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A NOTE ON EXTRACTION OF ORTHOGONAL POLYNOMIALS FROM GENERATING FUNCTION FOR RECIPROCAL OF ODD NUMBERS

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Motivated by a recent paper by Shashikala [*Indian J. Pure Appl. Math.* **48** (2) (2017), 177-185] on the extraction four sequences of orthogonal polynomials from generating function from reciprocal of odd numbers, in this note we identify the weight functions and the intervals of orthogonality of these sequences of polynomials. Two of these sequences can be expressed in terms of particular Jacobi polynomials transformed to $[0,1]$, and other two are non-classical polynomials also orthogonal on $[0,1]$.

Key words : Orthogonal polynomials; weight function; recurrence relation; continued fractions, Gauss's hypergeometric function.

1. Introduction and Preliminaries

Let $x \mapsto w(x)$ be a weight function on $[a,b]$, $a<b$, such that all moments $\mu_k = \int_a^b x^k w(x)\,\mathrm{d}x$, $k \in \mathbb{N}_0$, exist and are finite, and $\mu_0 > 0$. Then, there exists a unique sequence of monic polynomials $\{\pi_n(x)\}_{n=0}^{\infty}$ orthogonal on $[a,b]$ with respect to this weight function, i.e.,

$$(\pi_n,\pi_m) = \int_a^b \pi_n(x)\pi_m(x)w(x)\,\mathrm{d}x = \|\pi_n\|^2\delta_{n,m},$$

where $\delta_{n,m}$ is Kronecker's delta. These polynomials satisfy the three-term recurrence relation

$$\pi_{n+1}(x) = (x-\alpha_n)\pi_n(x) - \beta_n\pi_{n-1}(x), \quad n = 0,1,2\ldots, \tag{1.1}$$

with $\pi_0(x) = 1$ and $\pi_{-1}(x) = 0$, where $\alpha_n = \alpha_n(w)$ and $\beta_n = \beta_n(w)$ are recursion coefficients. The coefficient β_0 may be arbitrary, but is conveniently defined by $\beta_0 = \mu_0 = \int_a^b w(x)\,\mathrm{d}x$.

The same recursion coefficients α_k and β_k appear in the so-called *Jacobi continued fraction associated with the weight function* w,

$$F(x) = \int_a^b \frac{w(t)}{x-t}\,\mathrm{d}t \sim \frac{\beta_0}{x-\alpha_0-}\,\frac{\beta_1}{x-\alpha_1-}\,\cdots\,,$$

which is known as the *Stieltjes transform* of the weight function $x \mapsto w(x)$ (for details see [1, p. 15], [5, p. 114], [6]). For the n-th *convergent* of this continued fraction, we have

$$\frac{\beta_0}{x-\alpha_0-}\,\frac{\beta_1}{x-\alpha_1-}\,\cdots\,\frac{\beta_{n-1}}{x-\alpha_{n-1}} = \frac{\sigma_n(x)}{\pi_n(x)}\,, \tag{1.2}$$

where $\sigma_n(x)$ are the so–called *associated polynomials*, defined by

$$\sigma_n(x) = \int_a^b \frac{\pi_n(x)-\pi_n(t)}{x-t} w(t)\,\mathrm{d}t, \quad k \geq 0.$$

These polynomials satisfy the same three-term recurence relation (1.1), i.e.,

$$\sigma_{n+1}(x) = (x-\alpha_n)\sigma_n(x) - \beta_n\sigma_{n-1}(x), \quad n \geq 0, \tag{1.3}$$

only with starting values $\sigma_0(x) = 0$, $\sigma_{-1}(x) = -1$ (cf. [5, pp. 111-114]).

