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INDICES OF A FINITISTIC SPACE WITH MOD 2 COHOMOLOGY $\mathbb{R}P^n \times \mathbb{S}^2$

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Let $G = \mathbb{Z}_2$ act freely on a finitistic space X with mod 2 cohomology ring isomorphic to the product of a real projective space and 2-sphere $\mathbb{S}^2$. In this paper, we determine the Conner and Floyd's mod 2 cohomology index and the Volovikov's numerical index of X. Using these indices, we discuss the nonexistence of equivariant maps $X \to \mathbb{S}^n$ and $\mathbb{S}^n \to X$. The covering dimensions of the coincidence sets of continuous maps $X \to \mathbb{R}^k$ are also determined.

Key words : Free action; finitistic space; Leray-Serre spectral sequence; index; covering dimension.

1. INTRODUCTION

The classical Borsuk-Ulam theorem states that there is no equivariant map $\mathbb{S}^n \to \mathbb{S}^{n-1}$ relative to the antipodal actions on spheres. It is equivalent to say that the coincidence set $A(f) = \{x \in \mathbb{S}^n | f(x) = f(-x)\}$ of a continuous map $f : \mathbb{S}^n \to \mathbb{R}^n$ is nonempty. This lead to the study of nonexistence of a G-equivariant map between two G-spaces and the study of the size of the coincidence set of a given map. Yang [22] and Conner and Floyd [6] defined indices for a free involution on a paracompact Hausdorff space. Yang [22] and Bourgin [4] independently proved that the index of coincidence set of a continuous map $f : X \to \mathbb{R}^k$ must be greater than or equal to $n - k$, where n is the index of X. Fadell and Husseini [9, 10] and Voloviko [20, 21] have also introduced the notion of indices for a compact Lie group G-spaces and generalized the Bourgin-Yang and Borsuk-Ulam theorems. Coelho *et al*. [5] obtained various possibilities of the Volovikov's index of a free $G = \mathbb{Z}_p$, p a prime, or $\mathbb{S}^1$-action on the product of two spheres, and applied it to the study of nonexistence of a equivariant

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map from the product of two spheres into a sphere. For a free $G = \mathbb{Z}_2$ or $\mathbb{S}^1$-action on the spaces of type (a, b) has been studied in [11, 13]. The co-index of a free involution on projective spaces and nonexistence of a equivariant map from sphere into projective spaces have been studied in [14]. Singh [15, 16] obtained the co-index of a free involution on $X = \mathbb{R}P^n \times \mathbb{S}^1, \mathbb{C}P^n \times \mathbb{S}^1$ or $\mathbb{C}P^n \times \mathbb{S}^2$, and studied the nonexistence of a equivariant map from sphere into X. The co-index and Volvovikov's indices of $X = \mathbb{F}P^n \times \mathbb{S}^3$, where $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, for a free $G = \mathbb{Z}_p$, p a prime, has been studied recently by singh *et al.* in [17, 18].

This paper is intended to obtained Conner and Floyd's mod 2 cohomology index and the Volovikov's numerical index of free involutions on a finitistic space X with mod 2 cohomology ring isomorphic to the product of a real projective space and 2-sphere $\mathbb{S}^2$. As an application, we obtain some Borsuk-Ulam type result for X. The size of the coincidence set of a continuous map $f : X \to \mathbb{R}^k$ is also determined.

By $X \sim_2 Y$, we shall mean that the cohomology rings $H^*(X; \mathbb{Z}_2)$ and $H^*(Y; \mathbb{Z}_2)$ are isomorphic.

2. Preliminaries

In this section, we recall some basic definitions and results. The finitistic spaces have been most suitable spaces for the study of cohomological aspects of topological transformation group [1]. A paracompact Hausdorff space X is said to be *finitistic* if every open covering of X has a finite dimensional open refinement, where the dimension of a covering is one less than the maximum number of members of the covering which intersect non-trivially.

