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INDEX THEORY FOR SECOND ORDER LINEAR HAMILTONIAN SYSTEMS WITH L^1 COEFFICIENT MATRIX SATISFYING GENERALIZED PERIODIC BOUNDARY VALUE CONDITIONS

Tingting Hu

College of Information Technology, Nanjing University of Chinese Medicine,
Nanjing 210023, *P. R. China*
e-mail: tingtinghu0902@163.com

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In this paper, we will first establish an index theory for second order linear Hamiltonian systems with L^1 coefficient matrix satisfying generalized periodic boundary value conditions. And then we will investigate nontrivial solutions for asymptotically linear second-order Hamiltonian systems and obtain some new results.

Key words : Critical point; second order Hamiltonian system; index theory; solutions.

1. INTRODUCTION

In this paper we will first establish an index theory of the following second-order linear Hamiltonian system

$$\ddot{x}(t) + B(t)x(t) = 0, \tag{1.1}$$

$$x(1) = Mx(0), \dot{x}(1) = N\dot{x}(0), \tag{1.2}$$

where $B \in L^1([0,1], \mathcal{L}_s(\mathbf{R}^n))$, $M, N \in GL(n), M^T N = I_n$. And then under asymptotically linear conditions we investigate nontrivial solutions for the Hamiltonian system

$$\begin{aligned} &\ddot{x}(t) + V'(t,x) = 0, \\ &x(1) = Mx(0), \dot{x}(1) = N\dot{x}(0), \end{aligned} \tag{1.3}$$

where $V \in C^1([0,1] \times \mathbf{R}^n)$, and $V'(t,x)$ denotes the gradient of $V(t,x)$ with respect to x. When $M = N = I_n$, (1.2) reduces to

$$x(1) - x(0) = 0 = \dot{x}(1) - \dot{x}(0) \tag{1.4}$$

When $B \in C([0,1], \mathcal{L}_s(\mathbf{R}^n))$, some index theories have been established by Mawhin [20] and Long [18, 19] in order to investigate the problem (1.3) and (1.4). When $B \in L^\infty([0,1], \mathcal{L}_s(\mathbf{R}^n))$, an index theory has been established by Dong [6]. An index theory for operator equations has been established in [7], however this index theory does not apply to (1.1)-(1.2) with $B \in L^1([0,1], \mathcal{L}_s(\mathbf{R}^n))$. The paper will be organized as follows: we establish the index theory in section 2 and we investigate nontrivial solutions for second-order asymptotically linear Hamiltonian systems in section 3. For related materials we refer to [1, 4, 5, 9, 10, 12-17, 21-26, 28-35].

2. Index Theory for Second-order Linear Hamiltonian Systems

For any $B \in L^1([0,1], \mathcal{L}_s(\mathbf{R}^n))$, consider the following system:

$$\ddot{x}(t) + B(t)x(t) = 0, \tag{2.1}$$

$$x(1) = Mx(0), \dot{x}(1) = N\dot{x}(0). \tag{2.2}$$

Define

$$q_B(x,y) = \int_0^1 [\dot{x}(t)\cdot\dot{y}(t) - B(t)x(t)\cdot y(t)]dt, \forall x, y \in Z, \tag{2.3}$$

where $a \cdot b$ is the usual inner product in $\mathbf{R}^n$, and $Z \equiv \{x : [0,1] \to \mathbf{R}^n | x(t)$ is absolutely continuous on $[0,1], x(1) = Mx(0)$ and $\dot{x} \in L^2([0,1]; \mathbf{R}^n)\}$ is a Hilbert space with norm $||x|| \equiv (\int_0^1 |\dot{x}(t)|^2 + |x(t)|^2 dt)^{\frac{1}{2}}$. For any $x, y \in Z$ if $q_B(x,y) = 0$ we say that x and y are q_B-orthogonal. And for any two subspaces Z_1 and Z_2 of Z if $q_B(x,y) = 0$ for all $x \in Z_1, y \in Z_2$ we say that Z_1 and Z_2 are q_B-orthogonal. In what follows for any two symmetric matrices B_1 and B_2, we write $B_1 \leq B_2$ if $B_2 - B_1$ is positive semi-definite, and write $B_1 < B_2$ if $B_2 - B_1$ is positive definite. For any $B_1, B_2 \in L^1([0,1], \mathcal{L}_s(\mathbf{R}^n))$, we write $B_1 \leq B_2$ if $B_1(t) \leq B_2(t)$ for a.e. $t \in [0,1]$; write $B_1 < B_2$ if $B_1 \leq B_2$ and $B_1(t) < B_2(t)$ on a subset of $[0,1]$ with positive measure.

Proposition 2.1 — For any $B \in L^1([0,1], \mathcal{L}_s(\mathbf{R}^n))$, the space Z has a q_B-orthogonal decomposition

$$Z = Z^+(B) \bigoplus Z^0(B) \bigoplus Z^-(B)$$

such that q_B is positive definite, null and negative definite on $Z^+(B)$, $Z^0(B)$ and $Z^-(B)$ respectively. Moreover, $Z^0(B)$ and $Z^-(B)$ are finitely dimensional.

Proof : We first show

$$(x,y)_{\lambda_0} = \int_0^1 (\dot{x}(t)\cdot\dot{y}(t) - B(t)x(t)\cdot y(t))dt + \lambda_0 \int_0^1 x(t)\cdot y(t)dt, \forall x, y \in Z \tag{2.4}$$

is a new inner product in Z for $\lambda_0 > 0$ large enough. In fact, because $B(t)$ is symmetric, we obtain $(x,y)_{\lambda_0} = (y,x)_{\lambda_0}$. Since $B \in L^1([0,1], \mathcal{L}_s(\mathbf{R}^n))$, every element $b_{jk}(t) \in L^1[0,1]$. And for any constant $\epsilon > 0$ and there exists a positive constant c_ϵ such that for any $x \in H^1([0,1], \mathbf{R}^n)$, we have

$$|x(t)|^2 \leq c_\epsilon \|x\|_{L^2}^2 + \epsilon \|\dot{x}\|_{L^2}^2, t \in [0,1].$$