Recently Shashikala [8] has considered the series with coefficient as reciprocal of odd number,

$$T(x) = 1 + \frac{1}{3}x + \frac{1}{5}x^2 + \cdots + \frac{1}{2n+1}x^n + \cdots\,, \tag{1.4}$$

which has the following representation

$$T(x) = {}_2F_1\Big(\frac{1}{2}, 1; \frac{3}{2}; x\Big),$$

where

$${}_2F_1(a,b;c;x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k}\frac{x^k}{k!}$$

is the Gauss hypergeometric function, $(a)_0 = 1$, $(a)_k = a(a+1)\cdots(a+k-1) = \Gamma(a+k)/\Gamma(a)$, is Pochhammer's symbol, and $\Gamma(z)$ is Euler's gamma function.

Using the regular C-fraction of (1.4) (cf. [4]),

$$T(x) = \frac{1}{1+}\,\frac{-\frac{1}{3}x}{1+}\,\frac{-\frac{4}{15}x}{1+}\,\frac{-\frac{9}{35}x}{1+}\,\cdots\,\frac{-\frac{n^2}{4n^2-1}x}{1+}\,\cdots \tag{1.5}$$

and taking even and odd order convergents of (1.5), Shashikala [8] has obtained four sequences of monic orthogonal polynomials $\{Q_n^{(\nu)}(x)\}_{n=0}^{\infty}$, $\nu = 1,2,3,4$, which satisfy the three-term recurrence relation (1.1), with $Q_0^{(\nu)}(x) = 1$ and

$$Q_1^{(1)}(x) = x - \frac{1}{3}, \quad Q_1^{(2)}(x) = x - \frac{3}{5}, \quad Q_1^{(3)}(x) = x - \frac{4}{15}, \quad Q_1^{(4)}(x) = x - \frac{11}{21}.$$

In [8] these polynomials have been denoted by $q_n(x)$, $s_n(x)$, $r_n(x)$, $p_n(x)$, respectively, and their recurrence cofficients are:

$$\alpha_n^{(1)} = \frac{32n^3+24n^2-1}{(4n-1)(4n+1)(4n+3)}, \qquad \beta_n^{(1)} = \frac{(2n-1)^2(2n)^2}{(4n-3)(4n-1)^2(4n+1)}; \tag{1.6}$$

$$\alpha_n^{(2)} = \frac{32n^3+72n^2+48n+9}{(4n+1)(4n+3)(4n+5)}, \qquad \beta_n^{(2)} = \frac{(2n)^2(2n+1)^2}{(4n-1)(4n+1)^2(4n+3)}; \tag{1.7}$$

$$\alpha_n^{(3)} = \frac{32n^3+72n^2+48n+9}{(4n+1)(4n+3)(4n+5)}, \qquad \beta_n^{(3)} = \frac{(2n)^2(2n+1)^2}{(4n-1)(4n+1)^2(4n+3)}; \tag{1.8}$$

$$\alpha_n^{(4)} = \frac{32(n+1)^3+24(n+1)^2-1}{(4n+3)(4n+5)(4n+7)}, \quad \beta_n^{(4)} = \frac{(2n+1)^2(2n+2)^2}{(4n+1)(4n+3)^2(4n+5)}. \tag{1.9}$$

First two polynomials, $q_n(x)$ and $s_n(x)$, are classical orthogonal polynomials (cf. [5, pp. 121-146]) and they have been extracted from denominators, and other two, $r_n(x)$ and $p_n(x)$, are non-classical polynomials and extracted from numerators of (1.5).

Remark 1 : There is a mistake in [8, Eq. (10)]; $r_1(x)$ should be $x - 4/15$ (not $x - 4/5$).

According to the previous facts, $r_n(x)$ and $p_n(x)$ are the associated polynomials. In the next section we identify their weight functions and the intervals of orthogonality, as well as ones for the classical polynomials $q_n(x)$ and $s_n(x)$.

