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SELF-SIMILAR SOLUTIONS TO THE SPHERICALLY-SYMMETRIC EULER EQUATIONS WITH A TWO-CONSTANT EQUATION OF STATE

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This paper is concerned with self-similar flows of the multidimensional isentropic compressible Euler equations caused by the uniform expansion of a spherically-symmetric piston into the undisturbed fluid. Under the spherically-symmetric and self-similar assumptions, the problem can be reduced to a boundary value problem for a system of nonlinear ordinary differential equations. We consider the two-constant equation of state $p = A_1\rho^{\gamma_1} + A_2\rho^{\gamma_2}$ which arises in a number of various physical contexts and results the problem becomes more complicated than the case of polytropic gas equation of state. To deal with the difficulty, we first establish the global existence of smooth solutions to the boundary value problem for a new ODE system.

Key words : Euler equations; spherically-symmetric flows; self-similar solutions; shock wave.

1. Introduction

The three-dimensional isentropic compressible Euler equations are of the form

$$\begin{cases} \rho_t + \nabla\cdot(\rho U) = 0, \\ (\rho U)_t + \nabla\cdot(\rho U\otimes U) + \nabla p = 0, \end{cases} \qquad (t,\mathbf{x})\in\mathbb{R}^+\times\mathbb{R}^3, \tag{1.1}$$

where $\rho(t,\mathbf{x})$ is the density, $U(t,\mathbf{x}) = (u_1,u_2,u_3)(t,\mathbf{x})$ is the velocity, $p = p(\rho)$ is a given increasing function of ρ. If the flow is spherically symmetric, that is the solution has the following geometric structure

$$\Big(\rho(t,\mathbf{x}), U(t,\mathbf{x})\big) = (\rho(t,r), u(t,r)\frac{\mathbf{x}}{r}\Big)\,,$$

where $r = |\mathbf{x}|$ and $u(t,r)$ is a scalar function, then system (1.1) can be rewritten as

$$\begin{cases} \rho_t + (\rho u)_r = -\frac{2\rho u}{r}, \\ (\rho u)_t + (\rho u^2 + p)_r = -\frac{2\rho u^2}{r}, \end{cases} \qquad (t,r) \in \mathbb{R}^+ \times \mathbb{R}^+. \tag{1.2}$$

It is well known that the spherically-symmetric flow is one of the most important flows in gas dynamics and attracts much attention in recent years, since this kind of flows arises in many important physical situations such as supernovae formation in stellar dynamics, inertial confinement fusion and explosion waves in water, air, and other media, see the famous monographs Courant and Friedrichs [9] and Whitham [22]. In this paper, we are interested in the spherically-symmetric flows of the isentropic compressible Euler equations arising from the uniform expansion of a spherical piston and preceded by a shock front. This problem is often taken as the simplest way to model an explosion and has been studied by many authors. Under the self-similar assumption, the problem can be reduced to a system of nonlinear ordinary differential equations (ODE) with some boundary conditions imposed on the piston surface and the shock front. The study of such nonlinear ODE problems was initiated by Taylor [21] in numerical for the isentropic Euler equations. In [9], Courant and Friedrichs established the vector field of the ODE system and discussed the properties of the solution curves. A single second-order nonlinear ordinary differential equation in the velocity potential was first proposed by Lighthill [15]. He considered a simplified equation and obtained an approximate relation between the shock Mach number M and the nondimensional piston velocity α. This result was recently improved in [12]. We also refer the reader to [13, 14, 23, 24] for the construction of global axisymmetric solutions to the two-dimensional isentropic Euler equations.

The investigation of the global analysis of the multidimensional piston problem to the Euler equations for compressible flows was started by Chen [7]. He investigated a free boundary value problem of the second-order ordinary differential equation in the velocity potential and established the global existence of solutions to the piston problem for unsteady potential flows. For more relevant results, one may consult [5, 8, 10] and the references therein. In [19], Peng and Lien considered the multidimensional piston problem of the isentropic Euler equations with the polytropic gas equation of state by studying a nonlinear ODE system. Based on a careful analysis of the ODE system, they obtained the global existence of smooth solutions to the ODE problem together with some appropriate boundary conditions. The nonlinear stability of the spherically-symmetric flows was presented in [11].

We notice that studies on the multidimensional piston problem of the compressible Euler equations are almost limited to the polytropic gas, that is the equation of state takes the form $p(\rho) = A\rho^\gamma$

with constants $A > 0, \gamma \geq 1$, while the analyses for the general equation of state are absent. As a ladder step to study the case of the general equation of state, in the present paper we investigate a somewhat general case than the polytropic gas equation of state, i.e., we consider the two-constant equation of state

$$p(\rho) = A_1\rho^{\gamma_1} + A_2\rho^{\gamma_2}, \tag{1.3}$$

where $A_1, A_2, \gamma_1, \gamma_2$ are constants. When $\gamma_1 = \gamma_2 \geq 1$ and $A_1 + A_2 > 0$, (1.3) reduces to the polytropic gas equation of state. Most importantly, the equation of state (1.3) indeed arises in a number of various physical contexts and has more implications. For example, it can be taken as the sum of the fluid and magnetic pressures in the ideal magnetogasdynamics with a transverse magnetic field [20] and can correspond to the modified Chaplygin gas equation of state being used to describe the current accelerated expansion of the universe [1-4]. It is rather worthwhile to mention that the equation of state (1.3) can also be applied to regularize the polytropic gas equation of state to avoid the possibility of formation of cavitation of the solutions for the perturbation Euler equations, see, e.g., [6, 16-18]. In the present paper, we deal with the case $A_i > 0$ and $\gamma_i \geq 1$ $(i = 1, 2)$, the other cases will be considered in a later publication.