And hence

$$\begin{aligned}
&|\int_0^1 B(t)x(t)\cdot x(t)dt| = |\int_0^1 \sum_{j=1}^n \sum_{k=1}^n b_{jk}(t)x_j(t)x_k(t)dt| \\
&\leq \sum_{j=1}^n \sum_{k=1}^n \int_0^1 |b_{jk}(t)x_j(t)x_k(t)|dt \\
&\leq \sum_{j=1}^n \sum_{k=1}^n \int_0^1 |b_{jk}(t)|dt \cdot ||x||_0^2 \\
&\leq c_1\epsilon ||\dot{x}||_{L^2}^2 + c_1 c_\epsilon ||x||_{L^2}^2
\end{aligned}$$

for all $x \in Z$ and some $c_1 > 0$, where $\|x\|_0 = \max_{t\in[0,1]} |x(t)|$. Then

$$\begin{aligned}
0 \leq (1 - c_1\epsilon)||\dot{x}||_{L^2}^2 + (\lambda_0 - c_1 c_\epsilon)||x||_{L^2}^2 &\leq (x,x)_{\lambda_0} \\
&\leq (1 + c_1\epsilon)||\dot{x}||_{L^2}^2 + (\lambda_0 + c_1 c_\epsilon)||x||_{L^2}^2
\end{aligned}$$

if we choose $\epsilon > 0$ such that $1 - c_1\epsilon \geq \frac{1}{2}$ and $\lambda_0 > 0$ such that $\lambda_0 - c_1 c_\epsilon \geq \frac{1}{2}$. Hence $(\cdot,\cdot)_{\lambda_0}$ is an inner product and the associated the norm denoted by $||\cdot||_{\lambda_0}$ is equivalent to $||\cdot||$ in Z. Now we define a linear compact operator $K_{\lambda_0} : Z \to Z$ satisfying

$$\int_0^1 x(t)\cdot y(t)dt = (x, K_{\lambda_0} y)_{\lambda_0} \text{ for all} x, y \in Z. \tag{2.5}$$

If $\mu \in \sigma_p(K_{\lambda_0})$ and $e \in Z \setminus \{0\}$ such that $K_{\lambda_0} e = \mu e$, then (2.5) implies that $\mu\|e\|_{\lambda_0}^2 = \|e\|_{L^2}^2$ and hence $\mu > 0$. So $\sigma_p(K_{\lambda_0}) \subset (0, +\infty)$. By the spectral theory of self-adjoint compact operators, there exists a nonzero sequence $\mu_j \to 0$ and a basis $\{e_j\}_{j=1}^\infty \subset Z$ such that

$$(e_j, e_k)_{\lambda_0} = \delta_{jk}, K_{\lambda_0} e_j = \mu_j e_j. \tag{2.6}$$

From (2.4)-(2.6), we have

$$\mu_j (x, e_j)_{\lambda_0} = \int_0^1 x(t)\cdot e_j dt, \forall x \in Z. \tag{2.7}$$

For any $x \in Z$ with $x = \sum\limits_{j=1}^{\infty} c_j e_j$,

$$q_B(x,x) = \int_0^1 \dot{x}(t) \cdot \dot{x}(t) - B(t)x(t) \cdot x(t)dt = (x,x)_{\lambda_0} - \lambda_0(x,x)$$
$$= (x,x)_{\lambda_0} - \lambda_0(x, K_{\lambda_0}\iota x)_{\lambda_0} = \sum_{j=1}^{\infty} c_j^2 - \lambda_0(\sum_{j=1}^{\infty} c_j e_j, \sum_{j=1}^{\infty} \mu_j e_j c_j)_{\lambda_0}$$
$$= \sum_{j=1}^{\infty}(1 - \lambda_0\mu_j)c_j^2.$$

Hence the results hold if we define:

$$Z^+(B) = \{x = \sum c_j e_j | c_j = 0 \text{ if} 1 - \lambda_0\mu_j \leq 0\},$$
$$Z^0(B) = \{x = \sum c_j e_j | c_j = 0 \text{ if} 1 - \lambda_0\mu_j \neq 0\},$$
$$Z^-(B) = \{x = \sum c_j e_j | c_j = 0 \text{ if} 1 - \lambda_0\mu_j \geq 0\}$$

since $\mu_j \to 0$ as $j \to +\infty$.

Definition 2.1 — For any $B \in L^1([0,1], \mathcal{L}_s(\mathbf{R}^n))$, we define $i(B) = \dim Z^-(B), \nu(B) = \dim Z^0(B)$.

We call $i(B)$ and $\nu(B)$ the index and nullity of B respectively. In what follows we will discuss the properties of $(i(B), \nu(B))$.

Proposition 2.2 — (i) For any $B \in L^1([0,1], \mathcal{L}_s(\mathbf{R}^n))$, $\nu(B)$ is the dimension of the solution space of (2.1)-(2.2).

(ii) For any $B_1, B_2 \in L^1([0,1], \mathcal{L}_s(\mathbf{R}^n))$, if $B_1 \leq B_2$, then $i(B_1) \leq i(B_2), i(B_1) + \nu(B_1) \leq i(B_2) + \nu(B_2)$; if $B_1 < B_2$, then $i(B_1) + \nu(B_1) \leq i(B_2)$.

(iii) (Poincare's Inequality) For any $B \in L^1([0,1], \mathcal{L}_s(\mathbf{R}^n))$ with $i(B) = 0$ one has

$$\int_0^1 |\dot{x}(t)|^2 dt \geq \int_0^1 B(t)x(t) \cdot x(t)dt, \forall x \in Z.$$

And the equality holds if only if $x \in Z^0(B)$.

(iv) For any $B_1, B_2 \in L^1([0,1], \mathcal{L}_s(\mathbf{R}^n))$, if $B_1 \leq B_2, i(B_1) = i(B_2)$, and $\nu(B_1) = 0 = \nu(B_2)$, then $Z = Z^-(B_1) \bigoplus Z^+(B_2)$ and $(-q_B(x_1,x_1))^{\frac{1}{2}} + (q_B(x_2,x_2))^{\frac{1}{2}}$ is an equivalent norm on Z for $x = x_1 + x_2$ with $x_1 \in Z^-(B_1), x_2 \in Z^+(B_2)$.