2. Weight Functions and Intervals of Orthogonality

Since $\deg \sigma_n = n-1$, very often for monic associated polynomials $\hat{\sigma}_{n+1}(x)$ we use the notation $\pi_n^{[1]}(x)$, i.e.,

$$\pi_n^{[1]}(x) = \frac{1}{\beta_0}\int_a^b \frac{\pi_{n+1}(x)-\pi_{n+1}(t)}{x-t} w(t)\,\mathrm{d}t, \quad n \geq 0 \tag{2.1}$$

(see [5, p. 112]). Then, because of (1.3), we have

$$\pi_{n+1}^{[1]}(x) = (x-\alpha_{n+1})\pi_n^{[1]}(x) - \beta_{n+1}\pi_{n-1}^{[1]}(x), \quad \pi_0^{[1]}(x) = 1,\ \pi_{-1}^{[1]}(x) = 0, \tag{2.2}$$

and, according to Favard's theorem, these monic associated polynomials are orthogonal with respect to some weight function $x \mapsto w_1(x)$ on $[c, d]$. Grosjean [2, 3] developed a theory for finding an explicit expression for the weight $w_1(x)$ and a procedure for obtaining its interval of orthogonality $[c, d] \subset [a, b]$ (see also [9], [7], [5, pp. 112-113]). Namely, if $w(x)$ is a piecewise weight function on $[a, b]$, then $[c, d] = [a, b]$ and

$$w_1(x) = \frac{\beta_0 w(x)}{\left(\text{P.V.}\displaystyle\int_a^b \frac{w(t)\,\mathrm{d}t}{t-x}\right)^2 + (\pi w(x))^2}, \qquad a < x < b, \tag{2.3}$$

where $\beta_0 = \mu_0 = \int_a^b w(x)\,\mathrm{d}x$ and $\int_a^b w_1(x)\,\mathrm{d}x = \beta_1$, and P.V. means Cauchy principal value.

In the sequel we suppose the even weight function on $[-1,1]$, $w(-x) = w(x)$, for which the monic polynomials $\pi_n(x)$ are even or odd depending on the parity of n, i.e., $\pi_n(-x) = (-1)^n \pi_n(x)$. For such polynomials the coefficients α_n in (1.1) are equal to zero for each n. Also, we need the following result (see [5, pp. 102-103]) for polynomials defined by

$$p_n^{(1)}(x) = \pi_{2n}(\sqrt{x}) \quad \text{and} \quad p_n^{(2)}(x) = \frac{\pi_{2n+1}(\sqrt{x})}{\sqrt{x}}, \quad n = 0, 1, 2, \dots . \tag{2.4}$$

Theorem 1 — *The sequences of polynomials* $\{p_n^{(\nu)}(x)\}_{n\in\mathbb{N}_0}$, $\nu = 1, 2$, *are orthogonal* $[0,1]$ *with respect to the weight functions* $w^{(1)}(x) = w(\sqrt{x})/\sqrt{x}$ *and* $w^{(2)}(x) = \sqrt{x}w(\sqrt{x})$, *respectively, and they satisfy the recurrence relations*

$$p_{n+1}^{(\nu)}(x) = (x - a_n^{(\nu)})p_n^{(\nu)}(x) - b_n^{(\nu)} p_{n-1}^{(\nu)}(x), \quad n = 0, 1, \dots, \tag{2.5}$$

with $p_0^{(\nu)}(x) = 1$ *and* $p_{-1}^{(\nu)}(x) = 0$, *where* $a_0^{(1)} = \beta_1$, $a_0^{(2)} = \beta_1 + \beta_2$, *and for* $n \in \mathbb{N}$

$$a_n^{(1)} = \beta_{2n} + \beta_{2n+1}, \qquad b_n^{(1)} = \beta_{2n-1}\beta_{2n}$$

and

$$a_n^{(2)} = \beta_{2n+1} + \beta_{2n+2}, \qquad b_n^{(2)} = \beta_{2n}\beta_{2n+1}.$$