The aim of this paper is to establish the global existence of solutions for the uniformly expanding piston problem to the isentropic compressible Euler equations (1.1) with the equation of state (1.3). The approach of the paper is inspired by Peng and Lien [19]. However, their procedure depends strongly on the concrete form of the polytropic gas equation of state which results in it cannot be applied to the equation of state (1.3) and also the general equation of state. To deal with (1.2) with (1.3), we first consider a new system

$$\begin{cases} \rho_t + (\rho u)_r = -\frac{2\rho u}{r}, \\ (\rho u)_t + (\rho u^2 + p_1 + p_2)_r = -\frac{2\rho u^2}{r}, \qquad (t,r) \in \mathbb{R}^+ \times \mathbb{R}^+, \\ \varphi_t + (\varphi u)_r = -\frac{2\varphi u}{r}, \end{cases} \tag{1.4}$$

where $p_1 = A_1\rho^{\gamma_1}, p_2 = A_2\varphi^{\gamma_2}$ and $\varphi \geq 0$. Without loss of generality, we assume $\gamma_1 < \gamma_2$ throughout the paper. Clearly, if $\varphi \equiv \rho$, then system (1.4) reduces to (1.2) with (1.3). The problem can be reduced to a free boundary value problem for a system of nonlinear ordinary differential equations by the self-similar assumptions. We study carefully the ODE system to obtain the global existence of smooth solutions of the free boundary value problem. The original problem is then resolved by restricting $\varphi = \rho$ at time zero. We point out that this method is successfully applied to construct a class of global bounded weak axisymmetric solutions to the two-dimensional isentropic Euler equations with the equation of state (1.3) [13].

The rest of the paper is organized as follows. Section 2 is devoted to delivering some preliminaries and presenting the main results of the paper. The proof of our theorem is divided into two cases $\gamma_1 = 1$ and $\gamma_1 > 1$ by symmetry. The detailed proof of these two cases are provided in Section 3 and Section 4, respectively.

2. Formulation and Results

Suppose that the initial state is given by $(\rho, u, \varphi) = (\rho_0, 0, \varphi_0)$ with $\rho_0 > 0, \varphi_0 > 0$ and the piston is located at the origin. Starting from $t = 0$, the piston isotropically expands outward at a constant speed b_0, as a result, a shock wave forms and moves outward with constant velocity $\sigma_0 (> b_0)$. Set $c_0 = \sqrt{p_1'(\rho_0)}$ and denote $M = \sigma_0/c_0$. Here we emphasize that the constant c_0 (not the speed of sound for the undisturbed gas) thus chosen plays an important role in our paper. Then $M > M_0 := \sqrt{1 + p_2'(\varphi_0)/p_1'(\rho_0)} > 1$ by the fact that the flow in front of the shock wave is supersonic.

We introduce a self-similar variable

$$\xi = \frac{r}{\sigma_0 t} = \frac{r}{c_0 M t}$$

to fix the location of the shock at $\xi = 1$. Looking for self-similar solutions $(\rho, u, \varphi) = (\rho, u, \varphi)(\xi)$ of (1.4) gets the following system of ordinary differential equations

$$\begin{cases} \rho_\xi = \frac{2\rho u (c_0 M\xi - u)}{\xi[(c_0 M\xi - u)^2 - (p_1'(\rho) + \bar{\varphi})]}, \\ u_\xi = \frac{2u(p_1'(\rho) + \bar{\varphi})}{\xi[(c_0 M\xi - u)^2 - (p_1'(\rho) + \bar{\varphi})]}, \\ \bar{\varphi}_\xi = \frac{2(\gamma_2 - 1)\bar{\varphi} u (c_0 M\xi - u)}{\xi[(c_0 M\xi - u)^2 - (p_1'(\rho) + \bar{\varphi})]}, \end{cases} \tag{2.1}$$

where $\bar{\varphi} = \gamma_2 A_2 \varphi^{\gamma_2}/\rho$. Moreover, we introduce

$$u = c_0 M f(\xi), \quad \rho = \rho_0 h(\xi), \quad \bar{\varphi} = c_0^2 g(\xi). \tag{2.2}$$

