Indian J. Pure Appl. Math., **50**(1): 69-81, March 2019

DOI: 10.1007/s13226-019-0307-x

THE FIRST DIRICHLET EIGENVALUE OF THE LAPLACIAN IN A CLASS OF DOUBLY CONNECTED DOMAINS IN COMPLEX PROJECTIVE SPACE

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(*Received* 22 *June* 2017; *after final revision* 20 *January* 2018;
accepted 1 *March* 2018)

Let B_1 be an open ball of radius r_1 in the complex projective space. Let B_0 be a smaller open ball inside it. It is shown that first Dirichlet eigenvalue of the Laplacian on $B_1 \setminus \overline{B_0}$ is maximal if and only if the balls are concentric.

Key words : Laplace-Beltrami operator; extremum of first Dirichlet eigenvalues; maximum-principles.

1. Introduction and Statement of Main Result

The complex projective space $\mathbb{C}P^n$ is the set of all complex lines through origin in $\mathbb{C}^{n+1}$. It is a complex manifold of dimension n with the Fubini-Study metric $\langle \cdot, \cdot \rangle$ (see § 2). Let Δ be the Laplace-Beltrami operator of $\mathbb{C}P^n$. Let Ω be an open subset of $\mathbb{C}P^n$ such that $\bar{\Omega}$ is a smooth compact submanifold of $\mathbb{C}P^n$ with smooth boundary.

Consider the following Dirichlet eigenvalue problem on Ω :

$$\left.\begin{aligned} -\Delta u = \lambda u \quad & \text{on } \Omega, \\ u = 0 \quad & \text{on } \partial\Omega \end{aligned}\right\} \tag{1.1}$$

By Theorem 4.4, p.105 of [4], the eigenvalues of the Laplace-Beltrami operator $-\Delta$ are strictly positive. The eigenfunctions corresponding to the first eigenvalue λ_1 are proportional, they belong to $\mathcal{C}^\infty(\bar{\Omega})$ and they are strictly positive or strictly negative on Ω. Moreover,

$$\lambda_1 = \inf \{ \| \nabla\psi \|_{L^2}^2 : \psi \in H_0^1(\Omega) \ \& \ \| \psi \|_{L^2}^2 = 1 \}$$

Let $y_1 := y_1(\Omega)$ be the unique solution of (1.1) corresponding to the first eigenvalue $\lambda_1(=\lambda_1(\Omega))$ characterized by $y_1 > 0$ on Ω and $\int {y_1}^2 dV = 1$.

Fix two real numbers r_0 and r_1 such that $0 < r_0 < r_1 < \frac{\pi}{2}$. Let B_0 and B_1 be arbitrary balls of radii r_0 and r_1 respectively in $\mathbb{C}P^n$ such that $\bar{B_0} \subseteq B_1$. Consider the family $\mathcal{F} = \{\Omega := B_1 \setminus \bar{B_0}\}$ of domains in $\mathbb{C}P^n$. We study the maxima of the first eigenvalue $\lambda_1(\Omega)$ over $\mathcal{F}$.

We state the main result: Put $\Omega_0 := B(p, r_1) \setminus \overline{B(p, r_0)}$ for any $p \in \mathbb{C}P^n$ fixed arbitrarily.

Theorem — *The first Dirichlet eigenvalue $\lambda_1(\Omega)$ on $\mathcal{F}$ attains its maximum at Ω if and only if $\Omega = \Omega_0$ (i.e. the balls are concentric).*

Similar problems for Euclidean space, Space forms, rank-1 symmetric spaces of non-compact type were solved in the research papers [1-3] respectively.

2. GEOMETRY OF $\mathbb{C}P^n$

The standard Hermitian inner product on $\mathbb{C}^{n+1}$ is denoted by $\langle \cdot, \cdot \rangle_{\mathbb{C}^{n+1}}$. We identify $\mathbb{C}^{n+1}$ with $\mathbb{R}^{2n+2}$. The real part of $\langle \cdot, \cdot \rangle_{\mathbb{C}^{n+1}}$ is the Euclidean inner product of $\mathbb{R}^{2n+2}$, denoted by $\langle \cdot, \cdot \rangle_{R^{2n+2}}$. Let $S^{2n+1} = \{(z_0, z_1, \cdot\cdot, z_n) \in \mathbb{C}^{n+1} \mid \sum_{j=0}^{n} |z_j|^2 = 1\}$ denote the unit sphere of $\mathbb{R}^{2n+2}$.

