+3}(\xi)e^{-\xi^2}, \tag{49}$$

where $\xi = x/\sqrt{4t}$ and H_{k+3} is a Hermite polynomial of order $k+3$.

Consider, for $1 \leqslant p \leqslant \infty$,

$$\begin{aligned}||\partial_x^{k+3}G_t(.)||_p &= \frac{1}{\sqrt{\pi}(4t)^{\frac{k+4}{2}}}\left(\sqrt{4t}\int_{-\infty}^{\infty}|H_{k+3}(\xi)e^{-\xi^2}|^p d\xi\right)^{1/p} \\ &= O(t^{\frac{1}{2p}-\frac{k+4}{2}}) \quad \text{as} \quad t\to\infty. \end{aligned} \tag{50}$$

Using Young's inequality, we find that

$$||\partial_x^{k+3}(G_t * F(\phi_0))||_p \leqslant ||F(\phi_0)||_1 \, ||\partial_x^{k+3} G_t||_p \tag{51}$$

Therefore, in view of Lemma 2.2, Lemma 2.3 and the decay rate (50), the order of convergence in (6) follows from the Young's inequality (51). □

Repeating the proof of Theorem 1.1 with $(k+2)^{th}$ partial derivative of G_t in place of $(k+3)^{th}$ partial derivative, we obtain the following:

Remark 2 : Let $\phi(x,t)$ be a solution to heat equation (1) subject to (2) satisfying $(1+|x|)^{k+3+\epsilon}\phi_0 \in L^1(\mathbb{R})$. Then, for $1 \leqslant p \leqslant \infty$, we have

$$\| \int_{-\infty}^{\cdot} [\phi(y,t) - \phi^k(y,t)]dy \, \|_p = O(t^{-\frac{3+k}{2}+\frac{1}{2p}}), \;\; t \to \infty, \tag{52}$$

where ϕ^k is given in (7).

Remark 3 : It is important to notice that assuming the equality (28), we seek for expressions of a_0, a_1, b_1 and b_0 in equations (10)-(13). This, in turn, produces the conditional statement (7) from the proposed approximate solution (9). In other words, the moments, upto the order $(k+2)$, of $\phi(x,t)$ agree with those of $\phi^k(x,t)$ given in (7) as $t \to 0^+$.

Remark 4 : It is known that [10, 5] there exist at least two real solutions to the quartic equation (8) in all the cases except the following three cases:

1. $a < 0, \; c > \frac{a^2}{4}, \; \triangle > 0,$
2. $a \geq 0, \; \triangle > 0$
3. $\triangle = 0, \; c = \frac{a^2}{4}, b = 0,$

where $a = q - \dfrac{3p^2}{8}, b = r + \dfrac{p^3}{8} - \dfrac{pq}{2}, c = s - \dfrac{3p^4}{256} + \dfrac{p^2 q}{16} - \dfrac{pr}{4}$ and
$\triangle = 144ab^2c - 128a^2c^2 - 4a^3b^2 + 16a^4c - 27b^4 + 256c^3$

with

$$p = -\frac{4k(k+1)(k+2)M_{k-1}M_k}{k^2(k+1)(k+2)M_{k-1}^2},$$

$$q = \frac{12k(k+2)M_{k-1}M_{k+1}}{k^2(k+1)(k+2)M_{k-1}^2},$$

$$r = -\frac{24kM_{k-1}M_{k+2}}{k^2(k+1)(k+2)M_{k-1}^2},$$

$$s = \frac{12[2M_kM_{k+2}(k+1) - (k+2)M_{k+1}^2]}{k^2(k+1)^2(k+2)M_{k-1}^2}.$$

In case the coefficients of the equation (8) satisfy one of those three cases, we do not have higher order approximate solutions in the proposed form (9) and the work is left for future research. However, if $M_{k-1}M_{k+1} = \frac{k+1}{2k}M_k^2$ and $\frac{6k^2}{(k+2)(k+1)}M_{k-1}^2M_{k+2} = M_k^3$ in those three cases, we have higher order approximate solution given in the first statement of (7).