Putting the above into (2.1) yields

$$\begin{cases} h'(\xi) = \frac{2M^2 f h (\xi - f)}{\xi[M^2(\xi - f)^2 - (h^{\gamma_1 - 1} + g)]}, \\ f'(\xi) = \frac{2f(h^{\gamma_1 - 1} + g)}{\xi[M^2(\xi - f)^2 - (h^{\gamma_1 - 1} + g)]}, \\ g'(\xi) = \frac{2(\gamma_2 - 1)M^2 g f (\xi - f)}{\xi[M^2(\xi - f)^2 - (h^{\gamma_1 - 1} + g)]}. \end{cases} \tag{2.3}$$

The Rankine-Hugoniot jump conditions of (1.4) read that

$$\begin{cases} \sigma[\rho] = [\rho u], \\ \sigma[\rho u] = [\rho u^2 + p_1 + p_2], \\ \sigma[\varphi] = [\varphi u], \end{cases}$$

where σ is the speed of the discontinuity, $[q] := q_r - q_l$, q_r, q_l are the states on the two sides of a discontinuity line, from which and the Lax entropy condition and the assumption $u_0 = 0$ one has

$$\frac{\hat\rho}{\rho_0} = \frac{\hat\varphi}{\varphi_0}, \quad \hat u = \sigma_0\Big(1 - \frac{\rho_0}{\hat\rho}\Big), \quad \sigma_0^2\rho_0(\hat\rho - \rho_0) = \hat\rho\Big[p_1(\hat\rho) + p_2(\hat\varphi) - p_1(\rho_0) - p_2(\varphi_0)\Big], \tag{2.4}$$

where $(\hat\rho, \hat u, \hat\varphi)$ is the state behind the shock front satisfying $\hat\rho > \rho_0$. Combining (2.2) and (2.4), we find that $\hat h := \hat\rho/\rho_0 > 1$ solves the following algebraic equation

$$h^{\gamma_1+1} + A_0 h^{\gamma_2+1} - \Big(\gamma_1 M^2 + 1 + A_0\Big)h + \gamma_1 M^2 = 0, \tag{2.5}$$

where $A_0 = A_2\varphi_0^{\gamma_2}/(A_1\rho_0^{\gamma_1})$. Moreover, we claim that the algebraic equation (2.5) has only one root greater than one. In fact, we denote

$$H(h) := h^{\gamma_1+1} + A_0 h^{\gamma_2+1} - \Big(\gamma_1 M^2 + 1 + A_0\Big)h + \gamma_1 M^2,$$

and differentiate the above with respect to h to arrive

$$H'(h) = (\gamma_1+1)h^{\gamma_1} + A_0(\gamma_2+1)h^{\gamma_2} - \Big(\gamma_1 M^2 + 1 + A_0\Big),$$

$$H''(h) = \gamma_1(\gamma_1+1)h^{\gamma_1-1} + A_0\gamma_2(\gamma_2+1)h^{\gamma_2-1},$$

from which we see that $H(h)$ is a strictly convex function for $h > 0$. In addition, we find that $H(+\infty) = +\infty$, $H(1) = 0$ and

$$\begin{aligned} H'(1) &= (\gamma_1+1) + A_0(\gamma_2+1) - \Big(\gamma_1 M^2 + 1 + A_0\Big) \\ &= \gamma_1(1 - M^2) + \gamma_2 A_0 < \gamma_1(1 - M_0^2) + \gamma_2\frac{A_2\varphi_0^{\gamma_2}}{A_1\rho_0^{\gamma_1}} = 0, \end{aligned}$$

which imply that there exists a unique number $\hat h > 1$ such that $H(\hat h) = 0$.

Thus, the boundary conditions of system (2.3) on the shock front $\xi = 1$ are

$$\begin{cases} h(1) = \hat h, \\ f(1) = 1 - \frac{1}{\hat h}, \\ g(1) = \frac{\gamma_2 A_0}{\gamma_1}\hat h^{\gamma_2-1}. \end{cases} \tag{2.6}$$

Furthermore, we note that the path of the piston is $r = b_0 t$, which means that $\xi_b = b_0/\sigma_0 < 1$ is the piston location. On the other hand, the kinematic condition at the piston requires that the flow velocity on the piston surface is the same as the piston velocity, i.e., $u(\xi_b) = b_0$ or

$$f(\xi_b) = \xi_b. \tag{2.7}$$

Then our problem is to seek solutions of (2.3) with the boundary conditions (2.6)-(2.7) in $[\xi_b, 1]$ for any given $M > M_0$.

We state our main result as follows.