Let S^1 denote the multiplicative group of complex numbers of absolute value 1. Consider the natural action of S^1 on the unit sphere S^{2n+1} defined as follows: $\forall\ \lambda \in S^1$ and $\forall\ (z_0, z_1, \cdots, z_n) \in S^{2n+1}$

$$\lambda(z_0, z_1, \cdots, z_n) = (\lambda z_0, \lambda z_1, \cdots, \lambda z_n).$$

This action of S^1 defines an equivalence relation on S^{2n+1} and $\mathbb{C}P^n$ can be viewed as the resulting quotient space of S^{2n+1}. Let $[(z_0, z_1, \cdots, z_n)]$ denote the equivalence class of $(z_0, z_1, \cdots, z_n) \in S^{2n+1}$. Consider the quotient map $\pi : S^{2n+1} \longrightarrow \mathbb{C}P^n$ defined by

$$\pi(z_0, z_1, \cdots, z_n) = [(z_0, z_1, \cdots, z_n)],\ ((z_0, z_1, \cdots, z_n) \in S^{2n+1}).$$

Let $q \in \mathbb{C}P^n$ and $\tilde{q} \in \pi^{-1}(q)$ be arbitrary. The tangent space of $\pi^{-1}(q)$ at $\tilde{q}$ is called the vertical space at $\tilde{q}$, denoted by $\mathcal{V}_{\tilde{q}}$, i.e. $\mathcal{V}_{\tilde{q}} = \{t(i\tilde{q}) \,|\, t \in \mathbb{R}\}$. Let $\mathcal{H}_{\tilde{q}} = \{v \in \mathbb{C}^{n+1} : \langle v, \tilde{q} \rangle_{\mathbb{C}^{n+1}} = 0\}$. Thus we have an orthogonal decomposition

$$T_{\tilde{q}} S^{2n+1} = \mathcal{H}_{\tilde{q}} \oplus \mathcal{V}_{\tilde{q}}$$

and any vector $v \in \mathcal{H}_{\tilde{q}}$ is called a horizontal tangent vector of S^{2n+1} at $\tilde{q}$. Note that $d\pi : \mathcal{H}_{\tilde{q}} \longrightarrow T_q\mathbb{C}P^n$ is an isomorphism. For any $v, w \in T_q\mathbb{C}P^n$, $\exists$ unique $\tilde{v}$,$\tilde{w} \in \mathcal{H}_{\tilde{q}}$ such that $d\pi(\tilde{v}) = v$ and

$d\pi(\tilde{w}) = w$. Then the Fubini-Study metric $\langle \cdot, \cdot \rangle$ on $\mathbb{C}P^n$ is defined by

$$\langle v, w \rangle = \langle \tilde{v}, \tilde{w} \rangle_{\mathbb{R}^{2n+2}}.$$

Thus the map $\pi : S^{2n+1} \longrightarrow \mathbb{C}P^n$ is a Riemannian submersion.

If X is a tangent vector field on $\mathbb{C}P^n$ then there is a unique tangent vector field $\tilde{X}$ on S^{2n+1} such that $d\pi_z(\tilde{X}(z)) = X(\pi(z))$ & $\tilde{X}(z) \in \mathcal{H}_z \ \forall \ z \in S^{2n+1}$. The vector field $\tilde{X}$ is called the horizontal lift of X. Let $\bar{\nabla}, \nabla$ be the Levi-Civita connections on S^{2n+1} and CP^n respectively. Then

$$\overline{\nabla}_{\tilde{X}} \tilde{Y} = \widetilde{\nabla_X Y} + \frac{1}{2}[\tilde{X}, \tilde{Y}]^v$$

where $[\tilde{X}, \tilde{Y}]^v$ denotes the vertical part the vector $[\tilde{X}, \tilde{Y}]$. Hence

$$(\nabla_X Y)(q) = d\pi_{\tilde{q}}\left((\overline{\nabla}_{\tilde{X}} \tilde{Y})(\tilde{q})\right) \quad \forall q \in \mathbb{C}P^n, \ \forall \tilde{q} \in \pi^{-1}(q).$$

The Riemannian manifold $\mathbb{C}P^n$ is complete and its injectivity radius is $\frac{\pi}{2}$.

For any two points $p, q \in \mathbb{C}P^n$, $d(p, q)$ denotes the distance between them in $\mathbb{C}P^n$. Then $d(p, q) \leq \frac{\pi}{2}$. For any two points $z, w \in S^{2n+1}$, $d_{S^{2n+1}}(z, w)$ denotes the distance between them in S^{2n+1}.

Definition — A geodesic $c(s)$ in S^{2n+1} such that $c'(s) \in \mathcal{H}_{c(s)}$ is called a horizontal geodesic of S^{2n+1}.

The geodesics of $\mathbb{C}P^n$ are described by the following result:

Proposition 2.1 — [6].