The *Borel construction* [2] on X is defined as the orbit space $X_G = (X \times E_G)/G$, where G acts diagonally (and freely) on the product $X \times E_G$. The projection $X \times E_G \to E_G$ gives a fibration $X_G \to B_G$ with fiber X. We will use the Leray-Serre spectral sequence associated to the Borel fibration $X \stackrel{i}{\hookrightarrow} X_G \stackrel{\pi}{\to} B_G$. If $\pi_1(B_G)$ acts trivially on $H^*(X)$ then the system of local coefficient is simple and E_2-term of the spectral sequence of the fibration $X \stackrel{i}{\hookrightarrow} X_G \stackrel{\pi}{\to} B_G$ is given by

$$E_2^{k,l} = H^k(B_G) \otimes H^l(X).$$

Note that if X is connected, then the coefficient sheaf $\mathcal{H}^0(X)$ is constant with stalk $H^0(X) = \mathbb{Z}_2$. The homomorphisms $\pi^* : H^k(B_G) \to H^k(X_G)$ and $i^* : H^l(X_G) \to H^l(X)$ are the edge homomorphisms

$$H^k(B_G) = E_2^{k,0} \to E_3^{k,0} \to \cdots \to E_k^{k,0} \to E_{k+1}^{k,0} = E_\infty^{k,0} \subset H^k(X_G), \text{ and}$$

$$H^l(X_G) \to E_\infty^{0,l} = E_{l+1}^{0,l} \subset E_l^{0,l} \subset \cdots \subset E_2^{0,l} \subset H^l(X) \text{ respectively}.$$

For the results in spectral sequences, we refer [7, 12]. Next, we recall some well known results.

Proposition 2.1 — [19]. Let $G = \mathbb{Z}_2$ act freely on a paracompact Hausdorff space X. Then the Borel construction X_G is homotopy equivalent to the orbit space X/G.

Proposition 2.2 — [3]. Let $G = \mathbb{Z}_2$ act freely on a finitistic space X with $H^i(X) = 0, \forall\, i > n$. Then, $H^i(X/G) = 0$, for all $i > n$.

Proposition 2.3 — [3]. Let g be the generator of $G = \mathbb{Z}_2$ and X be a finitistic G-space. Suppose that $H^i(X) = 0$ for all $i > 2n$ and $H^{2n}(X) = \mathbb{Z}_2$. If $c \in H^n(X)$ is an element such that $cg^*(c) \neq 0$, then the fixed point set is non-empty. Moreover, the element $cg^*(c)$ is a permanent cocycle in the spectral sequence of $X_G \to B_G$.

We next recall some indices associated with a paracompact Hausdorff free G-space X, where $G = \mathbb{Z}_2$.

Definition 2.4 — [6]. The index of X is the the largest integer n such that there exist a G-equivariant map $\mathbb{S}^n \to X$ relative to antipodal action on $\mathbb{S}^n$, and is denoted by $index(X)$.

Definition 2.5 — [6]. The mod 2 cohomology index of X, $co\text{-}ind_2(X)$, is the largest integer n such that $w^n \neq 0$, where $w \in H^1(X/G)$ is the characteristic class of the principal G-bundle $X \to X/G$.

Conner and Floyd [6] has shown that

Proposition 2.6 — [6]. $index(X) \leq co\text{-}ind_2(X)$.

First Yang [22] has introduced a notion of the index for a compact G-space X. For $G = \mathbb{Z}_2$, the numerical index $in(X)$ of a G-space X introduced by Volovikov in 1992 may be defined as:

Definition 2.7 — [20]. The number $in(X)$ is the greatest integer d such that $H^j(B_G) \to H^j(X_G)$ is a monomorphism for all $j \leq d$.

If X is a compact Hausdorff free G-space X, then $in(X)$ and $Yang\text{-}index_2(X)$ [22] are equal. Now, we have relationship between Volovikov's index of a given space and coincidence set of a continuous map.

Proposition 2.8 — [21]. Let $G = \mathbb{Z}_2$ act freely on a space X. Let $f : X \to \mathbb{R}^k$ be any continuous map. Then

$$in(A(f)) \geq in(X) - k.$$

Volovikov [21] also gives another notion of a numerical index to a G-space X. We will denoted it by $i(X)$.

Definition 2.9 — [21]. The index $i(X)$ is the smallest r such that for some k, $d_r : E_r^{k-r,r-1} \longrightarrow E_r^{k,0}$ in the cohomology Leray-Serre Spectral sequence of the fibration $X \overset{i}{\hookrightarrow} X_G \overset{\pi}{\longrightarrow} B_G$ is nontrivial.

It is clear that $i(X) = r$ if $E_2^{k,0} = E_3^{k,0} = ... = E_r^{k,0}$ for all k and $E_r^{k,0} \neq E_{r+1}^{k,0}$ for some k. If $E_2^{*,0} = E_\infty^{*,0}$, then $i(X) = \infty$. Thus, $i(X)$ is either an integer greater than 1 or infinity. We have

Proposition 2.10 — [21]. $i(X) = in(X) + 1$.