PROOF : (i) Assume $x \in Z^0(B)$. By definition,

$$0 = q_B(x,y) = \int_0^1 \dot{x}\cdot\dot{y} - B(t)x(t)\cdot y(t)dt = \int_0^1 (\dot{x} + \int_0^t B(s)x(s)ds + C)\cdot \dot{y}(t)dt$$

for $y \in Z$ satisfying $y(1) = 0 = y(0)$ and $c_1 = x(0) - x(1) - \int_0^1\int_0^{t_1} B(s)x(s)dsdt_1$. Choosing $y \in Z$ satisfying $\dot{y}(t) = \dot{x} + \int_0^t B(s)x(s)ds + C, y(1) = 0 = y(0)$ yields $\dot{x} + \int_0^t B(s)x(s)ds + C = 0$. Therefore,

$$\ddot{x}(t) + B(t)x(t) = 0.$$

Combining this equation we obtain

$$\dot{x}(1)\cdot y(1) - \dot{x}(0)\cdot y(0) = 0$$

for $y \in Z$. Noticing $y \in Z$ is arbitrary and $y(1) = My(0)$ we have

$$\dot{x}(1) = N\dot{x}(0).$$

(ii) Set $k = i(B_1)$ and assume $k \geq 1$. Let $e_1, e_2, \cdots, e_k$ be a basis of $Z^-(B_1)$, and from Proposition 2.1, $e_j = e_j^+ \bigoplus e_j^0 \bigoplus e_j^-$ with $e_j^* \in Z^*(B_2)(* = +, 0, -)$. In order to prove $i(B_2) \geq k$, we only need to show that $\{e_j^-\}_{j=1}^k$ is linear independent. In fact, if not there exist not all zero constants $c_1, c_2, \cdots, c_k$ such that $\sum\limits_{j=1}^k c_j e_j^- = 0$. So $e = \sum\limits_{j=1}^k c_j e_j = \sum\limits_{j=1}^k c_j e_j^+ + \sum\limits_{j=1}^k c_j e_j^0 \in Z^+(B_2) \bigoplus Z^0(B_2)$, and we have $q_{B_2}(e,e) \geq 0$. On the other hand, $Z^-(B_1)$ is a linear subspace and $\{e_j\}_{j=1}^k \in Z^-(B_1)$. So $e = \{e_j\}_{j=1}^k \in Z^-(B_1) \setminus \{0\}$ and hence $q_{B_1}(e,e) < 0$, a contradiction.

(iii) For any $x \in Z$ with $x = \sum\limits_{j=1}^{\infty} c_j e_j$, from (2.4) and (2.7) we have

$$\int_0^1 |\dot{x}(t)|^2 dt = \int_0^1 B(t)x(t)\cdot x(t)dt + \sum_{j=1}^{+\infty}(1 - \lambda_0\mu_j)c_j^2.$$

Because $i(B) = 0$, by definition, $1 - \lambda_0\mu_j \geq 0$ for all $j \in \mathbf{N}^*$. So the inequality holds. And the equality is valid if and only if $c_j = 0$ as $1 - \lambda_0\mu_j \neq 0$.

(iv) Let $N = Z^-(B_1) \bigoplus Z^0(B_1)$, $M = Z^+(B_2) \bigoplus Z^0(B_2)$. Since $q_{B_2}(x,x) \geq 0, \forall x \in M, q_{B_1}(x,x) \leq 0, \forall x \in N$ and $q_{B_1}(x,x) \geq q_{B_2}(x,x)$ for all $x \in Z$ via $B_1 < B_2$, if $x \in N \bigcap M$ then $q_{B_1}(x,x) = 0 = q_{B_2}(x,x)$. It follows that $x \in Z^0(B_1) \bigcap Z^0(B_2)$ and $x(t) = 0$ on a subset of $[0,1]$ with positive measure, and hence $x = 0$ via (i). Thus $M \bigcap N = \{0\}$. So we need only to prove that $Z = N + M$. By definition it follows that $Z = M \bigoplus Z^-(B_2)$, i.e. for any $x \in Z$ there exists a unique pair $(x_1, x_2) \in M \times Z^-(B_2)$ such that $x = x_1 + x_2$. Let $\{e\}_{j=1}^k$ be a basis of N,$e_j = e_j^M + e_j^-$

with $e_j^M \in M, e_j^- \in Z^-(B_2), j = 1,2,3,\cdots,k = i(B_1) + \nu(B_1)$. Then e_j^- is a basis of $Z^-(B_2)$. Since $i(B_2) = i(B_1) + \nu(B_1) = k$, we show e_j^- are linear independent. In fact, otherwise there exist not all zero constants $c_1, c_2, \cdots, c_k$ such that $\sum_{j=1}^{k} c_j e_j^- = 0$. This leads to $\sum_{j=1}^{k} c_j e_j \in M \bigcap N$, a contradiction. The linear independence implies that there exist constants $\{\alpha_j\}_{j=1}^k$ such that $x_2 = \alpha_j e_j^-$. And hence $x = x_1 + x_2 = x_1 + \sum_{j=1}^{k} \alpha_j e_j^- = x_1 + \sum_{j=1}^{k} \alpha_j e_j - \sum_{j=1}^{k} \alpha_j e_j^M = \sum_{j=1}^{k} \alpha_j e_j + (x_1 - \sum_{j=1}^{k} \alpha_j e_j^M)$. Hence the result holds.

Now we prove the second part, for any $x_1 \in Z^-(B_1)$ with $x_1 = \sum_{\lambda\mu_j - 1 > 0} c_j e_j$,

$$-q_{B_1}(x_1, x_1) = \sum_{\lambda\mu_j - 1 > 0} (\lambda_0 \mu_j - 1) c_j^2.$$

Since $\mu_j \to 0$ as $j \to \infty$, there exist $c_3 > c_2 > 0$ such that

$$c_2 \|x_1\| \leq (-q_{B_1}(x_1, x_1))^{\frac{1}{2}} \leq c_3 \|x_1\|. \tag{2.8}$$

Similarly there exist c_4, c_5 such that for any $x_2 \in Z^+(B_2)$ we have

$$c_4 \|x_2\| \leq (q_{B_2}(x_2, x_2))^{\frac{1}{2}} \leq c_5 \|x_2\|. \tag{2.9}$$

Hence, $(-q_B(x_1, x_1))^{\frac{1}{2}} + (q_B(x_2, x_2))^{\frac{1}{2}}$ is a norm on Z and with this norm Z becomes a Banach space. Since for $x = x_1 + x_2$, we have

$$\begin{aligned} &\min\{c_2, c_4\} \|x\| \leq \min\{c_2, c_4\}(\|x_1\| + \|x_2\|) \\ &\leq (-q_{B_1}(x_1, x_1))^{\frac{1}{2}} + (q_{B_2}(x_2, x_2))^{\frac{1}{2}} \end{aligned}$$

$(-q_B(x_1, x_1))^{\frac{1}{2}} + (q_B(x_2, x_2))^{\frac{1}{2}}$ is an equivalent norm on Z. The proof is complete.