(1) Let $c : [0, 1] \longrightarrow S^{2n+1}$ be a geodesic such that its initial velocity vector $c'(0) \in \mathcal{H}_{c(0)}$. Then $c'(s) \in \mathcal{H}_{c(s)} \ \forall s \in [0, 1]$ and the curve $\pi o c$ is a geodesic of $\mathbb{C}P^n$ having the same length as c.

(2) Conversely, let $\sigma : [0, 1] \longrightarrow \mathbb{C}P^n$ be any geodesic. Then for any $\tilde{q} \in \pi^{-1}(\sigma(0))$, there exists a unique horizontal geodesic $\tilde{\sigma} : [0, 1] \longrightarrow S^{2n+1}$ such that $\tilde{\sigma}(0) = \tilde{q}, \ d\pi_{\tilde{q}}(\tilde{\sigma}'(0)) = \sigma'(0)$ & $\pi \circ \tilde{\sigma} = \sigma$. □

Example 2.2 : For each $0 \leq j \leq n$, let $e_j = (0, 0, \cdot\cdot, 1, 0, \cdot\cdot, 0)$ where 1 occurs in the $j+1$-th place. Then $e_0 := (1, 0, \cdots, 0) \in S^{2n+1}$, & $ie_1 = (0, i, 0, \ldots, 0)$ is a horizontal tangent vector of S^{2n+1} at e_0. Define

$$c(s) := (\cos s, i \sin s, 0, \ldots, 0) \ (s \in \mathbb{R}).$$

Then c is a horizontal geodesic in S^{2n+1} (with $c(0) = e_0$ & $c'(0) = ie_1$) and hence $\pi \circ c$ is geodesic in $\mathbb{C}P^n$. □

Let $q \in \mathbb{C}P^n$ be arbitrary and $\tilde{q} \in \pi^{-1}(q)$ be fixed. Let $\{v, w\} \subset T_q\mathbb{C}P^n$ be an orthonormal subset and let $P = \text{span}_{\mathbb{R}}\{v, w\}$ be a plane of $T_q\mathbb{C}P^n$. Then the Riemannian sectional curvature $K(P)$ of the plane P is $K(P) = 1 + 3\cos^2\theta$ where $\cos\theta = \langle v, iw\rangle$. In particular $1 \leq K(P) \leq 4$. □

Lemma 2.3 — Fix any two distinct points p_1, p_2 of $\mathbb{C}P^n$.

(1) Fix $\tilde{p_1} \in \pi^{-1}(p_1)$. Then there exists $\tilde{p_2} \in \pi^{-1}(p_2)$ such that $d_{S^{2n+1}}(\tilde{p_1}, \tilde{p_2}) = d(p_1, p_2)$. And given any such pair $(\tilde{p_1}, \tilde{p_2})$ of points in S^{2n+1}, $\exists$ a unique horizontal geodesic $\tilde{\sigma}$ in S^{2n+1} joining $\tilde{p_1}$ to $\tilde{p_2}$ having length $d(p_1, p_2)$.

(2) Suppose that $d(p_1, p_2) < \frac{\pi}{2}$. Fix any $\tilde{p_1} \in \pi^{-1}(p_1)$. Then $\exists$ unique $\tilde{p_2} \in \pi^{-1}(p_2)$ such that $d_{S^{2n+1}}(\tilde{p_1}, \tilde{p_2}) = d(p_1, p_2)$.

(3) Let $z \in \pi^{-1}(p_1)$ and $w \in \pi^{-1}(p_2)$ be arbitrary. Then

$$d(p_1, p_2) = \arccos|\langle z, w\rangle_{\mathbb{C}^{n+1}}|. \tag{2.1}$$

PROOF : Put $l := d(p_1, p_2)(\leq \frac{\pi}{2})$. Let $\sigma : [0, l] \longrightarrow \mathbb{C}P^n$ be a unit speed geodesic such that $\sigma(0) = p_1$ & $\sigma(l) = p_2$. By proposition 2.1 $\exists$ unique horizontal geodesic $\tilde{\sigma} : [0, l] \longrightarrow S^{2n+1}$ of length l such that $\tilde{\sigma}(0) = \tilde{p_1}$ & $\pi \circ \tilde{\sigma} = \sigma$. Put

$$\tilde{v} = \tilde{\sigma}'(0) \ \& \ \tilde{p_2} := \tilde{\sigma}(l). \tag{2.2}$$

Since the injectivity radius of S^{2n+1} is π and length$(\tilde{\sigma}) = l < \pi$, $\tilde{\sigma}$ is the unique geodesic of length l in S^{2n+1} joining $\tilde{p_1}$ to $\tilde{p_2}$. This proves (1).