Remark 5 : Let $M_{k-1} = 0$, $M_k \neq 0$ and $\phi(x,t)$ be a solution to heat equation (1) subject to (2) satisfying $(1+|x|)^{k+2+\epsilon}\phi_0 \in L^1(\mathbb{R})$, where k is any positive integer and ϵ is any small positive real number. Then, for $1 \leqslant p \leqslant \infty$, we have

$$||\phi(.,t) - \phi^k(.,t)||_p = O(t^{-\frac{3+k}{2}+\frac{1}{2p}}), \ \ t \to \infty, \tag{53}$$

where ϕ_k is given by

$$\phi^k(x,t) = \sum_{j=0}^{k-2}(-1)^j \frac{M_j}{j!}\partial_x^j G_t(x) + (-1)^k \frac{M_k}{k!}\partial_x^k G_t\left(x - \frac{M_{k+1}}{(k+1)M_k}\right). \tag{54}$$

2.1 *On viscous Burgers Equation*

We now mention the large time asymptotics for solutions of the viscous Burgers equation;

$$\begin{aligned} u_t + uu_x &= u_{xx}, \quad x \in \mathbb{R},\ t > 0, &(55)\\ u(x,0) &= u_0(x), \quad x \in \mathbb{R}, &(56)\end{aligned}$$

with the initial data u_0 satisfying

$$(1+|x|)^{k+3+\epsilon}u_0 \in L^1(\mathbb{R}), \tag{57}$$

for any positive integer k and a small real number $\epsilon > 0$.

***Theorem* 2.2** — *Let $u(x,t)$ be a solution to the Burgers equation (55) subject to (56) with u_0 satisfying $(1+|x|)^{k+3+\epsilon}u_0 \in L^1(\mathbb{R})$. Let*

$$M_r := \int_{\mathbb{R}} x^r \left[-\frac{1}{2}u_0(x)\exp\left(-\frac{1}{2}\int_{-\infty}^x u_0(y)dy\right)\right]dx, \quad r = 0,1,2,\ldots. \tag{58}$$

then there exists $T_k > 0$ such that

$$||u(.,t) - u^k(.,t)||_p = O(t^{-\frac{4+k}{2}+\frac{1}{2p}}), \ \ t \to \infty, \tag{59}$$

where $1 \leqslant p \leqslant \infty$ and

$$u^k(x,t) := -2\frac{\phi^k(x,t)}{1+\int_{-\infty}^x \phi^k(y,t)dy}, \quad x \in \mathbb{R}, \ \ t > 0, \tag{60}$$

is well defined for $x \in \mathbb{R},\ t \geq T_k$. *Here* ϕ_k *is given by (7).*

The proof of the Theorem 2.2 is omitted as it is easily proved using the Theorem 1.1, Remark 2 and standard arguments [14].

Remark 6 : Let $M_{k-1} = 0$, $M_k \neq 0$. Further, let $u(x,t)$ be a solution to the Burgers equation (55) subject to (56) with u_0 satisfying $(1+|x|)^{k+2+\epsilon}u_0 \in L^1(\mathbb{R})$. Then there exists $T_k > 0$ such that

$$||u(.,t) - u^k(.,t)||_p = O(t^{-\frac{3+k}{2}+\frac{1}{2p}}),\ \ t \to \infty, \tag{61}$$

where $1 \leqslant p \leqslant \infty$ and

$$u^k(x,t) := -2\frac{\phi^k(x,t)}{1+\int_{-\infty}^{x}\phi^k(y,t)dy}, \quad x \in \mathbb{R},\ \ t > 0, \tag{62}$$

is well defined for $x \in \mathbb{R},\ t \geq T_k$. Here ϕ_k is given by (54).

3. Comparison with Yanagisawa [14]'s and Kim and Ni [6]'s Asymptotic Approximate Solutions

We first list out the advantages and disadvantages of the proposed asymptotic approximate solution (9) by comparing with Yanagisawa [14]'s asymptotic approximate solution (5).