Coelho, Mattos and Santos proved the following:

Proposition 2.11 — [5]. Let G be compact Lie group and X, Y be Hausdorff, pathwise connected and paracompact free G-spaces. With a PID as the coefficient for the cohomology, suppose that $i(X) \geq m + 1$ for some natural $m \geq 1$ and $H^{k+1}(Y/G) = 0$ for some $1 \leq k \leq m$.

- (i) If $k = m$ and $\beta_m(B_G) < \beta_{m+1}(B_G)$, there is no G-equivariant map $f : X \to Y$, where β_i denotes the i^{th} Betti number.
- (ii) If $1 \leq k < m$ and $0 < \beta_{k+1}(B_G)$, then there is no G-equivariant map $f : X \to Y$.

We note that

1. $H^*(\mathbb{R}P^n \times \mathbb{S}^2) \cong \mathbb{Z}_2[a, b]/\langle a^{n+1}, b^2\rangle$, deg $a = 1$ and deg $b = 2$.

3. The Cohomology Algebra of the Orbit Space of a Free $\mathbb{Z}_2$-action on $\mathbb{R}P^n \times \mathbb{S}^2$

Consider a free involution on the product of a real projective space and 2-sphere. We determine the possible cohomology algebra of the orbit space. We first prove the following lemma.

Lemma 3.1 — Let $G = \mathbb{Z}_2$ act freely on a finitistic space $X \sim_2 \mathbb{R}P^n \times \mathbb{S}^2$, where $n \neq 3$. Then, $\pi_1(B_G)$ acts trivially on $H^*(X)$.

PROOF : Let g be the generator of $G = \pi_1(B_G)$. For $n = 1$, obviously, g^* acts trivially on $H^*(X)$. So assume that $n \geq 2$. By the naturality of the cup product, we get $g^*(a^k b) = g^*(a^k)g^*(b)$ and $g^*(a^k) = g^*(a)^k$, where $a \in H^1(X)$ and $b \in H^2(X)$ are generators of the cohomology algebra $H^*(X)$. Clearly, $g^*(a) = a$. If G acts nontrivially on $H^*(X)$, then we get $g^*(b) = a^2$ or $g^*(b) = a^2 + b$. If $g^*(b) = a^2$, then $g^*(a^n b) = a^{n+2} = 0$. This gives $a^n b = 0$, a contradiction. So $g^*(b) = a^2 + b$. For $n = 2$, we have $bg^*(b) \neq 0$, which contradicts Proposition 2.3. If $n \geq 4$ then $0 = g^*(b^2) = g^*(b)^2 = a^4$, a contradiction. This proves the lemma. □

Now, we obtain the cohomology algebra of the orbit space of free involutions on $X \sim_2 \mathbb{R}P^n \times \mathbb{S}^2$.

***Theorem* 3.2** — *Let $G = \mathbb{Z}_2$ act freely on a finitistic space $X \sim_2 \mathbb{R}P^n \times \mathbb{S}^2$, where $n \neq 3$. Then, $H^*(X/G)$ is isomorphic to one of the following graded commutative algebras:*

(1) $\mathbb{Z}_2[x, y, z]/\langle x^2, y^{\frac{n+1}{2}}, z^2\rangle$, *where* deg $x = 1$, deg $y = 2 =$deg z, *and n is odd.*

(2) $\mathbb{Z}_2[x, y]/\langle x^{n+3}, y^{n+1}+\alpha xy^n+\beta x^{n+1}, x^2y+\gamma x^3\rangle$, *where* deg $x = 1 =$ deg y *and* $\alpha, \beta, \gamma \in \mathbb{Z}_2$.

(3) $\mathbb{Z}_2[x, y]/\langle x^3, y^{n+1} + \alpha xy^n + \beta x^2y^{n-1}\rangle$, *where* deg $x = 1 =$deg y *and* $\alpha, \beta \in \mathbb{Z}_2$.

