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A VARIATIONAL APPROACH FOR ONE-DIMENSIONAL SCALAR FIELD PROBLEMS

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In this paper, we are interested in the existence of infinitely many weak solutions for a one-dimensional scalar field problem. By using variational methods, in an appropriate functional space which involves the potential V, we determine intervals of parameters such that our problem admits either a sequence of weak solutions strongly converging to zero provided that the nonlinearity has a suitable behavior at zero or an unbounded sequence of weak solutions if a similar behavior occurs at infinity.

Key words : One-dimensional scalar field problem; variational methods; infinitely many solutions.

1. Introduction and Preliminaries

The aim of this paper is to study the following one-dimensional scalar field problem

$$\begin{cases} -u'' + V(x)u = \lambda W(x) f(u), & x \in \mathbb{R} \\ u(x) \to 0 \quad \text{as} \quad |x| \to +\infty, \end{cases} \tag{P_λ}$$

where V, W are positive potentials, $f : \mathbb{R} \to \mathbb{R}$ is continuous and $\lambda > 0$ is a real parameter.

More precisely, the main goal here is to obtain some sufficient conditions to guarantee that, for suitable values of λ, problem (P_λ) admits either a sequence of weak solutions strongly converging to zero provided that the nonlinearity has a suitable behavior at zero (see Theorem 2.1) or an unbounded sequence of weak solutions if a similar behavior occurs at infinity (see Theorem 2.9).

We assume throughout, and without further mention, that the following conditions on the potentials V, W hold:

(V1) $V \in L^{\infty}_{loc}(\mathbb{R})$, $V_0 := \operatorname{ess\,inf}_{\mathbb{R}} V > 0$, and for any $M > 0$ and any $r > 0$ there holds:

$$\operatorname{meas}\left(\{x \in \mathbb{R} : |x-y| < r,\ V(x) \leq M\}\right) \to 0 \ \text{ as } \ |y| \to +\infty,$$

where "meas" denotes the Lebesgue measure in $\mathbb{R}$.

(W1) $W \in L^1(\mathbb{R}) \cap L^{\infty}(\mathbb{R})$, $W \geq 0$, $\|W\|_{\infty} > 0$.

Now, we introduce some notations that will be used in the sequel. Let H_V be the Hilbert space

$$H_V := \left\{ u \in H^1(\mathbb{R}) : \int_{\mathbb{R}} V(x)u^2 < +\infty \right\},$$

endowed with the inner product

$$\langle u, v \rangle_V := \int_{\mathbb{R}} \left(u'v' + V(x)uv \right),$$

for each $u, v \in H_V$. The induced norm will be denoted by $\|\cdot\|_V$. Due to Morrey's Theorem, the embedding $H_V \subset H^1(\mathbb{R}) \equiv W^{1,2}(\mathbb{R}) \hookrightarrow L^{\infty}(\mathbb{R})$ is continuous. Moreover, the embedding $H_V \hookrightarrow L^2(\mathbb{R})$ is compact, cf. Bartsch, Pankov and Wang [2]. In the sequel, we denote by $\kappa_{\infty} > 0$ the best Sobolev embedding constant for $H^1(\mathbb{R}) \hookrightarrow L^{\infty}(\mathbb{R})$; $\|\cdot\|_p$ denotes the usual norm of $L^p(\mathbb{R})$, $p \in [1, \infty]$.

Let $\sigma \in]0,1[$, and define the set

$$A_{\sigma} := \{\mu > 0 : W(x) > \sigma \|W\|_{\infty} \ \text{ a.e. } \ x \in [x_0 - \mu, x_0 + \mu] \text{ for some } x_0 \in \mathbb{R}\}.$$

Note that $A_{\sigma} \neq \emptyset$ while $\sup A_{\sigma}$ is finite and attained on some $\mu_{\sigma} \in A_{\sigma}$, due to assumption (W1). Hence, there exists $x_{\sigma} \in \mathbb{R}$ such that

$$W(x) > \sigma \|W\|_{\infty} \ \text{ a.e. } \ x \in [x_{\sigma} - \mu_{\sigma}, x_{\sigma} + \mu_{\sigma}].$$