***Theorem* 1** — *For any given $M > \max\{\sqrt{1 + p_2'(\varphi_0)/p_1'(\rho_0)}, b_0/c_0\}$, the ODE system (2.3) with the boundary conditions (2.6)-(2.7) has a unique positive smooth solution $(h(\xi), f(\xi), g(\xi))$ on $[\xi_b, 1]$. Moreover, $h(\xi), f(\xi), g(\xi)$ are decreasing functions for $\xi \in [\xi_b, 1]$.*

Based on Theorem 1, we can obtain the result for system (1.2). First we find by (2.1) that

$$\frac{\rho_\xi}{\rho} = \frac{\varphi_\xi}{\varphi},$$

from which one gets

$$\frac{\varphi(\xi)}{\rho(\xi)} = \frac{\hat{\varphi}}{\hat{\rho}}.$$

If we restrict $\varphi_0 = \rho_0$, then we obtain by (2.4) that $\hat{\varphi} = \hat{\rho}$, which means $\varphi(\xi) \equiv \rho(\xi)$ for $\xi \in [\xi_b, 1]$. Thus the vector function $(\rho, u) = (\rho_0 \widetilde{h}(\xi), c_0 M \widetilde{f}(\xi))$, defined on $[\xi_b, 1]$, solves the following problem

$$\begin{cases} \rho_\xi = \frac{2\rho u(c_0 M\xi - u)}{\xi[(c_0 M\xi - u)^2 - (p_1'(\rho) + p_2'(\rho))]}, \\ u_\xi = \frac{2u(p_1'(\rho) + p_2'(\rho))}{\xi[(c_0 M\xi - u)^2 - (p_1'(\rho) + p_2'(\rho))]}, \end{cases} \tag{2.8}$$

with the boundary conditions

$$\rho(1) = \hat{\rho}, \quad u(1) = c_0 M \left(1 - \frac{\rho_0}{\hat{\rho}}\right), \quad u(\xi_b) = b_0. \tag{2.9}$$

It is notice that system (2.8) is the corresponding system of (1.2) in terms of variable ξ. Therefore, we have the following theorem.

***Theorem* 2** — *Let ρ_0 be the density of the undisturbed gas and b_0 be the piston velocity. For any given $\sigma_0 > \max\{\sqrt{p_1'(\rho_0) + p_2'(\rho_0)}, b_0\}$, system (1.2) with (1.3) has a unique positive self-similar smooth solution $(\rho, u)(r/t)$ defined in $\mathbb{R}^+ \times [b_0 t, \sigma_0 t]$ such that $r = b_0 t$ is the piston surface and $r = \sigma_0 t$ is a shock wave front.*

3. Proof Theorem 1 for $\gamma_1 = 1$

In this section, we show Theorem 1 for the case $\gamma_2 > \gamma_1 = 1$. In this case, the ODE problem (2.3)

(2.6) can be written as

$$\begin{cases} h'(\xi) = \frac{2M^2 fh(\xi-f)}{\xi[M^2(\xi-f)^2-(1+g)]}, \\ f'(\xi) = \frac{2f(1+g)}{\xi[M^2(\xi-f)^2-(1+g)]}, \\ g'(\xi) = \frac{2(\gamma_2-1)M^2 gf(\xi-f)}{\xi[M^2(\xi-f)^2-(1+g)]}, \end{cases} \tag{3.1}$$

with the boundary values

$$\begin{cases} h(1) = \hat{h}, \\ f(1) = 1 - \frac{1}{\hat{h}}, \\ g(1) = \gamma_2 A_0 \hat{h}^{\gamma_2-1}, \end{cases} \tag{3.2}$$

where $\hat{h}$ is the only real root greater than 1 to the equation

$$h^2 + A_0 h^{\gamma_2+1} - \Big(M^2 + 1 + A_0\Big)h + M^2 = 0,$$

from which one has

$$M^2 = \frac{\hat{h}^2 + A_0\hat{h}^{\gamma_2+1} - (1+A_0)\hat{h}}{\hat{h}-1}. \tag{3.3}$$

We define an auxiliary function

$$I_1(\xi) = M^2[\xi - f(\xi)]^2 - [1 + g(\xi)].$$

Inserting (3.2) and (3.3) into the above leads to

$$I_1(1) = \frac{-(\hat{h}-1)^2 - A_0[\gamma_2\hat{h}^{\gamma_2+1} - (\gamma_2+1)\hat{h}^{\gamma_2} + 1]}{\hat{h}(\hat{h}-1)} < 0.$$

Here we used the fact that $\gamma_2 x^{\gamma_2+1} - (\gamma_2+1)x^{\gamma_2} + 1 > 0$ for $x > 1$. Therefore, we obtain by (3.1) and (3.2) that

$$\begin{aligned} &h(1) > 1, \quad 0 < f(1) < 1, \quad g(1) > 0, \\ &h'(1) < 0, \quad f'(1) < 0, \quad g'(1) < 0. \end{aligned} \tag{3.4}$$

Moreover, we differentiate the function $I_1(\xi)$ with respect to ξ to get

$$I_1'(\xi) = 2M^2(\xi - f)(1 - f') - g', \tag{3.5}$$

which along with (3.4) gives

$$I_1'(1) > 0. \tag{3.6}$$

We combine (3.4)-(3.6) and use the classical local existence theory of smooth solutions for the nonlinear ODE system to find that there exists a small positive number ε_0 such that the problem (3.1) (3.2) has a unique smooth solution $(h,f,g)(\xi)$ in the interval $[1-\varepsilon_0,1]$ satisfying

$$\begin{array}{l} h(\xi)>0, \quad \xi>f(\xi)>0, \quad g(\xi)>0, \\ h'(\xi)<0, \quad f'(\xi)<0, \quad g'(\xi)<0, \quad \forall\, \xi\in[1-\varepsilon_0,1]. \\ \qquad I_1(\xi)<0, \quad I_1'(\xi)>0 \end{array} \tag{3.7}$$

Let $\bar{\xi}$ be a constant in $(0,1-\varepsilon_0)$ such that the interval $(\bar{\xi},1]$ is the maximal interval of existence for the positive continuous solution to the problem (3.1)-(3.2). We next prove that there exists a number $\xi_b\in(\bar{\xi},1-\varepsilon_0)$ such that $f(\xi_b)=\xi_b$, which means the solution $(h,f,g)(\xi)$ satisfies the boundary condition (2.7) at $\xi=\xi_b$.