PROOF : As $\pi_1(B_G) = \mathbb{Z}_2$ acts trivially on $H^*(X)$, the fibration $X \hookrightarrow X_G \longrightarrow B_G$ has a simple system of local coefficients on B_G. Consequently, $E_2^{p,q} = H^p(B_G) \otimes H^q(X)$. Using Künneth formula, we obtain that for $n \geq 2$,

$$E_2^{p,q} = \begin{cases} \mathbb{Z}_2 & p \geq 0 \text{ and } q = 0, 1, n+1, n+2 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & p \geq 0 \text{ and } 2 \leq q \leq n \\ 0 & \text{otherwise.} \end{cases}$$

For $n = 1$, we have

$$E_2^{p,q} = \begin{cases} \mathbb{Z}_2 & p \geq 0 \text{ and } q = 0, 1, 2, 3 \\ 0 & \text{otherwise.} \end{cases}$$

Let $a \in H^1(X)$ and $b \in H^2(X)$ be generators of the cohomology algebra of $H^*(X)$. Since G acts freely on X, by Theorem 1.6 [3, p. 374], the spectral sequence does not collapse at the E_2-term. Note that there are following four possibilities for the homomorphisms:

(i) $d_2(1 \otimes a) \neq 0$ and $d_2(1 \otimes b) = 0$.

(ii) $d_2(1 \otimes a) \neq 0$ and $d_2(1 \otimes b) \neq 0$.

(iii) $d_2(1 \otimes a) = 0$ and $d_2(1 \otimes b) \neq 0$.

(iv) $d_2 = 0$ and $d_3(1 \otimes b) \neq 0$.

We consider each case separately.

Case (i) : We have $d_2(1 \otimes a) = t^2 \otimes 1$ and $d_2(1 \otimes b) = 0$. This implies that

$$d_2(1 \otimes a^k) = \begin{cases} t^2 \otimes a^{k-1} & \text{for } k \text{ odd} \\ 0 & \text{otherwise.} \end{cases}$$

If n is even then $0 = d_2\{(1\otimes a)(1\otimes a^n)\} = t^2\otimes a^n$, a contradiction. Thus, n must be odd.

It is easy to observe that $d_2 : E_2^{0,q+2} \longrightarrow E_2^{2,q+1}$ is an isomorphism for q odd, and the trivial homomorphism for q even. It gives

$$E_3^{p,q} = \begin{cases} \mathbb{Z}_2 & \text{if } p = 0,1;\ q = 0, n+1 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{if } p = 0,1;\ 2 \leq q \text{ even } \leq n-1 \ (n > 1) \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $E_3 = E_\infty$. The additive structure of $H^*(X_G)$ is given by

$$H^j(X_G) = \begin{cases} \mathbb{Z}_2 & j = 0, 1, n+1, n+2 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & 2 \leq j \leq n \ (n > 1) \\ 0 & \text{otherwise.} \end{cases}$$

The elements $1\otimes a^2$ and $1\otimes b$ are permanent cocycles. Hence, they determine elements $u \in E_\infty^{0,2}$ and $v \in E_\infty^{0,2}$, respectively. Let $x = \pi^*(t)$ be determined by $t\otimes 1 \in E_\infty^{1,0}$. Clearly, $x^2 = 0$, $u^{\frac{n+1}{2}} = 0$ and $v^2 = 0$. Thus, the total complex Tot$E_\infty^{*,*}$ is a graded commutative algebra given by $\mathbb{Z}_2[x, u, v]/\langle x^2, u^{\frac{n+1}{2}}, v^2\rangle$, where deg $x = 1$, deg $u = 2 =$ deg v.

Let $y \in H^2(X_G)$ and $z \in H^2(X_G)$ be such that $i^*(y) = a^2$ and $i^*(z) = b$, respectively. Clearly, $y^{\frac{n+1}{2}} = 0 = z^2$. Hence, $H^*(X_G)$ is the graded commutative algebra $\mathbb{Z}_2[x, y, z]/\langle x^2, y^{\frac{n+1}{2}}, z^2\rangle$, where deg $x = 1$, deg $y = 2$ =deg z, and n is odd. If $n = 1$ then $H^*(X_G) = \mathbb{Z}_2[x, z]/\langle x^2, z^2\rangle$, where deg $x = 1$, deg $z = 2$. The result follows from Proposition 2.1.

Case (ii) : Let $d_2(1\otimes a) \neq 0$ and $d_2(1\otimes b) \neq 0$. As $d_2(1\otimes b) \subset$ ker$(d_2 : E_2^{2,1} \to E_2^{4,0})$. This forces $d_2(1\otimes b) = 0$, a contradiction. Therefore, this case is not possible.