For every $a_n > 0$ define

$$w_n(x) := \begin{cases} 0, & |x - x_{\sigma}| > \mu_{\sigma}; \\ a_n, & |x - x_{\sigma}| \leq \frac{\mu_{\sigma}}{2}; \\ \frac{2a_n}{\mu_{\sigma}} \left(\mu_{\sigma} - |x - x_{\sigma}| \right), & \frac{\mu_{\sigma}}{2} < |x - x_{\sigma}| \leq \mu_{\sigma}. \end{cases} \tag{1.1}$$

One can see that $w_n \in H_V$ because $V \in L^{\infty}_{loc}(\mathbb{R})$. If we introduce

$$\alpha_{\sigma} := \left(\frac{4}{\mu_{\sigma}} + 2\mu_{\sigma} \sup_{|x - x_{\sigma}| \leq \mu_{\sigma}} V(x) \right)^{1/2},$$

then an easy calculation shows that

$$\|w_n\|_V \leq a_n \alpha_\sigma.$$

We say that a function $u \in H_V$ is a *weak solution* of problem (P_λ), if u satisfies

$$\int_{\mathbb{R}} u'(x)v'(x)dx + \int_{\mathbb{R}} V(x)u(x)v(x)dx - \lambda \int_{\mathbb{R}} W(x)f(u(x))v(x)dx = 0,$$

for every $v \in H_V$.

We just point out that Faraci and Kristály in [4, Theorem 2.1], using the same variational setting but different technical arguments, ensured the existence of infinitely many solutions for problem (P_1). In this context the authors assume, among others technical assumptions, that the following condition at zero holds:

$$-\infty < \liminf_{\xi \to 0^+} \frac{F(\xi)}{\xi^2} \leq \limsup_{\xi \to 0^+} \frac{F(\xi)}{\xi^2} = +\infty.$$

Instead of the above inequality, in Theorem 2.2 we require

$$\liminf_{\xi \to 0^+} \frac{\max_{|t| \leq \xi} F(t)}{\xi^2} < \frac{W_\sigma}{(\alpha_\sigma \kappa_\infty)^2 \|W\|_1} \limsup_{\xi \to 0^+} \frac{F(\xi)}{\xi^2},$$

where

$$W_\sigma := \int_{|x - x_\sigma| \leq \frac{\mu_\sigma}{2}} W(x)dx.$$

Also, in [4, Theorem 2.3], the authors assume that the following condition at infinity holds:

$$-\infty < \liminf_{\xi \to +\infty} \frac{F(\xi)}{\xi^2} \leq \limsup_{\xi \to +\infty} \frac{F(\xi)}{\xi^2} = +\infty.$$

Instead of the above inequality, we require here

$$\liminf_{\xi \to +\infty} \frac{\max_{|t| \leq \xi} F(t)}{\xi^2} < \frac{W_\sigma}{(\alpha_\sigma \kappa_\infty)^2 \|W\|_1} \limsup_{\xi \to +\infty} \frac{F(\xi)}{\xi^2}.$$

A more precise comparison with the cited result is explained in Remark 2.8. A special case of our main result reads as follows.

***Theorem* 1.1** — *Let $f : \mathbb{R} \to \mathbb{R}$ be a nonnegative and continuous function. Assume that*

$$\liminf_{\xi \to 0^+} \frac{F(\xi)}{\xi^2} = 0 \qquad \text{and} \qquad 0 < \limsup_{\xi \to 0^+} \frac{F(\xi)}{\xi^2} \leq +\infty.$$

Then, for every $\lambda > \dfrac{\alpha_\sigma^2}{(2\mu_\sigma) \limsup_{\xi \to 0^+} \frac{F(\xi)}{\xi^2}}$, *problem* (P_λ) *admits a sequence of nontrivial and nonnegative weak solutions* $\{u_n\}$ *which satisfy* $\lim_{n \to \infty} \|u_n\|_V = 0$. *In particular,* $\lim_{n \to \infty} \|u_n\|_\infty = 0$.