Proof of (2) : Let σ be as given in (1). If $l < \frac{\pi}{2}$ then σ is the unique unit speed geodesic of length l in $\mathbb{C}P^n$ joining p_1 to p_2. Let $\bar{p_2} \in \pi^{-1}(p_2)$ be any point such that $d_{S^{2n+1}}(\tilde{p_1}, \bar{p_2}) = l$. Let c be the unique unit speed geodesic of length l in S^{2n+1} joining $\tilde{p_1}$ to $\bar{p_2}$. Then $\pi \circ c$ is a distance minimizing curve in $\mathbb{C}P^n$ joining p_1 to p_2, and hence it is a geodesic of $\mathbb{C}P^n$. Since $\frac{\pi}{2}$ is the injectivity radius of $\mathbb{C}P^n$ and $l < \frac{\pi}{2}$, it follows that $\pi \circ c = \sigma$. Then by (2) of Proposition 2.1, $c = \tilde{\sigma}$ & hence $\tilde{p_2} = \bar{p_2}$. The proof (2) is now complete.

Proof of (3): Now by (2.2), $\tilde{p_2} = (\cos l)\tilde{p_1} + (\sin l)\tilde{v}$. Hence

$$\begin{aligned} \langle \tilde{p_1}, \tilde{p_2}\rangle_{\mathbb{C}^{n+1}} &= (\cos l)\langle \tilde{p_1}, \tilde{p_1}\rangle_{\mathbb{C}^{n+1}} + (\sin l)\langle \tilde{p_1}, \tilde{v}\rangle_{\mathbb{C}^{n+1}} \\ &= \cos l \quad (\because \tilde{v} \in \mathcal{H}_{\tilde{p_1}}) \end{aligned} \tag{2.3}$$

Then $\langle \tilde{p_2}, \tilde{p_1}\rangle_{\mathbb{C}^{n+1}} = \cos l$. Hence

$$\cos^2 l = \langle \tilde{p_1}, \tilde{p_2}\rangle_{\mathbb{C}^{n+1}}\langle \tilde{p_2}, \tilde{p_1}\rangle_{\mathbb{C}^{n+1}}.$$

Thus

$$\cos l = \sqrt{\langle \tilde{p_1}, \tilde{p_2} \rangle_{\mathbb{C}^{n+1}} \langle \tilde{p_2}, \tilde{p_1} \rangle_{\mathbb{C}^{n+1}}} = |\langle \tilde{p_1}, \tilde{p_2} \rangle_{\mathbb{C}^{n+1}}|. \tag{2.4}$$

So $d(p_1, p_2) = l = \arccos |\langle \tilde{p_1}, \tilde{p_2} \rangle_{\mathbb{C}^{n+1}}|$. From 2.4, it is clear that $d(p_1, p_2) = \arccos |\langle z, w \rangle_{\mathbb{C}^{n+1}}|$ $\forall\ z \in \pi^{-1}(p_1)$ and $w \in \pi^{-1}(p_2)$. This proves (3). □

3. A Tangent Vector Field of $\mathbb{C}P^n$

Consider the matrix

$$A = \begin{pmatrix} 0 & i & & \\ i & 0 & & \\ & & & \\ & & & \end{pmatrix}$$

in $M(n+1, \mathbb{C})$ where all the entries not shown are zero. Then $A \in u(n+1)$ and $e^{tA} \in U(n+1)\ \forall\ t \in \mathbb{R}$. For each $t \in \mathbb{R}$ define a map $\psi_t : \mathbb{C}P^n \longrightarrow \mathbb{C}P^n$ by

$$\psi_t([(z_0, z_1, \cdot\cdot, z_n)]) = [e^{tA}((z_0, z_1, \cdot\cdot, z_n))],$$

$$i.e.\ \ \psi_t([(z_0, z_1, \cdot\cdot, z_n)]) = [((\cos t)z_0 + i(\sin t)z_1, i(\sin t)z_0 + (\cos t)z_1, z_3, \cdots, z_n)]$$

$\forall\ ([(z_0, z_1, \cdot\cdot, z_n)] \in \mathbb{C}P^n$. Then each ψ_t is a well-defined map, and it is an isometry of $\mathbb{C}P^n$.