Now we start with the classical Gegenbauer (or ultraspherical) polynomials $C_n^\lambda(x)$ orthogonal on $[-1,1]$ with respect to the Gegenbauer weight function $w(x) = (1-x^2)^{\lambda-1/2}$, $\lambda > -1/2$ (cf. [5, p. 133]). The corresponding monic polynomials $\pi_n(x) = \widehat{C}_n^\lambda(x)$, satisfy the three-term recurrence relation (1.1), with the coefficients (see [5, p. 132])

$$\alpha_n = 0 \quad (n \in \mathbb{N}_0), \quad \beta_0 = \sqrt{\pi}\,\frac{\Gamma\left(\lambda+\frac{1}{2}\right)}{\Gamma(\lambda+1)}, \quad \beta_n = \frac{n(2\lambda+n-1)}{4(\lambda+n-1)(\lambda+n)} \quad (n \in \mathbb{N}), \tag{2.6}$$

except the case $\lambda = 0$, when $\beta_1 = 1/2$.

According to Theorem 1, the polynomials $p_n^{(1)}(x) = \widehat{C}_{2n}^\lambda(\sqrt{x})$ and $p_n^{(2)}(x) = \widehat{C}_{2n+1}^\lambda(\sqrt{x})/\sqrt{x}$ are orthogonal on $[0,1]$ with respect to the weight functions

$$x \mapsto \frac{(1-x)^{\lambda-1/2}}{\sqrt{x}} \quad \text{and} \quad x \mapsto \sqrt{x}(1-x)^{\lambda-1/2},$$

respectively, and satisfy the recurrence relation (2.5), with the recurrence coefficients

$$\begin{aligned} a_0^{(1)} &= \frac{1}{2(\lambda+1)}, \qquad a_n^{(1)} = \beta_{2n} + \beta_{2n+1} = \frac{4n^2 + 4\lambda n + \lambda - 1}{2(\lambda+2n-1)(\lambda+2n+1)}, \\ b_0^{(1)} &= \frac{\sqrt{\pi}\,\Gamma\left(\lambda+\frac{1}{2}\right)}{\Gamma(\lambda+1)}, \quad b_n^{(1)} = \beta_{2n-1}\beta_{2n} = \frac{n(2n-1)(\lambda+n-1)(2\lambda+2n-1)}{4(\lambda+2n-2)(\lambda+2n-1)^2(\lambda+2n)}, \end{aligned} \tag{2.7}$$

and

$$
\begin{aligned}
a_0^{(2)} &= \frac{3}{2(\lambda+2)}, \qquad a_n^{(2)} = \beta_{2n+1}+\beta_{2n+2} = \frac{3\lambda+4n^2+4(\lambda+1)n}{2(\lambda+2n)(\lambda+2n+2)},\\
b_0^{(2)} &= \frac{\sqrt{\pi}\Gamma\left(\lambda+\frac{1}{2}\right)}{2\Gamma(\lambda+2)}, \quad b_n^{(2)} = \beta_{2n}\beta_{2n+1} = \frac{n(2n+1)(\lambda+n)(2\lambda+2n-1)}{4(\lambda+2n-1)(\lambda+2n)^2(\lambda+2n+1)}.
\end{aligned} \tag{2.8}
$$

In fact, they are (monic) Jacobi polynomials transformed to the interval $[0,1]$, with parameters $(\lambda-1/2, \mp 1/2)$, i.e.,

$$
p_n^{(1)}(x) = \frac{n!}{(n+\lambda)_n} P_n^{(\lambda-1/2,-1/2)}(2x-1) \quad \text{and} \quad p_n^{(2)}(x) = \frac{n!}{(n+\lambda+1)_n} P_n^{(\lambda-1/2,1/2)}(2x-1),
$$

where $P_n^{(\alpha,\beta)}(x)$ are the classical Jacobi polynomials orthogonal with respect to the weight function $x \mapsto (1-x)^\alpha(1+x)^\beta$ on $[-1,1]$ (cf. [5, pp. 131-140]).