We emphasize that, if

$$\limsup_{\xi\to 0^+}\frac{F(\xi)}{\xi^2}<+\infty,$$

Theorem 1.1 is not directly obtainable by the statements of the previous mentioned contributions. In particular, we note that no symmetry assumption is requested on the nonlinearity term f.

Finally, for completeness, we cite the manuscripts [1, 3, 5, 7, 8] for some relevant contributions related to the subject of this work.

Our variational approach is mainly based on the quoted Ricceri's variational principle (see [9, Theorem 2.5]) that we recall here in a convenient form.

***Theorem* 1.2** — *Let X be a reflexive real Banach space, let $\Phi, \Psi : X \to \mathbb{R}$ be two Gâteaux differentiable functionals such that Φ is sequentially weakly lower semicontinuous, strongly continuous and coercive, and Ψ is sequentially weakly continuous. For every $r > \inf_X \Phi$, put*

$$\varphi(r) := \inf_{u\in\Phi^{-1}(]-\infty,r[)}\frac{\left(\sup_{v\in\Phi^{-1}(]-\infty,r])}\Psi(v)\right)-\Psi(u)}{r-\Phi(u)},$$

$$\gamma := \liminf_{r\to+\infty}\varphi(r), \qquad \textit{and} \qquad \delta := \liminf_{r\to(\inf_X\Phi)^+}\varphi(r).$$

Then the following properties hold:

(a) *If $\gamma < +\infty$, then for each $\lambda \in]0, 1/\gamma[$, the following alternative holds: either*

(a$_1$) *$I_\lambda := \Phi - \lambda\Psi$ possesses a global minimum, or*

(a$_2$) *there is a sequence $\{u_n\}$ of critical points (local minima) of I_λ such that*

$$\lim_{n\to+\infty}\Phi(u_n) = +\infty.$$

(b) *If $\delta < +\infty$, then for each $\lambda \in]0, 1/\delta[$, the following alternative holds: either*

(b$_1$) *there is a global minimum of Φ which is a local minimum of I_λ, or*

(b$_2$) *there is a sequence $\{u_n\}$ of pairwise distinct critical points (local minima) of I_λ that converges weakly to a global minimum of Φ, with $\lim_{n\to\infty}\Phi(u_n) = \inf_{u\in X}\Phi(u)$.*

In conclusion, we cite a recent monograph by Kristály, Rădulescu and Varga [6] as a general reference on variational methods adopted here.

2. MAIN RESULTS

In this section we establish the main abstract result of this paper. Let

$$B^0 := \limsup_{\xi \to 0^+} \frac{F(\xi)}{\xi^2}.$$

With the above notation we are able to prove the following result.

***Theorem* 2.1** — *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that*

(A1) $F(\xi) \geq 0$ *for all* $\xi \in \mathbb{R}^+$.

Assume that there exist two sequences $\{a_n\}$ *and* $\{b_n\}$ in $]0, +\infty[$, *with* $\lim_{n\to\infty} b_n = 0$, *such that*

(A2) *for some* $n_0 \in \mathbb{N}$ *one has* $a_n < \dfrac{b_n}{\alpha_\sigma \kappa_\infty}$ *for each* $n \geq n_0$;

(A3) $\mathcal{A}_0 := \lim_{n\to\infty} \dfrac{\|W\|_1 \max_{|\xi|\leq b_n} F(\xi) - W_\sigma F(a_n)}{b_n^2 - (\kappa_\infty a_n \alpha_\sigma)^2} < \dfrac{B^0 W_\sigma}{(\kappa_\infty \alpha_\sigma)^2}$.

Then, for each

$$\lambda \in \left] \frac{\alpha_\sigma^2}{2B^0 W_\sigma}, \frac{1}{2\kappa_\infty^2 \mathcal{A}_0} \right[,$$

problem (P_λ) *admits a sequence of nontrivial weak solutions* $\{u_n\}$ *which satisfy* $\lim_{n\to\infty} \|u_n\|_V = 0$. *In particular,* $\lim_{n\to\infty} \|u_n\|_\infty = 0$.