Define a vector field V on $\mathbb{C}P^n$ by $V([(z_0, z_1, \cdot\cdot, z_n)]) = \frac{d}{dt}\psi_t([(z_0, z_1, \cdot\cdot, z_n)])_{|t=0}$. Then $V([(z_0, z_1, \cdot\cdot, z_n)]) = d\pi_{\tilde{z}}(A(\tilde{z})) = d\pi_{\tilde{z}}((iz_1, iz_0, 0, \cdots, 0))$ where $\tilde{z} := (z_0, z_1, \cdots, z_n)$. The horizontal tangent vector field $\tilde{V}$ on S^{2n+1} corresponding to V is given as follows: At any $\tilde{z} = (z_0, z_1, \cdot, \cdot, z_n) \in \pi^{-1}([(z_0, z_1, \cdot\cdot, z_n)])$,

$$\tilde{V}(\tilde{z}) = A(\tilde{z}) - \langle A(\tilde{z}), \tilde{z} \rangle_{\mathbb{C}^{n+1}} \tilde{z}.$$

Thus

$$\tilde{V}(\tilde{z}) = (iz_1, iz_0, 0, \cdots, 0) - 2\text{Re}(z_0 \bar{z_1})(i\tilde{z}). \tag{3.5}$$

Lemma 3.1 — Let $\mu(s) : [0, l] \longrightarrow \mathbb{C}P^n$ be any geodesic in $\mathbb{C}P^n$. Then $\langle V(\mu(s)), \mu'(s) \rangle$ is a constant function on $[0, l]$.

Proof : Let $\tilde{\mu} : [0, l] \longrightarrow S^{2n+1}$ be a horizontal geodesic in S^{2n+1} such that $\pi \circ \tilde{\mu} = \mu$. Then

$$\begin{aligned} \langle V(\mu(s)), \mu'(s) \rangle &= \langle \tilde{V}(\tilde{\mu}(s)), \tilde{\mu}'(s) \rangle_{\mathbb{R}^{2n+2}} \\ &= \langle A(\tilde{\mu}(s)), \tilde{\mu}'(s) \rangle_{\mathbb{R}^{2n+2}}. \end{aligned}$$

Hence $\dfrac{d}{ds}\langle V(\mu(s)), \mu'(s) \rangle = \langle A(\tilde{\mu}'(s)), \tilde{\mu}'(s) \rangle_{\mathbb{R}^{2n+2}} = 0.$ □

4. Proof of the Main Theorem

Put $p := \pi(e_0)$ where $e_0 = (1, 0, \cdots, 0) \in S^{2n+1} \subset \mathbb{C}^{n+1}$. As seen in example 2.2, $c(s) = (\cos s, i \sin s, 0, \cdots 0)\ \forall\, 0 \le s \le \frac{\pi}{2}$ is a horizontal geodesic with $c(0) = e_0$ and $c'(0) = ie_1$. Then $\gamma := \pi oc$ is a geodesic in $\mathbb{C}P^n$.

For $0 < r_0 < r_1 < \frac{\pi}{2}$ (c.f § 1), fix a real number s_0 such that $0 \le s_0 < r_1 - r_0$. Put $\Omega_{s_0} := B(p, r_1) \setminus \overline{B(\gamma(s_0), r_0)}$. Let $B_0 := B(\gamma(s_0), r_0)$, $B_1 := B(p, r_1)$ and $\Omega := \Omega_{s_0}$. See figure 1.

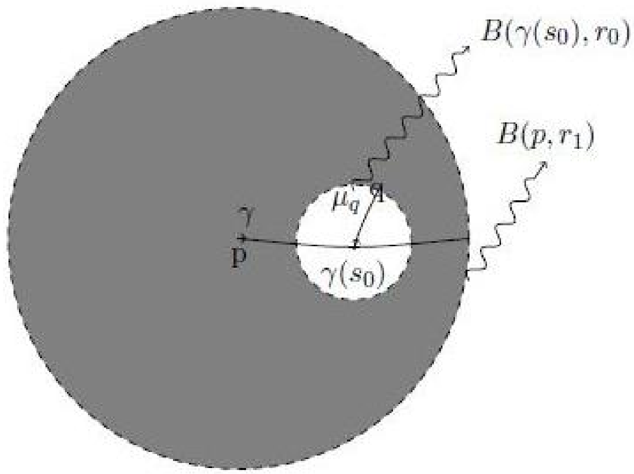

Figure 1: $B(p, r_1) \setminus \overline{B(\gamma(s_0), r_0)}$

Let $q \in \partial B_0$ be any point. Let $\mathbf{n}(q)$ be the outward unit normal of Ω on $\partial\Omega$ at $q \in \partial B_0$. Let μ_q be the unit speed geodesic in $\overline{B_0}$ such that $\mu_q(0) = \gamma(s_0)$ and $\mu_q(r_0) = q$. Put $v_q = \mu_q'(0)$. Then

$$\mu_q(s) = \exp_{\gamma(s_0)}(sv_q) \quad \forall 0 \le s \le r_0. \tag{4.1}$$

Then by Gauss's lemma, the outward unit normal of $\partial\Omega$ at $q \in \partial B_0$ is given by

$$\mathbf{n}(q) = -\frac{d}{ds}\mu_q(s)_{|s=r_0}. \tag{4.2}$$

Let

$$\partial B_0^+ = \{q \in \partial B_0 | \langle \mu_q'(0), \gamma'(s_0)\rangle > 0\}.$$

Lemma 4.1 — $\langle V(q), \mathbf{n}(q)\rangle < 0\ \forall\, q \in \partial B_0^+$.