Remark : For $B \in L^\infty([0,1], \mathcal{L}_s(\mathbf{R}^n))$, an index has been defined in [2, 3, 5, 8] concerning the system (1.3)-(1.4). The index defined in [20] is a special case of Definition 2.1.

3. NONTRIVIAL SOLUTIONS FOR ASYMPTOTICALLY LINEAR HAMILTONIAN SYSTEMS

Consider the problem

$$\ddot{x}(t) + V'(t, x) = 0, \tag{3.1}$$

$$x(1) = Mx(0), \dot{x}(1) = N\dot{x}(0), \tag{3.2}$$

where $V : [0,1] \times \mathbf{R}^n \to \mathbf{R}, V' : [0,1] \times \mathbf{R}^n \to \mathbf{R}^n$ are continuous, V' denotes the gradient of V with respect to x, $M \in GL(n), M^T N = I_n$. We assume that

(V_1) There exists a $B : [0,1] \times \mathbf{R}^n \to \mathcal{L}_s(\mathbf{R}^n)$ with $B(\cdot, x(\cdot)) \in L^1([0,1], \mathcal{L}_s(\mathbf{R}^n))$ for all $x \in C([0,1], \mathbf{R}^n)$ and there exist $B_1, B_2 \in L^1([0,1], \mathcal{L}_s(\mathbf{R}^n))$ such that

$$V'(t,x) = B(t,x)x + h(t,x), B_1(t) \leq B(t,x) \leq B_2(t)$$

for all $(t,x) \in [0,1] \times \mathbf{R}^n$, and $h : [0,1] \times \mathbf{R}^n \to \mathbf{R}^n$ is bounded.

(V_2) There exists a $B : [0,1] \times \mathbf{R}^n \to \mathcal{L}_s(\mathbf{R}^n)$ with $B(\cdot, x(\cdot)) \in L^1([0,1], \mathcal{L}_s(\mathbf{R}^n))$ for all $x \in C([0,1], \mathbf{R}^n)$ and there exists $B_2 \in L^1([0,1], \mathcal{L}_s(\mathbf{R}^n))$ such that

$$V'(t,x) = B(t,x)x + h(t,x), B(t,x) \leq B_2(t)$$

for all $(t,x) \in [0,1] \times \mathbf{R}^n$, and $h : [0,1] \times \mathbf{R}^n \to \mathbf{R}^n$ is bounded.

(V_3) There exists a $B_{02} \in L^1([0,1], \mathcal{L}_s(\mathbf{R}^n))$ such that

$$V(t,x) \leq \frac{1}{2} B_{02}(t)x \cdot x$$

for all $(t,x) \in [0,1] \times \mathbf{R}^n$ with $|x| \leq r$ and $r > 0$ is a constant.

(V_4) There exists a $B_{01} \in L^1([0,1], \mathcal{L}_s(\mathbf{R}^n))$ such that

$$V(t,x) \geq \frac{1}{2} B_{01}(t)x \cdot x$$

for all $(t,x) \in [0,1] \times \mathbf{R}^n$ with $|x| \leq r$ and $r > 0$ is a constant.

***Theorem* 3.1** — *Assume V satisfies (V_1) with $i(B_1) = i(B_2), \nu(B_2) = 0$, then (3.1)-(3.2) has one solution. If we further assume that (V_3) holds with $i(B_{02}) = \nu(B_{02}) = 0$, $i(B_1) \geq 1$ and $\int_0^1 V(t,0)dt = 0$, then (3.1)-(3.2) has one nonzero solution.*

***Theorem* 3.2** — *Assume V satisfies (V_2) with $i(B_2) = \nu(B_2) = 0$, then (3.1)-(3.2) has one solution. If we further assume that (V_4) holds with $i(B_{01}) \geq 1$ and $\int_0^1 V(t,0)dt = 0$, then (3.1)-(3.2) has one nonzero solution.*

PROOF OF THEOREM 3.1 : Assume (V_1) holds with $i(B_1) = i(B_2), \nu(B_2) = 0$. We prove (3.1)-(3.2) has one solution. To this end by the Leray-Schauder degree we only need to show that the possible solutions of the following problem are a priori bounded with respect to the norm $||\cdot||_0$ of $C([0,1], \mathbf{R}^n)$:

$$\ddot{x}(t) + \lambda B_2(t)x(t) + (1-\lambda)V'(t,x(t)) = 0, \tag{3.3}$$
$$x(1) = Mx(0), \dot{x}(1) = N\dot{x}(0)$$

where $\lambda \in (0,1)$. If not, there exists $\{x_k\} \subset Z$ with $||x_k||_0 \to +\infty, \{\lambda_k\} \subset (0,1)$ such that

$$\ddot{x}_k(t) + \lambda_k B_2(t)x_k(t) + (1-\lambda_k)V'(t, x_k(t)) = 0, \tag{3.4}$$

$$x_k(1) = Mx_k(0), \dot{x}_k(1) = N\dot{x}_k(0). \tag{3.5}$$

Set $y_k = x_k/||x_k||_0, \widetilde{B}_k(t) = \lambda_k B_2(t) + (1-\lambda_k)B(t, x_k)$ and $e_k(t) = (1-\lambda_k)(V'(t, x_k(t)) - B(t, x_k)x_k)||x_k||_0^{-1}$. By (V_1), we have $e_k \to 0$ in $L^2([0,1], \mathbf{R}^n)$, and (3.4)-(3.5) is equivalent to

$$\ddot{y}_k(t) + \widetilde{B}_k(t)y_k(t) + e_k(t) = 0, \tag{3.6}$$

$$y_k(1) = My_k(0), \dot{y}_k(1) = N\dot{y}_k(0). \tag{3.7}$$

Set $B_1 = (b_{ij}^{(1)}(t))_{n\times n}$, $B_2 = (b_{ij}^{(2)}(t))_{n\times n}$ and $\widetilde{B}(t, x_k) = (\widetilde{b}_{ij}^{(k)}(t))_{n\times n}$ for $k \in \mathbf{N}^*$. We have

$$b_{ii}^{(1)} \leq \widetilde{b}_{ii}^{(k)} \leq b_{ii}^{(2)}, \forall i = 1, 2, \cdots, n.$$

$$2b_{ij}^{(1)} + b_{ii}^{(1)} + b_{jj}^{(1)} \leq 2\widetilde{b}_{ij}^{(k)} + \widetilde{b}_{ii}^{(k)} + \widetilde{b}_{jj}^{(k)} \leq 2b_{ij}^{(2)} + b_{ii}^{(2)} + b_{jj}^{(2)}, \forall j \neq i.$$