Evidently, in the case $\lambda = 1/2$, the coefficients (2.7) and (2.8) reduce to ones for the polynomials $\{Q_n^{(\nu)}(x)\}_{n=0}^{\infty}$, $\nu = 1,2$, obtained in [8]. Namely,

$$
\begin{aligned}
a_n^{(1)}\big|_{\lambda=1/2} &= \alpha_n^{(1)} = \frac{8n^2+4n-1}{(4n-1)(4n+3)}, \quad b_n^{(1)}\big|_{\lambda=1/2} = \beta_n^{(1)} = \frac{(2n-1)^2(2n)^2}{(4n-3)(4n-1)^2(4n+1)},\\
a_n^{(2)}\big|_{\lambda=1/2} &= \alpha_n^{(2)} = \frac{8n^2+12n+3}{(4n+1)(4n+5)}, \quad b_n^{(2)}\big|_{\lambda=1/2} = \beta_n^{(2)} = \frac{(2n)^2(2n+1)^2}{(4n-1)(4n+1)^2(4n+3)}.
\end{aligned}
$$

Notice that the expressions for $\alpha_n^{(\nu)}$, $\nu = 1,2,3,4$, in (1.6)-(1.9), can be shortened by a common factor.

Thus, we have the following statement:

Proposition 1 — The polynomials $Q_n^{(1)}(x)$ and $Q_n^{(2)}(x)$ (i.e., $q_n(x)$ and $s_n(x)$) are orthogonal on $[0,1]$ with respect to the weight functions $x \mapsto 1/\sqrt{x}$ and $x \mapsto \sqrt{x}$, respectively, and they can be expressed in terms of Jacobi polynomials as

$$
Q_n^{(1)}(x) = \frac{n!}{\left(n+\frac{1}{2}\right)_n} P_n^{(0,-1/2)}(2x-1) \quad \text{and} \quad Q_n^{(2)}(x) = \frac{n!}{\left(n+\frac{3}{2}\right)_n} P_n^{(0,1/2)}(2x-1).
$$

We return now to the weight function $x \mapsto w_1(x)$ of the associated polynomial for the Gegenbauer polynomials $C_n^\lambda(x)$.

Taking $t = (z+x)/(1+xz)$, we have

$$
\text{P.V.} \int_{-1}^{1} \frac{(1-x^2)^{\lambda-1/2}\,\mathrm{d}t}{t-x} = (1-x^2)^{\lambda-1/2}\ \text{P.V.} \int_{-1}^{1} \frac{(1-z^2)^{\lambda-1/2}}{z(1+xz)^{2\lambda}}\,\mathrm{d}z.
$$

Since

$$
(1+xz)^{-2\lambda} = \sum_{k=0}^{\infty} \binom{-2\lambda}{k}(xz)^k = \sum_{k=0}^{\infty}(-1)^k \frac{(2\lambda)_k}{k!}(xz)^k
$$

and

$$\mu_k = \int_{-1}^{1} z^k (1-z^2)^{\lambda-1/2}\,\mathrm{d}z = \begin{cases} 0, & k \text{ is odd}, \\ \dfrac{\Gamma\left(\lambda+\frac{1}{2}\right)\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\lambda+1+\frac{k}{2}\right)}, & k \text{ is even}, \end{cases}$$

as well as $\mu_{-1} = \text{P.V.} \displaystyle\int_{-1}^{1} z^{-1}(1-z^2)^{\lambda-1/2}\,\mathrm{d}z = 0$, we get

$$\begin{aligned} \text{P.V.} \int_{-1}^{1} \frac{(1-x^2)^{\lambda-1/2}\,\mathrm{d}t}{t-x} &= -(1-x^2)^{\lambda-1/2}\Gamma\left(\lambda+\frac{1}{2}\right)\sum_{\nu=0}^{\infty} \frac{(2\lambda)_{2\nu+1}\Gamma\left(\nu+\frac{1}{2}\right)}{(2\nu+1)!\Gamma(\lambda+\nu+1)} x^{2\nu+1} \\ &= -2\lambda\beta_0(1-x^2)^{\lambda-1/2}\, x\, {}_2F_1\left(\lambda+\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; x^2\right), \end{aligned}$$

where β_0 is given in (2.6), so that the weight function (2.3) becomes

$$w_1(x) = \frac{\beta_0(1-x^2)^{1/2-\lambda}}{4\lambda^2\beta_0^2\left(x\,{}_2F_1\left(\lambda+\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; x^2\right)\right)^2 + \pi^2}, \quad x \in [-1,1].$$