Proof : Fix any $q \in \partial B_0^+$. By the Lemma 3.1, $\langle V(\mu_q(s)), \mu_q'(s)\rangle$ is a constant function on $[0, r_0]$. Now $V(\mu_q(0)) = \gamma'(s_0)$ and $\mathbf{n}(q) = -\mu_q'(r_0)$. Hence

$$\langle V(q), \mathbf{n}(q)\rangle = -\langle V(\mu_q(r_0)), \mu_q'(r_0)\rangle = -\langle V(\mu_q(0)), \mu_q'(0)\rangle = -\langle \gamma'(s_0), \mu_q'(0)\rangle.$$

Since $q \in \partial B_0^+$, $-\langle \gamma'(s_0), \mu_q'(0)\rangle < 0$. Hence the Lemma. □

Fix $c(s_0) \in \pi^{-1}(\gamma(s_0))$. Then by Lemma 2.3, $\exists$ unique $\tilde{q} \in \pi^{-1}(q)$ ($\tilde{q}$ depends on $c(s_0)$) such that $d(\tilde{q}, c(s_0)) = r_0$ and a unique horizontal geodesic $\tilde{\mu}_{\tilde{q}}$ in S^{2n+1} joining $c(s_0)$ to $\tilde{q}$ of length r_0. Let $\widetilde{v_q} := \tilde{\mu}'_{\tilde{q}}(0)$. Then

$$\tilde{\mu}_{\tilde{q}}(s) = (\cos s)c(s_0) + (\sin s)\widetilde{v_q} \quad \forall\, 0 \le s \le r_0 \tag{4.3}$$

and

$$\tilde{q} = (\cos r_0)c(s_0) + (\sin r_0)\widetilde{v_q}. \tag{4.4}$$

Recall that $\Omega = B_1 \setminus \overline{B_0}$. For any $z \in B(p, r_1)$ $(z \neq p)$, let η_z be the unit speed geodesic joining p to z with $p = \eta_z(0)$. Define

$$B^+(p, r_1) := \{z \in B(p, r_1) : \langle \eta_z'(0), \gamma'(0)\rangle > 0\}.$$

Lemma 4.2 — $d(\gamma(s_0), z) < \frac{\pi}{2} \quad \forall\, z \in\ B^+(p, r_1)$.

PROOF : Fix $\tilde{p} = (1, 0, \cdots, 0)$ in S^{2n+1}. Then by Lemma 2.3 there exists unique $\tilde{z}(\tilde{p})$ in $\pi^{-1}(z) \subset S^{2n+1}$ such that $d_{S^{2n+1}}(\tilde{p}, \tilde{z}(\tilde{p})) = d(p, z)$ and a unique unit speed horizontal geodesic $\tilde{\eta}_{\tilde{z}(\tilde{p})}$ of length $d(p, z)$ in S^{2n+1} joining $\tilde{p}$ to $\tilde{z}(\tilde{p})$. Let $a := d(p, z)$. Then $a < r_1 < \frac{\pi}{2}$.

Let $d_{S^{2n+1}}(c(s_0), \tilde{z}(\tilde{p})) =: x$. Let u be the velocity vector of the geodesic $\tilde{\eta}_{\tilde{z}(\tilde{p})}$ at $\tilde{p}$. Consider the spherical triangle $[\tilde{p}, \tilde{z}(\tilde{p}), c(s_0)]$ in S^{2n+1}. Let β be the angle of this triangle at $\tilde{p}$. Since $z \in B^+(p, r_1)$

$$\cos\beta = \langle u, ie_1\rangle_{\mathbb{R}^{2n+2}} = \langle \eta_z'(0), \gamma'(0)\rangle > 0. \tag{4.5}$$

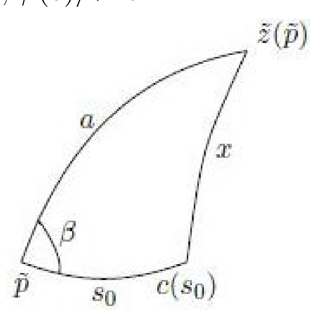

Figure 2: Spherical Triangle $[\tilde{p}, \tilde{z}(\tilde{p}), c(s_0)]$ in S^{2n+1}