Because $b_{ij}^{(1)}, \widetilde{b}_{ij}^{(k)}, b_{ij}^{(2)} \in L^1[0,1]$, from [11,Theorem 8.8], we have $\widetilde{b}_{ij}^{(k)} \rightharpoonup b_{ij}$ in $L^1[0,1]$ by going to subsequences if necessary. Setting $D_0 = (b_{ij})_{n\times n} \in L^1([0,1], \mathcal{L}_s(\mathbf{R}^n))$, we have

$$\int_0^1 \widetilde{B}(t, x_k)u(t)\cdot v(t)dt \to \int_0^1 D_0(t)u(t)\cdot v(t)dt, \forall u, v \in C([0,1], \mathbf{R}^n) \tag{3.8}$$

by going to subsequences if necessary, and

$$B_1 \leq D_0 \leq B_2. \tag{3.9}$$

We may assume $y_k \rightharpoonup y_0$ in Z and $y_k \to y_0$ in $C([0,1], \mathbf{R}^n)$. Taking the inner product in L^2 of both sides of (3.6) with y, taking the limit and considering (3.7)-(3.8) we have

$$\int_0^1 [\dot{y}_0(t)\cdot\dot{y}(t) - D_0(t)y_0(t)\cdot y(t)]dt = 0.$$

By the proof of Proposition 2.2(i),

$$\ddot{y}_0(t) + D_0(t)y_0(t) = 0, \tag{3.10}$$

$$y_0(1) = My_0(0), \dot{y}_0(1) = N\dot{y}_0(0). \tag{3.11}$$

By (3.9) and Proposition 2.2(ii), $\nu(D_0) = 0$. This contradicts the fact that $y_0 \neq 0$ and satisfies (3.10)-(3.11). Therefore, (3.1)-(3.2) has one solution.

Then we prove (3.1)-(3.2) has one nonzero solution under the further assumptions. Define

$$I(x)=\int_0^1\left[\frac{1}{2}|\dot{x}(t)|^2-V(t,x(t))\right]dt,\forall x\in Z. \tag{3.12}$$

Then it is easy to see that $I\in C^1(Z,\mathbf{R})$. We will use the Mountain Pass theorem to finish the proof. By assumptions, $I(0)=0$. Since $Z\subset H^1([0,1],\mathbf{R}^n)$, there exists a constant $c_1>0$ such that $|x(t)|\le c_1\|x\|$ for all $x\in Z$ and $t\in[0,1]$. Then for $x\in Z$ with $\|x\|=r/c_1$, we have $|x(t)|\le r$ and

$$I(x)\ge\frac{1}{2}q_{B_{02}}(x,x)\ge\frac{1}{2}c_2\|x\|^2=\frac{1}{2}c_2r^2/c_1^2>0$$

via Proposition 2.1(iv) and for some $c_2>0$. By (V_1),

$$\begin{aligned}V(t,x)&=\int_0^1V'(t,\theta x)\cdot xd\theta\\&=\int_0^1(B(t,\theta x)\theta x+h(t,\theta x))\cdot xd\theta\ge\int_0^1B_1(t)\theta x\cdot xd\theta-c_3|x|\\&=\frac{1}{2}B_1(t)x\cdot x-c_3|x|,\end{aligned} \tag{3.13}$$

where $|h(t,x)|\le c_3$ for all $(t,x)\in[0,1]\times\mathbf{R}^n$ and some $c_3>0$. Then

$$\begin{aligned}I(x)&=\int_0^1\frac{1}{2}|\dot{x}(t)|^2-V(t,x)dt\\&\le\int_0^1\frac{1}{2}|\dot{x}(t)|^2-\frac{1}{2}B_1(t)x\cdot xdt+c_3\|x\|\\&=\frac{1}{2}q_{B_1}(x,x)+c_3\|x\|<0\end{aligned}$$

for some $x\in Z^-(B_1)$ and $\|x\|$ is large enough via $i(B_1)>1$. We now verify I satisfies (PS). Suppose that there is a sequence $\{x_k\}\subset Z$ such that $I(x_k)$ is bounded and $I'(x_k)\to0$ in Z^*. Now it suffices to verify that $\{x_k\}$ has a convergent subsequence in Z. It is easy to see

$$I'(x)y=\int_0^1(\dot{x}(t)\cdot y(t)-V'(t,x(t))\cdot y(t))dt. \tag{3.14}$$

for all $x,y\in Z$.

We first show x_k is bounded in Z and it suffices to show x_k is bounded in $C([0,1],\mathbf{R}^n)$. If this is not the case, we may assume that $\|x_k\|_0\to\infty$. Set $y_k=x_k/\|x_k\|_0$. From (V_1),

$$V'(t,x_k(t))=C_k(t)x_k(t)+h_k(t), \tag{3.15}$$

where $\{h_k\}$ is bounded in $L^\infty([0,1],\mathbf{R}^n)$ and

$$B_1\le C_k\le B_2. \tag{3.16}$$

So

$$\int_0^1 \dot{y}_k(t)\dot{y}(t)dt = \|x_k\|_0^{-1} I'(x_k)y + \int_0^1 (C_k(t)y_k(t) + \|x_k\|_0^{-1} h_k(t)) \cdot y(t)dt \quad \forall y \in Z. \tag{3.17}$$