We first establish the following lemma.

Lemma 3.1 — Let $(h,f,g)(\xi)$ be the positive continuous solution for the problem (3.1)-(3.2). Assume that $\xi>f(\xi)$ in the interval $[\hat{\xi},1]$ for some number $\hat{\xi}\in(\bar{\xi},1-\varepsilon_0]$. Then it holds $f(\xi)>f(1)$ and $I_1(\xi)<0$ for all $\xi\in[\hat{\xi},1)$.

PROOF : We use the contradiction argument to prove this lemma. Assume that there exists a number $\xi^*\in[\hat{\xi},1)$ such that $f(\xi^*)=f(1)$ and $f(\xi)>f(1)$ for all $\xi\in(\xi^*,1)$. The proof is divided into two cases.

Case I : f is a differentiable function on $(\xi^*,1)$. In this case, we find by the Mean Value Theorem that there exists a point $\xi_m\in(\xi^*,1)$ such that $f'(\xi_m)=0$. On the other hand, we see by the second equation of (3.1) and the fact $f(\xi)>f(1)$ in $(\xi^*,1)$ that $f'(\xi)\neq 0$ for all $\xi\in(\xi^*,1)$, which leads to a contradiction.

Case II : f is not a differentiable function on $(\xi^*,1)$. This case indicates that there exists a point $\xi_n\in(\xi^*,1)$ such that $I_1(\xi_n)=0$ and $I_1(\xi)<0$ for all $\xi\in(\xi_n,1]$. Thanks to $f(\xi)>0, g(\xi)>0$ in $(\xi^*,1]$, it follows by (3.1) that

$$\lim_{\xi\to\xi_n^+}\frac{1}{I_1(\xi)}=-\infty, \quad \lim_{\xi\to\xi_n^+}f'(\xi)=-\infty, \quad \lim_{\xi\to\xi_n^+}g'(\xi)=-\infty,$$

which combined with (3.5) yields

$$\lim_{\xi\to\xi_n^+}I_1'(\xi)=+\infty,$$

which implies that there exists a small positive constant ε_1 such that $I_1'(\xi)>0$ for all $\xi\in(\xi_n,\xi_n+\varepsilon_1)$. However, due to the previous analysis, we have $I_1(\xi_n)=0$ and $I_1(\xi)<0$ in $(\xi_n,\xi_n+\varepsilon_1)$, which indicates by the Mean Value Theorem that there exists at least a point $\xi_* \in (\xi_n, \xi_n + \varepsilon_1)$ such that $I_1'(\xi_*) < 0$, a contradiction.

Thus, we have shown $f(\xi) > f(1)$ for all $\xi \in [\hat{\xi}, 1)$. Making use of the same argument, the details of which we omit here, we can obtain the conclusion that $I_1(\xi) < 0$ for all $\xi \in [\hat{\xi}, 1)$. □

PROOF OF THEOREM 1 : For $\gamma_1 = 1$, we prove that the theorem by contradiction. Suppose that $f(\xi) \neq \xi$ for all $\xi \in (\bar{\xi}, 1]$. Then it follows by the fact $f(1) < 1$ that $f(\xi) < \xi$ for all $\xi \in (\bar{\xi}, 1]$. According to Lemma 3.1, it suggests that $f(\xi) > f(1)$ and $I_1(\xi) < 0$ for all $\xi \in (\bar{\xi}, 1)$. Hence we have $f'(\xi) < 0, g'(\xi) < 0$ and $h'(\xi) < 0$ for all $\xi \in (\bar{\xi}, 1)$ and then the functions f, g, h are strictly monotone decreasing. One thus has

$$0 < f(1) \leq \lim_{\xi \to \bar{\xi}^+} f(\xi) \leq \bar{\xi} < 1,$$

which together with the ODE system (3.1) obtains that $\lim\limits_{\xi \to \bar{\xi}^+} h(\xi)$ and $\lim\limits_{\xi \to \bar{\xi}^+} g(\xi)$ are bounded. There-