Thus by cosine rule for spherical triangles, $\cos x = \cos a \cos s_0 + \sin a \sin s_0 \cos\beta$. Since $0 < a < \frac{\pi}{2}, 0 < s_0 < \frac{\pi}{2}$ and $\cos\beta > 0$ we get, $\cos x > 0$. Hence

$$0 < d_{S^{2n+1}}(c(s_0), \tilde{z}(\tilde{p})) = x < \frac{\pi}{2}. \tag{4.6}$$

It follows that $d(\gamma(s_0), z) \le x < \frac{\pi}{2}$. □

Let $z \in B^+(p, r_1)$ and $r := d(\gamma(s_0), z)$. Then by Lemma 4.2, $0 < r < \frac{\pi}{2}$. Let $\mu_z : [0, r] \longrightarrow B(p, r_1)$ be the unit speed geodesic joining $\gamma(s_0)$ to z with initial point $\mu_z(0) = \gamma(s_0)$ (c.f. 4.1). We can extend μ_z as a geodesic defined on whole of $\mathbb{R}$, and the extended geodesic is again denoted by μ_z. Put $v_z = \mu_z'(0)$.

Define

$$\mathcal{O} = \{z \in \Omega \cap B^+(p, r_1) : \langle \mu_z'(0), \gamma'(s_0) \rangle > 0\}.$$

See figure 3. Let $z \in \mathcal{O}$ and $z = \mu_z(r)$. By Lemma 2.3, $\exists\, \tilde{z} \in \pi^{-1}(z)$ such that $d(\tilde{z}, c(s_0)) = r$ and a unique horizontal geodesic $\tilde{\mu}_{\tilde{z}} : [0, r] \longrightarrow B(p, r_1)$ in S^{2n+1} joining $c(s_0)$ to $\tilde{z}$ of length r. Let $\tilde{v}_z := \tilde{\mu}_{\tilde{z}}'(0)$. Then

$$\tilde{z} = (\cos r) c(s_0) + (\sin r) \tilde{v}_z \tag{4.7}$$

Then w.r.t. the basis $\{c'(s_0), e_2, e_3 \cdots, e_n\}$ of $\mathcal{H}_{c(s_0)}$ at $c(s_0)$,

$$\tilde{v}_z = \langle \tilde{v}_z, c'(s_0) \rangle_{\mathbb{C}^{n+1}} c'(s_0) + \sum_{2 \le j \le n} \langle \tilde{v}_z, e_j \rangle_{\mathbb{C}^{n+1}} e_j$$

and hence

$$\langle \tilde{z}, e_0 \rangle_{\mathbb{C}^{n+1}} = \cos r \cos s_0 - \sin r \sin r_0 \langle \tilde{v}_z, c'(s_0) \rangle_{\mathbb{C}^{n+1}}.$$

Let $z' := \mu_z(-r)$. See figure 3. By Lemma 2.3 $\exists\, \tilde{z'} \in \pi^{-1}(z')$ such that $d(\tilde{z'}, c(s_0)) = r$. Then

$$\tilde{z'} = \cos(-r) c(s_0) + \sin(-r) \tilde{v}_z = (\cos r) c(s_0) + \sin(r)(-\tilde{v}_z).$$

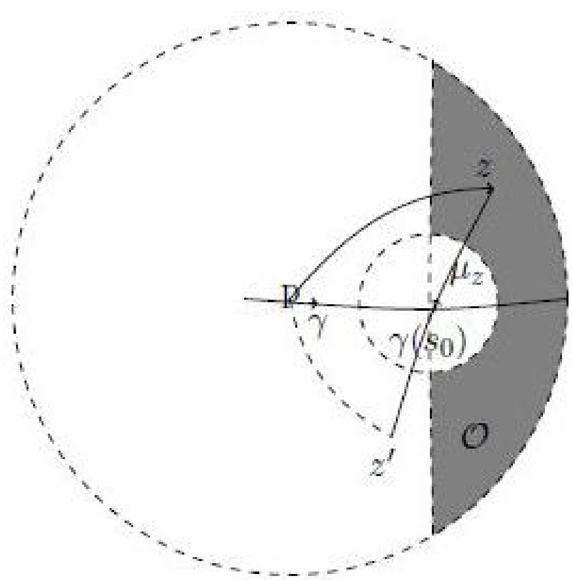

Figure 3: Domain $\mathcal{O}$

Lemma 4.3 — For $z \in \mathcal{O}$, $d(z', p) < d(z, p)$.