PROOF : Our aim is to apply Theorem 1.2(b) to problem (P_λ). Let the functionals $\Phi, \Psi : H_V \to \mathbb{R}$ be defined by

$$\Phi(u) := \frac{1}{2}\|u\|_V^2, \qquad \Psi(u) := \int_{\mathbb{R}} W(x)F(u(x))dx,$$

and put

$$I_\lambda(u) := \Phi(u) - \lambda\Psi(u),$$

for every $u \in H_V$. Clearly, Φ is sequentially weakly lower semicontinuous, strongly continuous and coercive. Since H_V is continuously embedded into $L^\infty(\mathbb{R})$ and compactly into $L^2(\mathbb{R})$, it is possible to prove in a standard way the sequentially weakly continuity of Ψ. Moreover, Φ and Ψ are continuously Gâteaux differentiable with derivative given by

$$\Phi'(u)(v) = \langle u, v\rangle_V,$$

and

$$\Psi'(u)(v) = \int_{\mathbb{R}} W(x)f(u(x))v(x)dx,$$

for every $u, v \in H_V$.

Fix λ as in the conclusion. First of all, we show that $\lambda < 1/\delta$. To this end, write

$$r_n := \frac{1}{2}\left(\frac{b_n}{\kappa_\infty}\right)^2 \qquad (\forall\, n \in \mathbb{N}).$$

Since $\|u\|_\infty \leq \kappa_\infty \|u\|_V$ for every $u \in H_V$, thus

$$\|u\|_\infty \leq b_n, \qquad (\forall\, n \in \mathbb{N})$$

for every $u \in H_V$ such that $\Phi(u) < r_n$. Then, for every $n \in \mathbb{N}$, it follows that

$$\varphi(r_n) \leq \inf_{\Phi(u)<r_n} \frac{\|W\|_1 \max\limits_{|\xi|\leq b_n} F(\xi) - \int_{\mathbb{R}} W(x)F(u(x))dx}{\frac{1}{2}\left(\frac{b_n}{\kappa_\infty}\right)^2 - \frac{1}{2}\|u\|_V^2}.$$

Now, for every $n \in \mathbb{N}$ let $w_n \in H_V$ the function as given in (1.1). Therefore, $\|w_n\|_V \leq a_n\alpha_\sigma$. Hence, by $(\mathrm{A2})$, one has $\Phi(w_n) < r_n$ for each $n \geq n_0$. Moreover, by $(\mathrm{A1})$, we also have

$$\Psi(w_n) \geq W_\sigma F(a_n),$$

for each $n \in \mathbb{N}$. Therefore, it follows that

$$\varphi(r_n) \leq \frac{\|W\|_1 \max\limits_{|\xi|\leq b_n} F(\xi) - W_\sigma F(a_n)}{\frac{1}{2}\left(\frac{b_n}{\kappa_\infty}\right)^2 - \frac{1}{2}(a_n\alpha_\sigma)^2},$$

for every $n \geq n_0$. Hence, bearing in mind $(\mathrm{A3})$, we can write

$$0 \leq \delta \leq \lim_{n\to\infty} \varphi(r_n) \leq 2\kappa_\infty^2 \mathcal{A}_0 < +\infty.$$

Taking into account the above relation, since $\lambda < \frac{1}{2\kappa_\infty^2\mathcal{A}_0}$, we also have $\lambda < 1/\delta$.

Now, we claim that the functional I_λ does not have a local minimum at zero. Since $1/\lambda < \frac{2W_\sigma}{\alpha_\sigma^2}B^0$, there exist a sequence $\{\eta_n\}$ of positive numbers and $\tau > 0$ such that $\lim\limits_{n\to\infty} \eta_n = 0$ and

$$\frac{1}{\lambda} < \tau < \frac{2W_\sigma}{\alpha_\sigma^2}\frac{F(\eta_n)}{\eta_n^2},$$

for each $n \in \mathbb{N}$ large enough. For all $n \in \mathbb{N}$, let $s_n \in H_V$ defined by

$$s_n(x) := \begin{cases} 0, & |x - x_\sigma| > \mu_\sigma; \\ \eta_n, & |x - x_\sigma| \leq \frac{\mu_\sigma}{2}; \\ \frac{2\eta_n}{\mu_\sigma}\left(\mu_\sigma - |x - x_\sigma|\right), & \frac{\mu_\sigma}{2} < |x - x_\sigma| \leq \mu_\sigma. \end{cases}$$