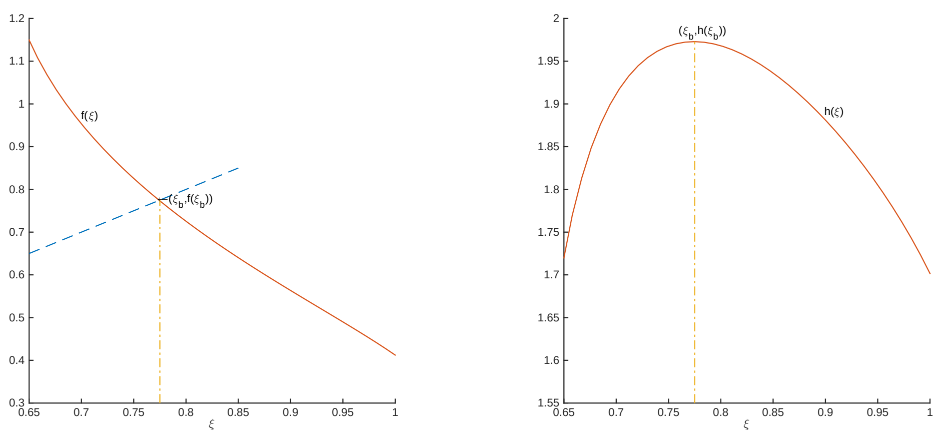

Figure 1: The numerical solution of problem (2.3) (2.6). Here $\gamma_1 = 1, \gamma_2 = 2, A_1 = \frac{1}{\gamma_1}, A_2 = \frac{1}{\gamma_2}, \rho_0 = \varphi_0 = 1, M = 2$.

fore, the positive continuous solution $(h, f, g)(\xi)$ for the problem (3.1) (3.2) can be extended to the larger interval $(\bar{\xi} - \varepsilon_2, 1]$ for some constant $\varepsilon_2 > 0$, which contradicts the fact that $(\bar{\xi}, 1]$ is the maximal interval of existence of the positive continuous solution. For the uniqueness of ξ_b, we assume that there exists another point $\xi_d \in (\bar{\xi}, \xi_b)$ such that $f(\xi_d) - \xi_d = 0$ and $f(\xi) - \xi > 0$ for all $\xi \in (\xi_d, \xi_b)$. Then it follows that $f'(\xi_d) > 1$. However, by the equation of f, one has $f'(\xi_d) = -2$ which yields a contradiction.

In summary, we have $f(\xi) < \xi$ for all $\xi \in (\xi_b, 1]$ and $f(\xi_b) = \xi_b$ and then have by Lemma 3.1 $I_1(\xi) < 0$, from which we have the strictly monotonic decreasing property of $f(\xi), h(\xi)$ and $g(\xi)$. See Fig. 1 for the numerical solution of problem (2.3) (2.6) with $\gamma_1 = 1$. So far, we have finished the proof of Theorem 1 for $\gamma_1 = 1$.

4. Proof Theorem 1 for $\gamma_1 > 1$

In this section, we prove Theorem 1 for the case $\gamma_2 > \gamma_1 > 1$. We introduce

$$H(\xi) = h(\xi)^{\gamma_1 - 1}, \quad \gamma_1 > 1,$$

and reduce system (2.3) to

$$\begin{cases} H'(\xi) = \frac{2(\gamma_1-1)M^2 f H(\xi-f)}{\xi[M^2(\xi-f)^2-(H+g)]}, \\ f'(\xi) = \frac{2f(H+g)}{\xi[M^2(\xi-f)^2-(H+g)]}, \\ g'(\xi) = \frac{2(\gamma_2-1)M^2 g f(\xi-f)}{\xi[M^2(\xi-f)^2-(H+g)]}, \end{cases} \tag{4.1}$$

with the boundary conditions

$$\begin{cases} H(1) = \hat{h}^{\gamma_1-1}, \\ f(1) = 1 - \frac{1}{\hat{h}}, \\ g(1) = \frac{\gamma_2 A_0}{\gamma_1}\hat{h}^{\gamma_2-1}, \end{cases} \tag{4.2}$$

where $\hat{h}$ is the unique solution greater than one of equation (2.5). Then we have

$$M^2 = \frac{\hat{h}^{\gamma_1+1} + A_0\hat{h}^{\gamma_2+1} - (1+A_0)\hat{h}}{\gamma_1(\hat{h}-1)}. \tag{4.3}$$

Define an auxiliary function

$$I(\xi) = M^2[\xi - f(\xi)]^2 - [H(\xi) + g(\xi)].$$

Making use of (4.2) and (4.3) gives by the fact $\hat{h} > 1$

$$I(1) = -\frac{[\gamma_1\hat{h}^{\gamma_1+1} - (\gamma_1+1)\hat{h}^{\gamma_1} + 1] + A_0[\gamma_2\hat{h}^{\gamma_2+1} - (\gamma_2+1)\hat{h}^{\gamma_2} + 1]}{\gamma_1\hat{h}(\hat{h}-1)} < 0.$$

Therefore, it holds

$$\begin{aligned} &H(1) > 1, \quad 0 < f(1) < 1, \quad g(1) > 0, \\ &H'(1) < 0, \qquad f'(1) < 0, \qquad g'(1) < 0. \end{aligned} \tag{4.4}$$

Moreover, differentiating the function $I(\xi)$ with respect to ξ leads to

$$I'(\xi) = 2M^2[\xi - f(\xi)](1 - f'(\xi)) - [H'(\xi) + g'(\xi)]. \tag{4.5}$$

The local existence of the positive continuous solution to problem (4.1)-(4.2) follows from the classical existence theory of the nonlinear ODE system. We still use the interval $(\bar{\xi}, 1]$ to denote the maximal interval of existence in $(0, 1]$ for the positive continuous solution of problem (4.1)-(4.2). Then we have the following lemma.