Case (iii) : Let $d_2(1\otimes a) = 0$ and $d_2(1\otimes b) \neq 0$. Thus $d_2(1\otimes b) = t^2\otimes a$. So, we have $d_2(1\otimes a^k b) = t^2\otimes a^{k+1}$. It gives that

$$E_3^{p,q} = \begin{cases} \mathbb{Z}_2 & (p \geq 0;\ q = 0, n+2) \text{ or } (p = 0,1;\ 1 \leq q \leq n) \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $d_r = 0$ for all $3 \leq r \leq n+2$. If $d_{n+3}(1\otimes a^n b) = 0$ then two lines of spectral sequence survive to infinity which contradicts Proposition 2.2. So, let $d_{n+3}(1\otimes a^n b) = t^{n+3}\otimes 1$. Consequently,

$E_{n+4} = E_\infty$. The cohomology groups $H^j(X_G)$ are given by

$$H^j(X_G) = \begin{cases} \mathbb{Z}_2 & j = 0, n+2 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & j = 1, n+1 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 & 2 \leq q \leq n \ (n > 1) \\ 0 & \text{otherwise.} \end{cases}$$

Since the element $1 \otimes a$ is a permanent cocycle, it determines $u \in E_\infty^{0,1}$. Let $x = \pi^*(t)$ be determined by $t \otimes 1 \in E_\infty^{1,0}$. Clearly, $x^{n+3} = u^{n+1} = ux^2 = 0$. It follows that the total complex $\text{Tot}E_\infty^{*,*}$ is a graded commutative algebra given by $\mathbb{Z}_2\left[x, u\right] / \langle x^{n+3}, u^{n+1}, ux^2\rangle$, where deg $x = 1 =$ deg u.

Choose $y \in H^1(X_G)$ such that $i^*(y) = a$. We have $y^{n+1} + \alpha x y^n + \beta x^{n+1} = x^2 y + \gamma x^3 = 0$, where $\alpha, \beta, \gamma \in \mathbb{Z}_2$. Therefore, $H^*(X_G)$ is the graded commutative algebra

$$\mathbb{Z}_2[x, y]/\langle x^{n+3}, y^{n+1} + \alpha x y^n + \beta x^{n+1}, x^2 y + \gamma x^3\rangle,$$

where deg $x = 1 =$ deg y and $\alpha, \beta, \gamma \in \mathbb{Z}_2$. This realises the possibility (2) of theorem.

Case (iv) : Let $d_2 = 0$ and $d_3(1 \otimes b) \neq 0$. Then, $d_3(1 \otimes b) = t^3 \otimes 1$. Thus, we have $d_3(1 \otimes a^k b) = t^3 \otimes a^k$. It gives that

$$E_4^{p,q} = \begin{cases} \mathbb{Z}_2 & 0 \leq p \leq 2;\ 0 \leq q \leq n \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $d_r = 0$ for $r \geq 4$. We observe that the cohomology groups $H^j(X_G)$ are the same as in case (iii). Since the element $1 \otimes a$ is a permanent cocycle, it determines $u \in E_\infty^{0,1}$. Let $x = \pi^*(t)$ be determined by $t \otimes 1 \in E_\infty^{1,0}$. Clearly, $x^3 = u^{n+1} = 0$. It follows that the total complex $\text{Tot}E_\infty^{*,*}$ is a graded commutative algebra given by $\mathbb{Z}_2\left[x, u\right] / \langle x^3, u^{n+1}\rangle$, where deg $x = 1 =$ deg u.

Choose $y \in H^1(X_G)$ such that $i^*(y) = a$. We have $y^{n+1} + \alpha x y^n + \beta x^2 y^{n-1} = 0$, where $\alpha, \beta \in \mathbb{Z}_2$. Therefore, $H^*(X_G)$ is the graded commutative algebra

$$\mathbb{Z}_2[x, y]/\langle x^3, y^{n+1} + \alpha x y^n + \beta x^2 y^{n-1}\rangle,$$

where deg $x = 1 =$ deg y and $\alpha, \beta, \in \mathbb{Z}_2$. This realises the possibility (3) of theorem. □

Example 3.3 : For n odd, the orbit space of any free involution on $\mathbb{R}P^n$ is mod 2 $\mathbb{S}^1 \times \mathbb{C}P^{\frac{n-1}{2}}$ (see [14]). Thus, the orbit space of the diagonal action on $\mathbb{R}P^n \times \mathbb{S}^2$ is mod 2 cohomology $(\mathbb{S}^1 \times \mathbb{C}P^{\frac{n-1}{2}}) \times$

$\mathbb{S}^2$, where $\mathbb{Z}_2$ acts trivially on 2-sphere $\mathbb{S}^2$. This realises possibility (1) of Theorem. Also, the orbit space of the diagonal action of $\mathbb{Z}_2$ on $\mathbb{R}P^n \times \mathbb{S}^2$ with trivial action on $\mathbb{R}P^n$ and antipodal action on $\mathbb{S}^2$ is $\mathbb{R}P^n \times \mathbb{R}P^2$. This realises possibility (3) of the above theorem when $\alpha = \beta = 0$.