Note that $\lambda\tau > 1$. Then, we obtain

$$\begin{aligned} I_\lambda(s_n) &= \Phi(s_n) - \lambda\Psi(s_n) \\ &\leq \frac{1}{2}(\eta_n\alpha_\sigma)^2 - \lambda W_\sigma F(\eta_n) \\ &< \frac{(\eta_n\alpha_\sigma)^2}{2}(1-\lambda\tau) \\ &< 0 = \Phi(0) - \lambda\Psi(0) \end{aligned}$$

for every $n \in \mathbb{N}$ large enough. Thus, the fact that $\|s_n\|_V \to 0$ as $n \to \infty$ implies that I_λ does not have a local minimum at zero. This, together with the fact that zero is the only global minimum of Φ, shows that the functional I_λ does not have a local minimum at the unique global minimum of Φ. Therefore, by Theorem 1.2(b), there exists a sequence $\{u_n\}$ of pairwise distinct critical points of I_λ converging to zero in H_V. This completes the proof. □

Now, we point out some consequences of Theorem 2.1. First, by setting

$$A_0 := \liminf_{\xi\to 0^+} \frac{\max_{|t|\leq\xi} F(t)}{\xi^2},$$

we get the following result.

***Theorem* 2.2** — *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that assumption* (A1) *in Theorem 2.1 holds. Assume also that*

(A4) $A_0 < \dfrac{W_\sigma}{(\alpha_\sigma\kappa_\infty)^2\|W\|_1} B^0.$

Then, for each

$$\lambda \in \left]\frac{\alpha_\sigma^2}{2B^0 W_\sigma}, \frac{1}{2\kappa_\infty^2 A_0 \|W\|_1}\right[,$$

problem (P_λ) *admits a sequence of nontrivial weak solutions* $\{u_n\}$ *which satisfy* $\lim_{n\to\infty}\|u_n\|_V = 0$.

PROOF : Let $\{b_n\}$ be a sequence of positive numbers which goes to zero such that

$$\lim_{n\to\infty} \frac{\max_{|\xi|\leq b_n} F(\xi)}{b_n^2} = A_0.$$

Taking $a_n = 0$ for every $n \in \mathbb{N}$, by Theorem 2.1 the conclusion follows. □

Remark 2.3 : When f is a nonnegative function, condition (A4) becomes

(A4′) $A_0' := \liminf_{\xi\to 0^+} \dfrac{F(\xi)}{\xi^2} < \dfrac{W_\sigma}{(\alpha_\sigma\kappa_\infty)^2\|W\|_1} B^0.$

In this case, $(\mathrm{A4}')$ ensures that for each

$$\lambda \in \left]\frac{\alpha_\sigma^2}{2B^0 W_\sigma}, \frac{1}{2\kappa_\infty^2 A_0' \|W\|_1}\right[,$$

problem (P_λ) admits a sequence of nontrivial and nonnegative weak solutions $\{u_n\}$ which satisfy $\lim_{n\to\infty} \|u_n\|_V = 0$.

Remark 2.4 : Theorem 1.1 in Introduction immediately follows by Theorem 2.2.

A special case of Theorem 2.2 is the following.

Corollary 2.5 — Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that assumption $(\mathrm{A1})$ in Theorem 2.1 holds. Assume that

$$A_0 < \frac{1}{2\kappa_\infty^2 \|W\|_1} \qquad \text{and} \qquad B^0 > \frac{\alpha_\sigma^2}{2W_\sigma}.$$

Then, problem (P_1) admits a sequence of nontrivial weak solutions $\{u_n\}$ which satisfy $\lim_{n\to\infty} \|u_n\|_V = 0$.

The next result is a consequence of Theorem 2.1 and guarantees the existence of infinitely many solutions to (P_λ) for each λ which lies in a precise half-line.