Lemma 4.1 — Let $(H, f, g)(\xi)$ be the positive continuous solution for the problem (4.1)-(4.2). Assume that $\xi > f(\xi)$ in the interval $(\hat{\xi}, 1]$ for some point $\hat{\xi} \in [\bar{\xi}, 1)$. Then $I(\xi) < 0$ for all $\xi \in (\hat{\xi}, 1]$.

PROOF : We show the lemma by contradiction. Suppose that there exists a point $\xi_* \in (\hat{\xi}, 1)$ such that $I(\xi_*) = 0$ and $I(\xi) < 0$ for all $\xi \in (\xi_*, 1]$. Then one arrives

$$\lim_{\xi \to \xi_*^+} \frac{1}{I(\xi)} = -\infty,$$

from which and the system (4.1) and the positivity of solution $(H, f, g)(\xi)$ in $(\bar{\xi}, 1]$, we obtain

$$\lim_{\xi \to \xi_*^+} f'(\xi) = -\infty, \quad \lim_{\xi \to \xi_*^+} H'(\xi) = -\infty, \quad \lim_{\xi \to \xi_*^+} g'(\xi) = -\infty,$$

which together with (4.5) gives

$$\lim_{\xi \to \xi_*^+} I'(\xi) = +\infty.$$

By a similar argument as in Lemma 3.1, we can get a contradiction and then finish the proof of the lemma. □

Furthermore, we have

Lemma 4.2— Let $(H, f, g)(\xi)$ be the positive continuous solution for the problem (4.1)-(4.2). If $\xi > f(\xi)$ in the interval $(\hat{\xi}, 1]$ for some number $\hat{\xi} \in [\bar{\xi}, 1)$, then $f(\xi) > f(1), H(\xi) > H(1)$ and $g(\xi) > g(1)$ for all $\xi \in (\hat{\xi}, 1]$.

PROOF : The proof is divided into three cases:

Case I : There exists a point $\xi_1 \in (\hat{\xi}, 1)$ such that $f(\xi_1) = f(1), H(\xi_1) \geq H(1), g(\xi_1) \geq g(1)$ and $f(\xi) > f(1), H(\xi) > H(1)$, $g(\xi) > g(1)$ for all $\xi \in (\xi_1, 1]$;

Case II : There exists a point $\xi_2 \in (\hat{\xi}, 1)$ such that $f(\xi_2) \geq f(1), H(\xi_2) = H(1), g(\xi_2) \geq g(1)$ and $f(\xi) > f(1), H(\xi) > H(1)$, $g(\xi) > g(1)$ for all $\xi \in (\xi_2, 1]$;

Case III : There exists a point $\xi_3 \in (\hat{\xi}, 1)$ such that $f(\xi_3) \geq f(1), H(\xi_3) \geq H(1), g(\xi_3) = g(1)$ and $f(\xi) > f(1), H(\xi) > H(1)$, $g(\xi) > g(1)$ for all $\xi \in (\xi_3, 1]$.

We only show Case I and the others are symmetric arguments. In this case, there exists a point $\xi_{1*} \in (\xi_1, 1)$ such that $f'(\xi_{1*}) > 0$ by the facts $f(1) > 0$ and $f'(1) < 0$. On the other hand, we see by Lemma 4.1 that $f(\xi) > 0, H(\xi) > 0$, $g(\xi) > 0$ and $I(\xi) < 0$ for all $\xi \in (\xi_1, 1]$. Then we get by the equation of f in (4.1) that $f'(\xi) < 0$ for all $\xi \in (\xi_1, 1]$, which leads to a contradiction. □

Finally, we establish the following lemma.

Lemma 4.3 — Let $(H, f, g)(\xi)$ be the positive continuous solution for the problem (4.1)-(4.2). Then there exists a unique $\xi_b \in (\bar{\xi}, 1)$ such that $f(\xi_b) = \xi_b$.