1. In the work of Yanagisawa [14], if $M_k \neq 0$ and $M_{k+2}M_k \leqslant \frac{k+2}{2(k+1)}[M_{k+1}]^2$, then time shift $(t_k)_+ = 0$ and hence Yanagisawa's approximation (5) converges to the true solution ϕ of the problem (1)-(2) with an order of convergence $O(t^{-\frac{3+k}{2}+\frac{1}{2p}})$ as $t \to \infty$. Whereas the asymptotic approximate solution (7) converges to the true solution ϕ of the problem (1)-(2) with an order of convergence $O(t^{-\frac{4+k}{2}+\frac{1}{2p}})$ as $t \to \infty$.

   Following example explains the same for $k = 3$.

   Example 3.1 : Consider a discontinuous initial data

$$\phi_0(x) = \begin{cases} (-x)^{-7}, & -\infty < x < -\frac{1}{2}, \\ 0, & -\frac{1}{2} \leqslant x \leqslant 1, \\ x^{-7}, & 1 < x < \infty. \end{cases} \tag{63}$$

   for the problem (1)-(2).

   Firstly, it is to be observed that ϕ_0 has moments up to 5^{th} order only. They are

$$M_0 = \frac{65}{6},\ M_1 = \frac{-31}{5},\ M_2 = \frac{17}{4},\ M_3 = \frac{-7}{3},\ M_4 = \frac{5}{2} \text{ and } M_5 = -1.$$

It is easy to see that $M_5 M_3 < \frac{5}{8} M_4^2$.

Hence, we obtain

$$||\phi(.,t) - \phi^3(.,t)||_p = O(t^{-\frac{6}{2}+\frac{1}{2p}}), \ \ t \to \infty,$$

where $\phi^3(x,t)$ is an asymptotic approximation of Yanagisawa [14]. On the other hand, choosing

$$a_0 = \frac{17}{8}, \quad b_0 =$$
$$\frac{1}{765}\left\{-140 + \sqrt{-37775 + 90\beta} + \sqrt{10\left[-7555 - 9\beta + 161509\sqrt{\frac{5}{-7555 + 18\beta}}\right]}\right\},$$

with $\beta = 17^{\frac{2}{3}} \times 333235^{\frac{1}{3}}$ and

$$a_1 = \frac{7}{18} + \frac{17}{8} b_0,$$
$$b_1 = \frac{3}{2}\left[\frac{51 b_0^2 - 5}{153 b_0 + 28}\right]$$

in asymptotic approximate solution (9), one gets

$$||\phi(.,t) - \phi^3(.,t)||_p = O(t^{-\frac{7}{2}+\frac{1}{2p}}), \ \ t \to \infty.$$

2. Suppose that $M_k = 0$. We then have to look for the largest integer $s(< k)$ so that $M_s \neq 0$ and so Yanagisawa [14]'s approximation (5) converges to the true solution ϕ either with an order of convergence $O(t^{-\frac{4+s}{2}+\frac{1}{2p}})$ when $(t_k)_+ > 0$ or $O(t^{-\frac{3+s}{2}+\frac{1}{2p}})$ when $(t_k)_+ = 0$. However, it is possible to construct an approximation (9) which converges to true solution ϕ with an order of convergence $O(t^{-\frac{4+k}{2}+\frac{1}{2p}})$ as $t \to \infty$.

Following example illustrates it with $k = 3$.

Example 3.2 : Consider a discontinuous initial data defined by

$$\phi_0(x) = \begin{cases} (-x)^{-7}, & -\infty < x < -1, \\ 0, & -1 \leqslant x \leqslant 1, \\ x^{-7}, & 1 < x < \infty. \end{cases} \tag{64}$$

for the problem (1)-(2).