By taking $n = 1$ in the above Theorem, we get

Corollary 3.4 — Let $G = \mathbb{Z}_2$ act freely on a finitistic space $X \sim_2 \mathbb{S}^1 \times \mathbb{S}^2$. Then $H^*(X/G)$ is isomorphic to one of the following graded algebras:

(1) $\mathbb{Z}_2[x, z]/\langle x^2, z^2\rangle$, where deg $x = 1$, deg $z = 2$.

(2) $\mathbb{Z}_2[x, y]/\langle x^4, y^2 + \alpha xy + \beta x^2, x^2y + \gamma x^3\rangle$, where deg $x = 1 =$ deg y and $\alpha, \beta \in \mathbb{Z}_2$.

(3) $\mathbb{Z}_2[x, y]/\langle x^3, y^2 + \alpha xy + \beta x^2\rangle$, where deg $x = 1 =$ deg y and $\alpha, \beta, \gamma \in \mathbb{Z}_2$.

Above corollary can also be obtained by taking $m = 1$ and $n = 2$ in Therorem 2 of [8].

Note that $\mathbb{R}P^3$ is homeomorphic to $SO(3)$. Now, for the case $n = 3$, we have

***Theorem* 3.5** — *Let $G = \mathbb{Z}_2$ act freely on a finitistic space $X \sim_2 SO(3) \times \mathbb{S}^2$.*

1. *If $\pi_1(B_G)$ acts nontrivially on $H^*(X)$ then $H^*(X/G)$ is given by*

   $\mathbb{Z}_2[x, u, v, w]/\langle x^2, u^2, v^2, w^2, xu, xv, uv, uw, vw\rangle$, *where* deg $x = 1$, deg $u = 2$, deg $v = 3$ *and* deg $w = 4$.

2. *If $\pi_1(B_G)$ acts trivially on $H^*(X)$ then $H^*(X/G)$ is given by*

   (i) $\mathbb{Z}_2[x, y, z]/\langle x^2, y^2, z^2\rangle$, *where* deg $x = 1$, deg $y = 2 =$ deg z.

   (ii) $\mathbb{Z}_2[x, y]/\langle x^6, y^4 + \alpha xy^3 + \beta x^4, x^2y + \gamma x^3\rangle$, *where* deg $x = 1 =$ deg y *and* $\alpha, \beta, \gamma \in \mathbb{Z}_2$.

   (iii) $\mathbb{Z}_2[x, y]/\langle x^3, y^4 + \alpha xy^3 + \beta x^2y^2\rangle$, *where* deg $x = 1 =$ deg y *and* $\alpha, \beta \in \mathbb{Z}_2$.

PROOF : *Case* (1). Let g be the generator of $G = \pi_1(B_G)$. Then, $g^*(a) = a$ and as g^* acts nontrivially on $H^*(X)$, we must have $g^*(b) = b + a^2$. Clearly, for $q \neq 2$ and 3, $g^* : H^q(X) \rightarrow H^q(X)$ is the indentity isomorphism. Let $\tau = 1 - g^*$ and $\sigma = 1 + g^*$. Note that $\sigma = \tau$. Recall that [3] the E_2-term of the Leray-Serre spectral sequence of the fibration $X \hookrightarrow X_G \rightarrow B_G$ is given by

$$E_2^{p,q} = \begin{cases} \ker\tau & \text{for } p = 0 \\ \ker\tau/\text{Im}\sigma & \text{for } p > 0 \text{ even} \\ \ker\sigma/\text{Im}\tau & \text{for } p > 0 \text{ odd.} \end{cases}$$