Corollary 2.6 — Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that assumption $(\mathrm{A1})$ in Theorem 2.1 holds. Assume that there exist a real sequence $\{a_n\}$ and a sequence $\{b_n\}$ in $]0, +\infty[$, with $\lim_{n\to\infty} b_n = 0$, such that $(\mathrm{A2})$ holds. Further, let

(A5) $W_\sigma F(a_n) = \|W\|_1 \max_{|\xi| \le b_n} F(\xi)$ for each $n \in \mathbb{N}$.

If $B^0 > 0$, then, for each $\lambda > \frac{\alpha_\sigma^2}{2B^0 W_\sigma}$, problem (P_λ) admits a sequence of nontrivial weak solutions $\{u_n\}$ which satisfy $\lim_{n\to\infty} \|u_n\|_V = 0$.

PROOF : By $(\mathrm{A5})$ we obtain $\mathcal{A}_0 = 0$. Hence, since $B^0 > 0$, condition $(\mathrm{A3})$ of Theorem 2.1 holds and the proof is complete.

The following theorem is a significative consequence of Theorem 2.1 that is not directly obtainable by the statements of [4].

***Theorem* 2.7** — *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that assumption $(\mathrm{A1})$ in Theorem 2.1 holds. Suppose that there exist two sequences $\{a_n\}$ and $\{b_n\}$* in $]0, +\infty[$, with $a_n < b_n$ *for every $n \ge \nu$, and $\lim_{n\to\infty} b_n = 0$, such that*:

(A6) $\lim_{n\to\infty} \dfrac{b_n}{a_n} = +\infty\,;$

(A7) $\max_{t\in[a_n,b_n]} f(t) \leq 0$ *for every* $n \geq \nu$;

(A8) $\dfrac{\alpha_\sigma^2}{W_\sigma} < 2B^0 < +\infty$.

Then, problem (P_1) *admits a sequence of nontrivial and nonnegative weak solutions* $\{u_n\}$ *which satisfy* $\lim_{n\to\infty} \|u_n\|_V = 0$.

PROOF : Our aim is to apply Theorem 2.1. If $\{a_n\}$ and $\{b_n\}$ are two real sequences satisfying our assumptions, we have that there exists $n_0 \geq \nu$ such that

$$\frac{a_n}{b_n} < \frac{1}{\alpha_\sigma \kappa_\infty},$$

for every $n \geq n_0$. Hence condition $(\mathrm{A2})$ in Theorem 2.1 holds. Now, in order to prove also condition $(\mathrm{A3})$, let us define the following real sequence $\{w_n\}$ given by

$$w_n := \|W\|_1 \frac{\max_{|\xi|\leq b_n} F(\xi)}{a_n^2} - W_\sigma \frac{F(a_n)}{a_n^2},$$

for every $n \geq n_0$. At this point, hypothesis $(\mathrm{A7})$ yields

$$\max_{|\xi|\leq b_n} F(\xi) = \max_{|\xi|\leq a_n} F(\xi). \tag{2.1}$$

Thus, since $\frac{W_\sigma}{\|W\|_1} \leq 1$ and $F(a_n) \geq 0$, and bearing in mind (2.1), we can write

$$\begin{aligned} \frac{\max_{|\xi|\leq b_n} F(\xi)}{a_n^2} &= \frac{\max_{|\xi|\leq a_n} F(\xi)}{a_n^2} \\ &\geq \frac{F(a_n)}{a_n^2} \\ &\geq \frac{W_\sigma}{\|W\|_1}\frac{F(a_n)}{a_n^2}, \end{aligned}$$

for every $n \geq n_0$. Hence, since $w_n \geq 0$ for every $n \geq n_0$, we have

$$0 \leq \limsup_{n\to\infty} w_n.$$

Further, by $(\mathrm{A8})$ we clearly have

$$0 \leq \limsup_{\xi\to 0^+} \frac{F(\xi)}{\xi^2} < +\infty, \tag{2.2}$$

and consequently (note that $a_n \searrow 0^+$ as $n \to \infty$) we obtain

$$0 < \limsup_{n\to\infty} \frac{F(a_n)}{a_n^2} < +\infty. \tag{2.3}$$

On the other hand, let $\xi_n \in]0, a_n]$ be a sequence such that $F(\xi_n) := \max_{|\xi|\leq a_n} F(\xi)$ for every $n \geq n_0$. Thus

$$\begin{aligned} \limsup_{n\to\infty} \frac{\max_{|\xi|\leq b_n} F(\xi)}{a_n^2} &= \limsup_{n\to\infty} \frac{\max_{|\xi|\leq a_n} F(\xi)}{a_n^2} \\ &= \limsup_{n\to\infty} \frac{F(\xi_n)}{a_n^2} \\ &\leq \limsup_{n\to\infty} \frac{F(\xi_n)}{\xi_n^2}. \end{aligned}$$