In the Legendre case $(\lambda = 1/2)$, it reduces to

$$w_1(x)\big|_{\lambda=1/2} = \frac{2}{4\left(\tanh^{-1} x\right)^2 + \pi^2}, \quad x \in [-1,1], \tag{2.9}$$

and the corresponding orthogonal polynomials $\pi_n^{[1]}(x)$ satisfy the three-term recurrence relation (2.2), with coefficients given by (2.6) for $\lambda = 1/2$, i.e.,

$$\pi_{n+1}^{[1]}(x) = x\pi_n^{[1]}(x) - \frac{(n+1)^2}{(2n+1)(2n+3)}\pi_{n-1}^{[1]}(x), \quad n = 0, 1, \ldots\ . \tag{2.10}$$

The weight function (2.9) is even and $\pi_n^{[1]}(-x) = (-1)^n \pi_n^{[1]}(x)$. Applying Theorem 1 we get two sequences of polynomials $\widehat{p}_n^{(1)}(x) = \pi_{2n}^{[1]}(\sqrt{x})$ and $\widehat{p}_n^{(2)}(x) = \pi_{2n+1}^{[1]}(\sqrt{x})/\sqrt{x}$, which are orthogonal on $[0,1]$ with respect to the weight functions

$$x \mapsto \widehat{w}_1(x) = \frac{1}{\sqrt{x}} \frac{2}{4\left(\tanh^{-1}\sqrt{x}\right)^2 + \pi^2} \quad \text{and} \quad x \mapsto \widehat{w}_2(x) = \frac{2\sqrt{x}}{4\left(\tanh^{-1}\sqrt{x}\right)^2 + \pi^2},$$

respectively. Starting from (2.10), i.e.,

$$\widehat{\beta}_n = \beta_{n+1}\big|_{\lambda=1/2} = \frac{(n+1)^2}{(2n+1)(2n+3)}, \quad n = 0, 1, \ldots,$$

and using Theorem 1, we obtain their recurrence coefficients, $\widehat{a}_n^{(\nu)}$ and $\widehat{b}_n^{(\nu)}$, $\nu = 1, 2$, in the following form

$$\widehat{a}_0^{(1)} = \widehat{\beta}_1 = \frac{4}{15}, \quad \widehat{a}_n^{(1)} = \widehat{\beta}_{2n} + \widehat{\beta}_{2n+1} = \frac{8n^2+12n+3}{(4n+1)(4n+5)},$$

$$\widehat{b}_n^{(1)} = \widehat{\beta}_{2n-1}\widehat{\beta}_{2n} = \frac{(2n)^2(2n+1)^2}{(4n-1)(4n+1)^2(4n+3)},$$

and

$$\widehat{a}_0^{(2)} = \widehat{\beta}_1 + \widehat{\beta}_2 = \frac{11}{21}, \quad \widehat{a}_n^{(2)} = \widehat{\beta}_{2n+1} + \widehat{\beta}_{2n+2} = \frac{8n^2+20n+11}{(4n+3)(4n+7)},$$
$$\widehat{b}_n^{(2)} = \widehat{\beta}_{2n}\widehat{\beta}_{2n+1} = \frac{(2n+1)^2(2n+2)^2}{(4n+1)(4n+3)^2(4n+5)}.$$

As before, we can see that these coefficients coincide with those of polynomials $\{Q_n^{(\nu)}(x)\}_{n=0}^{\infty}$, $\nu = 3, 4$, given in (1.8) and (1.9) and obtained in [8]. Namely,

$$\widehat{a}_n^{(1)} = \alpha_n^{(3)}, \quad \widehat{b}_n^{(1)} = \beta_n^{(3)} \quad \text{and} \quad \widehat{a}_n^{(2)} = \alpha_n^{(4)}, \quad \widehat{b}_n^{(2)} = \beta_n^{(4)},$$

so that we have the following statement:

Proposition 2 — The polynomials $Q_n^{(3)}(x)$ and $Q_n^{(4)}(x)$ (i.e., $r_n(x)$ and $p_n(x)$) are orthogonal on $[0,1]$ with respect to the weight functions $x \mapsto \widehat{w}_1(x)$ and $x \mapsto \widehat{w}_2(x)$, respectively.