This means that y_k is bounded in Z. We may assume $y_k \rightharpoonup y_0$ in Z and $y_k \to y_0$ in $C([0,1], \mathbf{R}^n)$. By definition, $\|y_k\|_0 = 1$ and hence $\|y_0\|_0 = 1$. By (3.8), there exists $C_0 \in L^1([0,1], \mathcal{L}_s(\mathbf{R}^n))$ such that

$$\int_0^1 C_k(t)u(t) \cdot v(t)dt \to \int_0^1 C_0(t)u(t) \cdot v(t)dt \ \ \forall u, v \in C([0,1], \mathbf{R}^n), \tag{3.18}$$

by going to subsequences if necessary, and

$$B_1 \le C_0 \le B_2 \tag{3.19}$$

Since $I'(x_k) \to 0$, from (3.14)-(3.18) it follows that

$$\int_0^1 [\dot{y}_0(t) \cdot \dot{y}(t) - C_0(t)y_0(t) \cdot y(t)]dt = 0, \ \ \forall y \in Z$$

Then $y = y_0(t)$ is a nontrivial solution of the following problem

$$\begin{aligned} &\ddot{y} + C_0(t)y = 0 \\ &y(1) = My(0), \dot{y}(1) = Ny(0). \end{aligned}$$

From Proposition 2.2 (ii) and (3.19), we have $\nu(C_0) = 0$, a contradiction. Therefore x_k is bounded in Z.

Suppose that $x_k \rightharpoonup x_0$ in Z and $x_k \to x_0$ in $C([0,1], \mathbf{R}^n)$. Taking limits in (3.17) yields

$$\int_0^1 [\dot{x}_0(t) \cdot \dot{y}(t) - V'(t, x_0(t)) \cdot y(t)]dt = 0$$

for all $y \in Z$, and hence

$$\int_0^1 (\dot{x}_k(t) - \dot{x}_0(t)) \cdot \dot{y}(t)dt = \int_0^1 (V'(t, x_k(t)) - V'(t, x_0(t))) \cdot y(t)dt + I'(x_k)y.$$

It follows that

$$\begin{aligned} &||x_k - x_0|| = \sup_{||y|| \le 1} |\int_0^1 (\dot{x}_k(t) - \dot{x}_0(t)) \cdot \dot{y}(t)dt + \int_0^1 (x_k(t) - x_0(t)) \cdot y(t)dt| \\ &\le ||I'(x_k)|| + \|V'(\cdot, x_k(\cdot)) - V'(\cdot, x_0(\cdot)\|_{L^2} + \|x_k - x_0\|_{L^2} \to 0, \end{aligned}$$

and hence $x_k \to x_0$ in Z. Therefore I satisfies (PS). Now the Mountain Pass Theorem applies and I has a critical point with positive critical value. Because $I(0) = 0$, the critical point is not zero and the associated solution of (3.1)-(3.2) is not zero. □

In order to prove Theorem 3.2 we need the following lemma.

Lemma 3.1 — [27, Theorem 2.7]. Let Z be a real Hilbert space and $I \in C^1(Z, \mathbf{R})$ satisfies (PS). If I is bounded from below. Then $c \equiv \inf_Z$ is a critical value of I.

PROOF OF THEOREM 3.2 : Assume V satisfies (V_2). From the above, we have I is bounded from below, then we first show that I satisfies (PS). Assume $\{x_k\} \subset Z$ such that $I(x_k)$ is bounded and $I'(x_k) \to 0$ in Z^*. We can assume $||I'(x_k)|| \leq 1$. From (3.14) and (V_2), we have

$$\int_0^1 |\dot{x}_k|^2 dt \leq ||x_k|| + \int_0^1 B_2 x_k \cdot x_k dt + c_3 ||x_k||,$$

and hence

$$q_{B_2}(x_k, x_k) \leq (c_3 + 1)\|x_k\|.$$

By Proposition 2.2(iv) and $i(B_2) = 0 = \nu(B_2)$, $||x_k||$ is bounded. Assume $x_k \rightharpoonup x_0$ in Z and $x_k \to x_0$ in $C([0,1], \mathbf{R}^n)$. As in the proof of Theorem 3.1, we have $x_k \to x_0$ in Z. Then $\inf_{x \in Z} I(x)$ is a critical value from Lemma 3.1.

If (V_4) holds with $i(B_{01}) > 1$,

$$I(x) = \int_0^1 (\frac{1}{2}|\dot{x}(t)|^2 - V(t,x))dt \leq \frac{1}{2}\int_0^1 (|\dot{x}(t)|^2 - B_{01}(t)x \cdot x)dt = \frac{1}{2} q_{B_{01}}(x,x) < 0,$$

as $\|x|| \leq r/c_1$ and $x \in Z^-(B_{01}) \backslash \{0\}$. Then $I(0) = 0 > \inf_Z I = I(x_3)$ and $x_3 \neq 0$. □

Remark : In [6, 8], nonzero solutions were obtained under the assumptions that $i(B_{02}) - i(B_1)$ is odd or $i(B_{01})$ is odd. Here we weaken the requirement.

Now we investigate (3.1)-(3.2) under the assumption that $V(t,x)$ is even with respect to x. We need the following assumptions

(V_5) There exists a $\bar{B}_0 : [0,1] \times \mathbf{R}^n \to \mathcal{L}_s(\mathbf{R}^n)$ with $\bar{B}(\cdot, x(\cdot)) \in L^1([0,1], \mathcal{L}_s(\mathbf{R}^n))$ for all $x \in C([0,1], \mathbf{R}^n)$ and there exist $B_{03}, B_{04} \in L^1([0,1], \mathcal{L}_s(\mathbf{R}^n))$ such that

$$V'(t,x) = \bar{B}_0(t,x)x, B_{03} \leq \bar{B}_0(t,x) \leq B_{04}(t)$$

for all $(t,x) \in [0,1] \times \mathbf{R}^n$ with $|x| \leq r$ for some $r > 0$.

(V_6) $V(t,-x) = V(t,x)$ for all $(t,x) \in [0,1] \times \mathbf{R}^n$.

We have the following two theorems.