The above relations and (2.2) yield

$$0 \leq \limsup_{n\to\infty} \frac{\max_{|\xi|\leq b_n} F(\xi)}{a_n^2} \leq \limsup_{n\to\infty} \frac{F(\xi_n)}{\xi_n^2} < +\infty.$$

Hence, there exists a constant β such that

$$0 \leq \limsup_{n\to\infty} w_n = \beta. \tag{2.4}$$

Then, by $(\mathrm{A}6)$ and (2.4), we have

$$\mathcal{A}_0 = \limsup_{n\to\infty} w_n \left(\frac{b_n^2}{a_n^2} - (\kappa_\infty \alpha_\sigma)^2\right)^{-1} = 0.$$

In conclusion, condition $(\mathrm{A}3)$ holds. Finally, bearing in mind $(\mathrm{A}8)$, one has

$$1 \in \left]\frac{\alpha_\sigma^2}{2B^0 W_\sigma}, +\infty\right[.$$

Thanks to Theorem 2.1, the theorem is proved.

Remark 2.8 : We observe that, in contrast to Theorem 2.7, studying problem (P_1), one of the key assumptions requested by Faraci and Kristály, is that

$$B^0 := \limsup_{\xi\to 0^+} \frac{F(\xi)}{\xi^2} = +\infty\,;$$

see [4, Theorem 2.1]. Moreover, we do not assume here that

$$\lim_{n\to\infty} \frac{\max_{\xi\in[0,a_n]} F(\xi)}{b_n^2} = 0$$

or

$$\limsup_{n\to\infty} \frac{\max_{\xi\in[0,a_n]} F(\xi)}{b_n^2} < \frac{\min\{1, V_0\}}{2\kappa_\infty^2 \|W\|_1};$$

see [4, Theorem 2.1 and Remark 6].

Now, put

$$B^{\infty} := \limsup_{\xi \to +\infty} \frac{F(\xi)}{\xi^2}.$$

Using Theorem 1.2(a) and arguing as in the proof of Theorem 2.1, we can obtain the following multiplicity result.

***Theorem* 2.9** — *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that assumption* $(\mathrm{A1})$ *in Theorem 2.1 holds. Assume that there exist two sequences* $\{c_n\}$ *and* $\{d_n\}$ in $]0, +\infty[$, *with* $\lim_{n\to\infty} d_n = +\infty$, *such that*

(A9) *for some* $n_0 \in \mathbb{N}$ *one has* $c_n < \dfrac{d_n}{\alpha_\sigma \kappa_\infty}$ *for each* $n \geq n_0$;

(A10) $$\mathcal{A}_\infty := \lim_{n\to\infty} \frac{\|W\|_1 \max_{|\xi|\leq d_n} F(\xi) - W_\sigma F(c_n)}{d_n^2 - (\kappa_\infty c_n \alpha_\sigma)^2} < \frac{B^{\infty} W_\sigma}{(\kappa_\infty \alpha_\sigma)^2}.$$

Then, for each

$$\lambda \in \left] \frac{\alpha_\sigma^2}{2B^{\infty} W_\sigma}, \frac{1}{2\kappa_\infty^2 \mathcal{A}_\infty} \right[,$$

problem (P_λ) *admits a sequence* $\{u_n\} \subset H_V$ *of weak solutions such that* $\lim_{n\to\infty} \|u_n\|_V = +\infty$.

Remark 2.10 : Applying Theorem 2.9, results similar to Theorems 1.1, 2.2 and 2.7 and Corollaries 2.5 and 2.6 can be obtained. We omit the discussions here.

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