It is easy to check that kerτ = Imσ for all $p > 0$ and $q = 2, 3$. In this case, $E_2^{p,q} = 0$. Also, the ker $\tau : H^q(X) \to H^q(X)$ is $\langle a^q \rangle$, we have $E_2^{0,q} = \mathbb{Z}_2$ ($q = 2, 3$). Since the coefficient sheaf $\mathcal{H}^q(X)$, $q \neq 2, 3$, is constant, we write $E_2^{p,q} = H^p(B_G) \otimes H^q(X)$. Let the generators $\alpha \in E_2^{0,2}$ and $\beta \in E_2^{0,3}$ be represented by a^2 and a^3, respectively. By the second part of Proposition 2.3, the element $1 \otimes a^2b$ is a permanent cocycle. If $d_2(1 \otimes a) = 0$, then the element $t^2 \otimes a^2b$ is also a permanent cocycle. This implies that $H^6(X_G) \neq 0$ which contradicts our hypothesis that G acts freely on X. Therefore, $d_2(1 \otimes a) = t^2 \otimes 1$. Then, $d_2(1 \otimes a^3b) = t^2 \otimes a^2b$ and $d_2(\beta) = 0$. Clearly, $d_2(\alpha) = 0$. It gives that

$$E_3^{p,q} = \begin{cases} \mathbb{Z}_2 & \text{for } (p = 0;\ q = 0, 2, 3, 4) \text{ and } (p = 1;\ q = 0, 4) \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $E_\infty = E_3$. Therefore, the additive structure of $H^j(X_G)$ is given by

$$H^j(X_G) \cong \begin{cases} \mathbb{Z}_2 & 0 \leq j \leq 5 \\ 0 & \text{otherwise.} \end{cases}$$

Since the elements α, β and $1 \otimes a^2b$ are permanent cocycles. So they determine nonzero elements $u \in E_\infty^{0,2}$, $v \in E_\infty^{0,3}$ and $w' \in E_\infty^{0,4}$. Let $x = \pi^*(t)$ be determined by $t \otimes 1 \in E_\infty^{1,0}$. Clearly, $x^2 = u^2 = v^2 = w'^2 = xu = xv = uv = uw' = vw' = 0$. Therefore, $TotE_\infty^{*,*}$ is the graded algebra

$$\mathbb{Z}_2[x, u, v, w']/\langle x^2, u^2, v^2, w'^2, xu, xv, uv, uw', vw' \rangle,$$

where deg $x = 1$, deg $u = 2$, deg $v = 3$ and deg $w' = 4$. Choose a nonzero element $w \in H^4(X_G)$ such that $i^*(w) = a^2b$. Clearly, we have $w^2 = uw = vw = 0$. Thus, the cohomology algebra $H^*(X/G)$ is given by

$$\mathbb{Z}_2[x, u, v, w]/\langle x^2, u^2, v^2, w^2, xu, xv, uv, uw, vw \rangle,$$

where deg $x = 1$, deg $u = 2$, deg $v = 3$ and deg $w = 4$.

Case (2) : In this case $E_2^{p,q} = H^p(B_G) \otimes H^p(X)$ for all p and q. The proof is similar to the proof of Theorem 3.2. □

Remark 3.6 : The orbit space X/G of case (1) in the above theorem is mod 2 cohomology $(\mathbb{S}^1 \times \mathbb{S}^4) \vee \mathbb{S}^2 \vee \mathbb{S}^3$.

4. Indices of $X \sim_2 \mathbb{R}P^n \times \mathbb{S}^2$

In this section, we determine the Conner and Floyd's mod 2 cohomology index and Volovikov's numercal index of X.

***Theorem* 4.1** — *If $G = \mathbb{Z}_2$ acts freely on a finitistic space $X \sim_2 \mathbb{R}P^n \times \mathbb{S}^2$ then co-$index_2(X)$ has one of the following values:*

(a) 1 (b) $n+2$ or (c) 2.

PROOF : Let η be homotopy inverse of homotopy equivalence between X_G and X/G. Then, the composition maps $X/G \xrightarrow{\eta} X_G \xrightarrow{\pi} B_G$ is a classifying map of the principle G-bundle $X \to X/G$, where $X \hookrightarrow X_G \xrightarrow{\pi} B_G$ is the borel fibration. The image of characteristic class of universal G-bundle $E_G \to B_G$ under the homomorphism

$$\pi^* : H^1(B_G) \to H^1(X_G) = H^1(X/G),$$

is the characteristic class of principal G-bundle $X \to X/G$. Thus, Theorems 3.2 and 3.5, co-$ind_2(X)$ is 1 or 2 or $n+2$. □

By Definition 2.9 and Theorems 3.2 and 3.5, we also obtain the following:

***Theorem* 4.2** — *If $G = \mathbb{Z}_2$ acts freely on a finitistic space $X \sim_2 \mathbb{R}P^n \times \mathbb{S}^2$, then $i(X)$ has one of the following values:*

(a) 2 (b) $n+3$ or (c) 3.