***Theorem* 3.3** — *Assume V satisfies $(V_1), (V_5)$ and (V_6) with $i(B_1) = i(B_2), i(B_{03}) = i(B_{04}), \nu(B_{04}) = \nu(B_2) = 0$. Then (3.1)-(3.2) has at least $|i(B_{03}) - i(B_2)|$ distinct pairs of nonzero solutions.*

***Theorem* 3.4** — *Assume V satisfies $(V_2), (V_5)$ and (V_6) with $i(B_2) = \nu(B_2) = \nu(B_{04}) = 0, i(B_{03}) = i(B_{04})$. Then (3.1)-(3.2) has at least $i(B_{03})$ distinct pairs of nonzero solutions.*

We need the following lemmas.

Lemma 3.2 — [2, Theorem 4.3.4]. Let $I \in C^1(Z, \mathbf{R})$ be even, satisfy (PS) and $I(0) = 0$. Assume I satisfies

(i) there are a $m-$dimensional subspace Z_1 and a constant $r > 0$ such that

$$\sup_{x \in Z_1 \cap \partial U_r} I(x) < 0,$$

(ii) there is a $j-$codimensional subspace Z_2 such that

$$\inf_{x \in Z_2} I(x) > -\infty.$$

Then I has at least $m - j$ distinct pairs of critical points provided $m - j > 0$.

Lemma 3.3 — [2, Theorem 4.3.6]. Let $I \in C^1(Z, \mathbf{R})$ be even, satisfy (PS) and $I(0) = 0$. Assume I satisfies

(i) there is a $j-$codimensional subspace Z_1 and positive constants r and α such that $I(x) \geq \alpha$ for all $x \in Z_1 \cap \partial U_r$, and

(ii) there are a $m-$dimensional subspace Z_2 and a constant $R > 0$ such that $I(x) \leq 0$ for all $x \in Z_2 \setminus U_R$. Then I has at least $m - j$ distinct pairs of critical points provided $m - j > 0$.

PROOF OF THEOREM 3.3 : Without loss of generality we assume that $V(t, 0) \equiv 0$. Otherwise we use $V(t, x) - V(t, 0)$ instead of $V(t, x)$. Define

$$I(x) = \int_0^1 \left[\frac{1}{2} |\dot{x}(t)|^2 - V(t, x(t)) \right] dt, \forall x \in Z. \tag{3.20}$$

Then it is easy to see that $I \in C^1(Z, \mathbf{R})$, $I(0) = 0$ and from (V_6), $I(-x) = I(x)$. As in the proof of Theorem 3.1, I satisfies (PS). Now we will finish the proof in two cases.

Case 1 : $i(B_{03}) - i(B_2) > 0$. By (V_1),

$$\begin{aligned} I(x) &= \int_0^1 \frac{1}{2}|\dot{x}(t)|^2 - \int_0^1 V(t,x)dt \\ &\geq \int_0^1 \frac{1}{2}|\dot{x}(t)|^2 - \frac{1}{2}B_2(t)x\cdot xdt - c_3\|x\| \\ &= \frac{1}{2}q_{B_2}(x,x) - c_3\|x\| \end{aligned}$$

is bounded from below on $x \in Z^+(B_2)$ via Proposition 2.2(iv). From (V_5), we obtain

$$\begin{aligned} I(x) &= \int_0^1 \frac{1}{2}|\dot{x}(t)|^2 - \int_0^1 V(t,x)dt \\ &\leq \int_0^1 \frac{1}{2}|\dot{x}(t)|^2 - \frac{1}{2}B_{03}(t)x\cdot xdt \\ &= \frac{1}{2}q_{B_{03}}(x,x) \leq -\frac{c_4}{2}\|x\|^2 = -\frac{c_4 r^2}{2c_1^2} \end{aligned}$$

for all $x \in Z^-(B_{03})$ and $\|x\| = r/c_1$ since $|x(t)| \leq c_1\|x\|$ for $x \in Z, t \in [0,1]$ as before. In Lemma 3.2 we choose $Z_2 = Z^+(B_2), Z_1 = Z^-(B_{03})$. Then $i(B_{03}) = \dim Z_1$ and $i(B_2) =$ the codimension of Z_2. Then I has $i(B_{03}) - i(B_2)$ distinct pairs critical points.

Case 2 : $i(B_2) - i(B_{03}) > 0$. As in Case 1,

$$I(x) \leq \frac{1}{2}q_{B_1}(x,x) + c_3\|x\| \leq 0$$

as $x \in Z^-(B_1)$ and $\|x\|$ is large enough via Proposition 2.2(iv). And

$$I(x) \geq \frac{1}{2}q_{B_{04}}(x,x) \geq \frac{1}{2}c_5\|x\|^2 = \frac{c_5 r^2}{2c_1^2}$$

as $x \in Z^+(B_{04})$ and $\|x\| = r/c_1$. By Lemma 3.3, I has $i(B_1) - i(B_{04}) = i(B_2) - i(B_{03})$ distinct pairs of critical points. □

PROOF OF THEOREM 3.4 : Similar to the proof of Theorem 3.3 in Case 1, we only need to show I satisfies (PS). And in the proof of Theorem 3.2, we have proved it. □

Finally we give two three solution theorems. We need the following assumption.

(V_7) $V \in C^2([0,1]\times \mathbf{R}^n), V'(t,0) \equiv 0, B_0(t) \equiv V''(t,0)$.

***Theorem* 3.5** — *If V satisfies (V_2) and (V_7) with $i(B_2) = \nu(B_2) = 0, i(B_0) \geq 1$, then (3.1)-(3.2) has at least a nontrivial solution. Moreover, if $\nu(B_0) = 0$, then (3.1)-(3.2) has at least two nontrivial solutions.*

***Theorem* 3.6** — *If V satisfies (V_1) and (V_7) with $i(B_1) = i(B_2), \nu(B_2) = 0, i(B_1) \notin [i(B_0), i(B_0)+\nu(B_0)]$, then (3.1)-(3.2) has at least a nontrivial solution. Moreover, if $\nu(B_0) = 0, |i(B_1) - i(B_0)| \geq 2n$, then (3.1)-(3.2) has at least two nontrivial solution.*

In order to prove the two Theorems, we need the following lemmas.

Lemma 3.4 — [3, Theorem 5.1.35]. Assume that Z is a Hilbert space, $I \in C^2(Z, \mathbf{R})$ satisfies (PS) and is bounded from below, and x_0 is a non-degenerate non-minimum point of I with finite Morse index. Then I has at least three distinct critical points.