It can be easily seen that $-\infty < M_j < \infty$ for $0 \leqslant j \leqslant 5$ and M_j is not defined for $j \geqslant 6$. In particular, we find that

$$M_0 = \frac{1}{3},\ M_1 = 0,\ M_2 = \frac{1}{2},\ M_3 = 0,\ M_4 = 1 \text{ and } M_5 = 0. \tag{65}$$

Further, notice that $M_5 M_3 < \frac{5}{8} M_4^2$. Therefore, we get

$$||\phi(.,t) - \phi^2(.,t)||_p = O(t^{-\frac{5}{2}+\frac{1}{2p}}),\ \ t \to \infty,$$

where $\phi^2(x,t)$ is an asymptotic approximation (5) of Yanagisawa [14]. On the other hand, choosing

$$a_0 = \frac{1}{4},\ a_1 = \frac{b_0}{4},\ b_0 = \sqrt{-1 + \frac{2}{\sqrt{3}}},\ b_1 = \frac{3b_0^2 - 1}{6b_0},$$

in the asymptotic approximate solution (9), one gets

$$||\phi(.,t) - \phi^3(.,t)||_p = O(t^{-\frac{7}{2}+\frac{1}{2p}}),\ \ t \to \infty.$$

3. If $M_{k-1} = 0$, $M_k \neq 0$ and $M_{k+2} M_k > \frac{k+2}{2(k+1)} [M_{k+1}]^2$, then higher rate of convergence of the approximation to the true solution ϕ is obtained by considering the asymptotic approximate solution of Yanagisawa [14] only, not the one in the form (9).

4. If $M_{k-1} = 0$, $M_k \neq 0$ and $M_{k+2} M_k < \frac{k+2}{2(k+1)} [M_{k+1}]^2$, then Yanagisawa approximate solution (5) as well as the proposed approximate solution (54) would give the same rate of convergence, namely $O\left(t^{-\frac{3+k}{2}+\frac{1}{2p}}\right)$ as $t \to \infty$.

We now list out the advantages of the proposed asymptotic approximate solution (9) by comparing with Kim and Ni [6]'s asymptotic approximate solution (5) in the following cases:

1. Suppose $(1+|x|)^{2n+1+\epsilon} \phi_0 \in L^1(\mathbb{R})$ with $M_{2n-3} M_{2n-1} = \dfrac{2n-1}{4n-4} M_{2n-2}^2$, $\dfrac{6(2n-2)^2}{(2n)(2n-1)} M_{2n-3}^2 M_{2n} = M_{2n-2}^3, M_{2n-3} \neq 0$, where $n > 1$ and $0 < \epsilon < 1$. Then we can construct an asymptotic solution Φ to (1)-(2) via Kim and Ni [6]' work as if

$$||\phi(.,t) - \Phi(.,t)||_p = O(t^{-\frac{2n+1}{2}+\frac{1}{2p}}),\ \ t \to \infty,\ \ 1 \leq p \leq \infty, \tag{66}$$

whereas the proposed asymptotic approximate solution (9) converges with a higher decay rate in L^p-norm, namely, $O(t^{-(n+1)+\frac{1}{2p}})$ as $t \to \infty$.

2. Suppose $(1+|x|)^{2n+1+\epsilon}\phi_0 \in L^1(\mathbb{R})$ with $M_{2n-3}=0$, $M_{2n-2} \neq 0, M_{2n}M_{2n-2} = \dfrac{2n}{4n-2}M_{2n-1}^2$, where $n > 1$ and $0 < \epsilon < 1$. Then we can construct an asymptotic solution Φ to (1)-(2) via Kim and Ni [6]' work as if

$$||\phi(.,t)-\Phi(.,t)||_p = O(t^{-\frac{2n+1}{2}+\frac{1}{2p}}), \quad 1 \leq p \leq \infty, \tag{67}$$

whereas the proposed asymptotic approximate solution (9) converges with a higher decay L^p-rate, namely, $O(t^{-(n+1)+\frac{1}{2p}})$ as $t \to \infty$.

Remark 7 : It is to be noted that, if ϕ_0 is sign changing initial profile, we use the generalization of truncated moment problem [7] to construct $\Phi(x,t)$ satisfying (66)-(67).

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