5. APPLICATIONS

As an application of the results established in the previous section, we prove Borsuk-Ulam type theorems for a finitistic space X with mod 2 cohomology ring isomorphic to $\mathbb{R}P^n \times \mathbb{S}^2$.

By Proposition 2.6 and Theorem 4.1, we obtain the following.

***Theorem* 5.1** — *Let $X \sim_2 \mathbb{R}P^n \times \mathbb{S}^2$ be a finitistic free G-space and the unit n-sphere $\mathbb{S}^n$ be equipped with antipodal action. Then there is no G-equivariant map $\mathbb{S}^k \to X$*

(a) for $k \geq 2$ if co-$ind_2(X) = 1$.

(b) for $k \geq n+3$ if co-$ind_2(X) = n+2$.

(c) for $k \geq 3$ if co-$ind_2(X) = 2$.

Remark 5.2 : If $G = \mathbb{Z}_2$ acts freely on a finitistic space $X \sim_2 SO(3) \times \mathbb{S}^2$ and $\pi_1(B_G)$ acts nontrivially on $H^*(X)$ then by Theorem 3.5, the co-$ind_2(X)$ is 1, and hence there is no G-equivariant map $\mathbb{S}^k \to X$ for $k \geq 2$.

***Theorem* 5.3** — *Let $G = \mathbb{Z}_2$ act freely on a finitistic space $X \sim_2 \mathbb{R}P^n \times \mathbb{S}^2$ and Y be a path connected, paracompact Hausdorff space. Then there is no G-equivariant map $X \to Y$*

(a) *if* $i(X) = n+3$ *and* $H^k(Y/G) = 0$ *for some* $2 \leq k \leq n+2$.

(b) *if* $i(X) = 3$ *and* $H^k(Y/G) = 0$ *for* $k = 2$.

PROOF : By Theorem 4.2, $i(X)$ has one of values 2, $n+3$ or 3. Note that i-th Betti number $\beta_i(B_G;\mathbb{Z}_2) > 0$ for $i \geq 0$. The result follows from Proposition 2.11. □

Taking Y in the above theorem to be the unit k-sphere $\mathbb{S}^k$, we obtain the following result.

Corollary 5.4 — Let $X \sim_2 \mathbb{R}P^n \times \mathbb{S}^2$ be a finitistic free G-space and the unit n-sphere $\mathbb{S}^k$ be equipped with free involutions. Then there is no G-equivariant map $X \to \mathbb{S}^k$

(a) if $i(X) = n+3$ and $k \leq n+1$.

(b) if $i(X) = 3$ and $k = 1$.

Now, we obtained the size of coincidence set.

***Theorem* 5.5** — *Let* $G = \mathbb{Z}_2$ *act freely on a finitistic space* $X \sim_2 \mathbb{R}P^n \times \mathbb{S}^2$. *Let* $f : X \to \mathbb{R}^k$ *be any continuous map. Then*

1. *cov.dim.*$A(f) \geq n+2-k$ *if* $i(X) = n+3$ *and* $n+2 \geq k$.
2. *cov.dim.*$A(f) \geq 2-k$ *if* $i(X) = 3$ *and* $2 \geq k$.

PROOF : By Proposition 2.10 and Theorem 4.2, we have $in(X) = n+2$ or 2 according as $i(X) = n+3$ or 3. If $i(X) = n+3$ and $n+2 \geq k$, then by Proposition 2.8, we have $inA(f) \geq in(X) - k$, this gives $inA(f) \geq n+2-k$. By Definition 2.7 of Volovikov's index, we have *cohom.dim*$A(f)_G \geq n+2-k$. Since G acts freely on a finitistic space $A(f)$, so $A(f)_G$ and $A(f)/G$ are homotopy equivalent. By the Gysin sequence of principal G-bundle $A(f) \to A(f)/G$, we have *cohom.dim* $A(f) =$ *cohom.dim* $A(f)/G \geq n+2-k$. The case (1) follows from the fact that *cov.dim* $A(f) \geq$ *cohom.dim* $A(f)$.

Similarly, we prove the case (2). □

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