PROOF : Assume the contrary: if $f(\xi) \neq \xi$ for any $\xi \in (\bar{\xi}, 1]$, then it follows by $f(1) < 1$ that $f(\xi) < \xi$ for all $\xi \in (\bar{\xi}, 1]$. According to Lemma 4.2 and Lemma 4.3, we know that $I(\xi) < 0$, $f(\xi) > f(1)$, $H(\xi) > H(1)$ and $g(\xi) > g(1)$ for all $\xi \in (\bar{\xi}, 1]$ which mean by system (4.1) that $f'(\xi) < 0, H'(\xi) < 0$ and $g'(\xi) < 0$. Thus the functions f, H and g are strictly decreasing on the interval $(\bar{\xi}, 1]$. Then we use Lemma 4.2 again to obtain $f(1) < f(\xi) < \xi \leq 1$ for any $\xi \in (\bar{\xi}, 1]$, which means that

$$f(1) \leq \bar{\xi} < 1.$$

By the definition of $\bar{\xi}$ and the uniform boundedness of f on $(\bar{\xi}, 1]$, there must have

$$\lim_{\xi \to \bar{\xi}^+} H(\xi) = +\infty, \quad \text{or} \quad \lim_{\xi \to \bar{\xi}^+} g(\xi) = +\infty.$$

If $H(\xi) \to +\infty$ as $\xi \to \bar{\xi}^+$, then there exists a point $\xi_* \in (\bar{\xi}, 1)$ and a constant $K > 1$ such that

$$H + g - M^2(\xi - f)^2 \geq K$$

holds for all $\xi \in (\bar{\xi}, \xi_*]$. Then we obtain

$$H'(\xi) = \frac{2(\gamma_1 - 1)M^2 H f(\xi - f)}{\xi[M^2(\xi - f)^2 - (H + g)]} \geq -\frac{2(\gamma_1 - 1)M^2}{K\bar{\xi}} H =: -kH$$

for all $\xi \in (\bar{\xi}, \xi_*]$. Integrating the above from ξ to ξ_* suggests

$$H(\xi) \leq H(\xi_*) e^{k(\xi_* - \xi)},$$

for which, we get by taking $\xi \to \bar{\xi}^+$ to arrive

$$\lim_{\xi \to \bar{\xi}^+} H(\xi) \leq H(\xi_*) e^{k(\xi_* - \bar{\xi})} < +\infty,$$

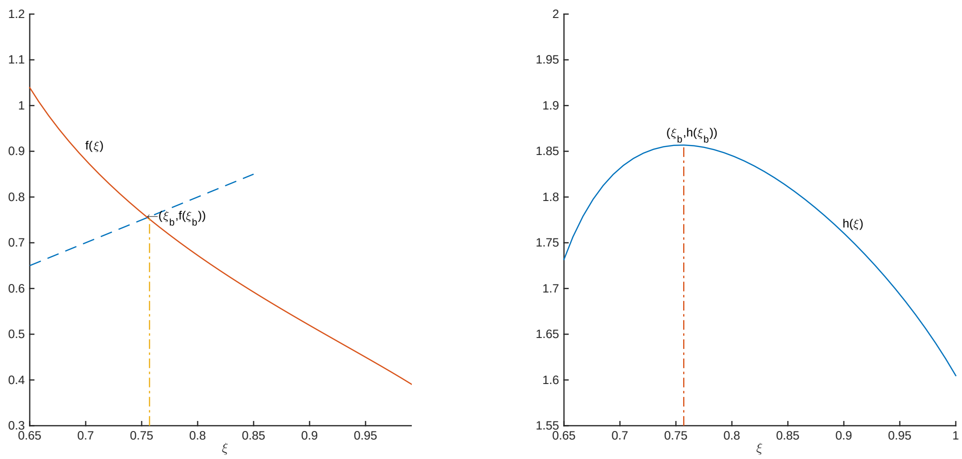

Figure 2: The numerical solution of problem (2.3) (2.6). Here $\gamma_1 = \frac{5}{3}, \gamma_2 = 2, A_1 = \frac{1}{\gamma_1}, A_2 = \frac{1}{\gamma_2}, \rho_0 = \varphi_0 = 1, M = 2$.

which achieves a contradiction. If $g(\xi) \to +\infty$ as $\xi \to \bar{\xi}^+$, we can obtain a contradiction by a similar procedure.

The proof of the uniqueness of ξ_b is the same as the case $\gamma_1 = 1$ in Section 3, so we omit it here.□

PROOF OF THEOREM 1 : For $\gamma_1 > 1$, it is easily seen that the positive continuous vector function $(H(\xi), f(\xi), g(\xi))$, defined on the interval $[\xi_b, 1]$, satisfies the ODE system (4.1) and the boundary conditions (4.2) at $\xi = 1$. Moreover, the function $f(\xi)$ satisfies the boundary condition (2.7) at $\xi = \xi_b$ by Lemma 4.3. In addition, we have $\xi > f(\xi)$ for all $\xi \in (\xi_b, 1]$ which along with Lemma 4.1 and system (4.1) yields $f'(\xi) < 0, H'(\xi) < 0$ and $g'(\xi) < 0$ on $(\xi_b, 1]$. Therefore, $f(\xi), H(\xi)$ and $g(\xi)$ are strictly decreasing functions of ξ. By the definition of $H(\xi)$ and $\gamma_1 > 1$, the function $h(\xi)$ is also strictly decreasing of ξ. Obviously, the vector function $(h(\xi), f(\xi), g(\xi))$ is the desired solution of problem (2.3) with the boundary conditions (2.6) and (2.7). See Fig. 2 for the numerical solution of problem (2.3) (2.6) with $\gamma_1 > 1$.

In summary, we have completed the proof of Theorem 1 and subsequently established Theorem 2.

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