Some other interesting cases could be for $\lambda = 0$ and $\lambda = 1$, but they are much simpler than the previous one for $\lambda = 1/2$.

In the case $\lambda = 0$ we start from the recurrence relation for the monic Chebyshev polynomials of the first kind $\pi_n(x) = 2^{1-n}T_n(x)$, $n \geq 1$,

$$\pi_{n+1}(x) = x\pi_n(x) - \beta_n\pi_{n-1}(x), \quad \pi_0(x) = 1,\ \pi_{-1}(x) = 0, \tag{2.11}$$

where $\beta_0 = \pi$, $\beta_1 = 1/2$, and $\beta_n = 1/4$, $n \geq 2$. These polynomials are orthogonal on $[-1,1]$ with respect to the weight function $x \mapsto (1-x^2)^{-1/2}$.

An application of Theorem 1 gives two well-known sequences of polynomials orthogonal on $[0,1]$ with respect to the weight functions $x \mapsto 1/\sqrt{x(1-x)}$ and $x \mapsto \sqrt{x/(1-x)}$, $2^{1-2n}T_{2n}(\sqrt{x})/\sqrt{x}$ and $2^{-2n}\sqrt{x}\,T_{2n+1}(\sqrt{x})$, which satisfy the recurrence relation (2.5), with the coefficients

$$a_n^{(1)} = \frac{1}{2}\ (n \geq 0), \quad b_0^{(1)} = \pi,\ b_1^{(1)} = \frac{1}{8},\ b_n^{(1)} = \frac{1}{16}\ (n \geq 2)$$

and

$$a_0^{(2)} = \frac{3}{4}\ a_n^{(2)} = \frac{1}{2}\ (n \geq 1), \quad b_0^{(1)} = \frac{\pi}{2},\ b_n^{(1)} = \frac{1}{16}\ (n \geq 1),$$

respectively.

In the case $\lambda = 1$, we start again from the same recurrence relation (2.11), of course, in this case for the monic Chebyshev polynomials of the second kind $\pi_n(x) = 2^{-n}U_n(x)$, orthogonal on $[-1,1]$ with respect to the weight function $x \mapsto (1-x^2)^{1/2}$. The recurrence coefficients are $\beta_0 = \pi/2$ and $\beta_n = 1/4$, $n \geq 1$. The corresponding sequences of polynomials, $2^{-2n}U_{2n}(\sqrt{x})$

and $2^{-2n-1}U_{2n+1}\,(\sqrt{x})\,/\,(\sqrt{x})$, obtained by the application of Theorem 1, are orthogonal on $[0,1]$ with respect to the weight functions $x \mapsto \sqrt{(1-x)/x}$ and $x \mapsto \sqrt{x(1-x)}$, and they satisfy the recurrence relation (2.5), with the coefficients

$$a_0^{(1)} = \frac{1}{4},\;\; a_n^{(1)} = \frac{1}{2}\;\; (n \ge 1), \quad b_0^{(1)} = \frac{\pi}{2},\;\; b_n^{(1)} = \frac{1}{16}\;\; (n \ge 1)$$

and

$$a_n^{(2)} = \frac{1}{2}\;\; (n \ge 0), \quad b_0^{(1)} = \frac{\pi}{8},\;\; b_n^{(1)} = \frac{1}{16}\;\; (n \ge 1),$$

respectively.

It is easy to see that the associated polynomials on $[-1,1]$ in the both cases ($\lambda = 0$ and $\lambda = 1$) are the (monic) Chebyshev polynomials of the second kind.

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