Lemma 3.5 — Assume V satisfies (V_1). Then

$$H_q(Z, I_a; \mathbf{R}) \cong \delta_{q\gamma}\mathbf{R}$$

for $-a > -I(0)$ large enough, where $\gamma = i(B_1)$.

Lemma 3.6 — [8, Proposition 5.5.2]. Assume $I \in C^2(X, R)$ and satisfies (PS), and $I''(x)$ is Fredholm with finite Morse index for all critical point $x \in X$ and $I'(0) = 0$. Suppose there is a positive integer γ such that $\gamma \notin [m^-(I''(0)), m^0(I''(0))+m^-(I''(0))]$ and $H_q(X, I_a; \mathbf{R}) \cong \delta_{q\gamma}\mathbf{R}$ for some regular value $a < I(0)$. Then I has a critical point $x_0 \neq 0$. Moreover, if 0 is a non-degenerate critical point, and $m^0(I''(x_0)) \leq |\gamma - m^-(I''(0))|$, then I has another critical point $x_1 \neq x_0, 0$.

PROOF OF THEOREM 3.5 : By Proposition 2.2(ii) there exists $\epsilon > 0$ such that $i(B_0 - \epsilon) = i(B_0)$. From (V_7), there exists $r_1 > 0$ such that

$$V''(t, x) \geq B_0(t) - \epsilon I_n \tag{3.21}$$

for $|x| \leq r_1$. Then for $|x| \leq r_1$,

$$\begin{aligned} V(t,x) \quad &= V(t,x) - V(t,0) = \int_0^1 V'(t, \theta_1 x) \cdot x d\theta_1 = \int_0^1 (V'(t, \theta_1 x) \cdot x - V'(t,0)) d\theta_1 \\ &= \int_0^1 \int_0^1 V''(t, \theta\theta_1 x)\theta x \cdot x d\theta_1 d\theta \geq \frac{1}{2}(B_0(t) - \epsilon I_n) x \cdot x. \end{aligned}$$

Hence the first part of Theorem 3.5 is similar to the second part of Theorem 3.2. In fact, by (V_2), $I(x)$ is bounded from below and satisfies (PS). Then there is $x_1 \in Z$ such that $I(x_1) = \inf_Z I$ via Lemma 3.1. By (3.21), $I(x) \leq \frac{1}{2} q_{B_0 - \epsilon I_n}(x, x) < 0$ for $x \in Z^-(B_0 - \epsilon I_n) \backslash \{0\}$ with $\|x\| = r_1/c_1$. Hence, $I(x_1) < 0 = I(0)$ and $x_1 \neq 0$.

We only prove the second part. If $\nu(B_0) = 0$, then $x = 0$ is a non-degenerate critical point. In fact, $I''(0)$ defines a bounded symmetric bilinear form on Z and corresponds to a bounded symmetric operator on Z denoted also by itself. It suffices to prove $I''(0)$ has a bounded inverse. For any $x, y \in Z$,

$$I''(0)(x,y) = \int_0^1 [\dot{x}(t)\cdot\dot{y}(t) - B_0(t)x(t)\cdot y(t)]dt.$$

If $I''(0)x = 0$, then for all $y \in Z$,

$$\int_0^1 [\dot{x}(t)\cdot\dot{y}(t) - B_0(t)x(t)\cdot y(t)]dt = 0.$$

Then

$$\ddot{x}(t) + B_0(t)x(t) = 0,$$
$$x(1) = Mx(0), \dot{x}(1) = N\dot{x}(0).$$

Because $\nu(B_0) = 0, x = 0$. The only thing left to do is to prove that for any $z \in Z, I''(0)x = z$ has one solution in Z. Because $I''(0) = I - K$ where $K : Z \to Z$ is defined by

$$(Kx, y) = \int_0^1 B_0(t)x \cdot y dt, \forall x, y \in Z.$$

So K is compact and as before $\ker(I''(0)) = \{0\}$, thus $R(I''(0)) = Z$. Then I has a third critical point other than $0, x_1$ via Lemma 3.4. □

PROOF OF THEOREM 3.6 : Note that $i(B_0) = m^-(I''(0)), \nu(B_0) = m^0(I''(0))$ and $\nu(B) \le 2n$ for all B. Set $\gamma = i(B_1)$ and $m^0(I''(0)) \le 2n \le |i(B_1) - i(B-0)| = |\gamma - m^-(I''(0))|$. Then the results follow from Lemma 3.5 and Lemma 3.6. □

PROOF OF LEMMA 3.5 : The proof is similar to [8] and [10], so we only sketch it here. From Proposition 2.2(iv), we have $Z = Z_1 \bigoplus Z_2$ where we set $Z_1 = Z^-(B_1), Z_2 = Z^+(B_2)$. First for any $x = x_1 + x_2$ with $x_1 \in Z_1, x_2 \in Z_2$, there exists $R_0 > 0$ such that

$$I'(x)(x_2 - x_1) > 1, \ \ \forall\, x \in Z$$

with $\|x_2\| > R_0$ or $\|x_1\| > R_0$. Set $\mathcal{M} = Z_1 \bigoplus (Z_2 \bigcap \bar{U}_{R_0})$. Then for a regular value a of $I(x)$,

$$H_q(Z, I_a; \mathbf{R}) \cong H_q(\mathcal{M}, \mathcal{M} \cap I_a; \mathbf{R}).$$

Then we prove that there exist $R_1 > 0$ and a regular value a_1 with $-a_1 > -I(0)$ large enough of $I(x)$ such that $(Z_2 \bigcap \bar{U}_{R_0}) \oplus (Z_1 \backslash U_{R_1})$ is a strong deformation retract of $\mathcal{M} \bigcap I_{a_1}$. Hence,

$$\begin{aligned} H_q(\mathcal{M}, \mathcal{M} \bigcap I_{a_1}; \mathbf{R}) &\cong H_q((Z_2 \bigcap \bar{U}_{R_0}) \bigoplus Z_1, (Z_2 \bigcap \bar{U}_{R_0}) \bigoplus (Z_1 \backslash U_{R_1}); \mathbf{R}) \\ &\cong H_q(Z_1 \bigcap \bar{U}_{R_1}, \partial(Z_1 \bigcap U_{R_1}); \mathbf{R}) \\ &\cong \delta_{q\gamma}\mathbf{R} \end{aligned}$$

So the proof is complete. □

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