PROOF : Fix $z \in \mathcal{O}$. Recall that $\exists\, \tilde{z} \in \pi^{-1}(z)$ such that $d(\tilde{z}, c(s_0)) = r$. Put $\tilde{z} = (z_0, z_1, \cdots, z_n)$. Then

$$\begin{aligned}\langle \tilde{z}, c'(s_0)\rangle_{\mathbb{R}^{2n+2}} &= (\sin r)\langle \tilde{v}_z, c'(s_0)\rangle_{\mathbb{R}^{2n+2}}\\ &= \sin r\langle \mu_z'(0), \gamma'(s_0)\rangle\\ &> 0 \qquad (\because\ \sin r > 0 \text{ and } z \in \mathcal{O}).\end{aligned}$$

Put $b_z := \langle \tilde{v}_z, c'(s_0)\rangle_{\mathbb{C}^{n+1}}$. Since $z \in \mathcal{O}$, Re $b_z > 0$. From Lemma 2.3,

$$\begin{aligned}d(z,p) &= \arccos(|\langle \tilde{z}, e_0\rangle_{\mathbb{C}^{n+1}}|) \quad (\because p_1 = \pi(e_0))\\ &= \arccos(|\cos r\cos s_0 - \sin r\sin s_0 b_z|)\end{aligned} \tag{4.8}$$

$$\begin{aligned}|\cos r\cos s_0 + b_z\sin r\sin s_0|^2 &= (\cos r\cos s_0 + \mathrm{Re}(b_z)\sin r\sin s_0)^2 + (\mathrm{Im}(b_z)\sin r\sin s_0)^2\\ &> (\cos r\cos s_0 - \mathrm{Re}(b_z)\sin r\sin s_0)^2 + (\mathrm{Im}(b_z)\sin r\sin s_0)^2\\ &= |\cos r\cos s_0 - b_z\sin r\sin s_0|^2.\end{aligned} \tag{4.9}$$

Thus

$$|\cos r\cos s_0 + b_z\sin r\sin s_0| > |\cos r\cos s_0 - b_z\sin r\sin s_0|. \tag{4.10}$$

Thus

$$\begin{aligned}d(z',p) &= \arccos|\langle \tilde{z'}, e_0\rangle_{\mathbb{C}^{n+1}}|\\ &= \arccos|\cos r\cos s_0 + \sin r\sin s_0\, b_z|\\ &< \arccos|\cos r\cos s_0 - \sin r\sin s_0\, b_z| \text{ (from } 4.10)\\ &= d(z,p).\end{aligned} \tag{4.11}$$

Hence the lemma follows. □

Lemma 4.4 — For any $q \in \partial B_0$

$$\langle V(q'), \mathbf{n}(q')\rangle = -\langle V(q), \mathbf{n}(q)\rangle \quad \forall q \in \partial B_0.$$

PROOF : By Lemma 3.1, $\langle V(\mu_q(s)), \mu_q'(s)\rangle$ is a constant function for all s, $\mathbf{n}(q) = -\mu_q'(r_0)$ and $\mathbf{n}(q') = \mu_q'(-r_0)$. Hence the result. □

A cylindrical symmetry of the domain Ω : Consider the matrix

$$B := \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & -1 & & \\ & & & \ddots & \\ & & & & -1 \end{pmatrix}$$

in $U(n+1)$ where all the entries not shown are zero. Define a map $\phi_B : \mathbb{C}P^n \longrightarrow \mathbb{C}P^n$ by

$$\phi_B([(z_0, z_1, z_2 \cdots, z_n)]) = [B(z_0, z_1, z_2 \cdots, z_n)] \forall \ [(z_0, z_1, \cdots, z_n)] \in \mathbb{C}P^n.$$

Lemma 4.5 — The map ϕ_B is an isometry of Ω, $\phi \circ \gamma = \gamma$ and ϕ_B reverses any normal geodesic along γ.

PROOF : Since $B \in U(n+1)$, ϕ_B is an isometry of $\mathbb{C}P^n$. Define T_B denote the linear map of $\mathbb{C}^{n+1}$ given by left multiplication by the matrix B. Then $T_B(e_j) = e_j$ for $j = 0, 1$ and hence $T_B(c(s)) = c(s) \ \forall \ s$. Thus ϕ_B keeps the geodesic γ invariant. Hence $\phi_B(B(p, r_1)) = B(p, r_1)$ and $\phi_B(B(\gamma(s_0), r_0)) = B(\gamma(s_0), r_0)$. Thus $\phi_B(\Omega) = \Omega$ and ϕ_B is an isometry of Ω.

Now the constant vector fields $e_2, \cdot\cdot, e_n$ of $\mathbb{C}^{n+1}$ are horizontal, parallel vector fields of S^{2n+1} along the curve $c(s)$. Define $E_j(\gamma(s)) := d\pi_{c(s)}(e_j) \ (2 \leq j \leq n)$. Then each $E_j \ (2 \leq j \leq n)$ is a parallel vector field of $\mathbb{C}P^n$ along the geodesic γ. Since $T(e_j) = -e_j \ (2 \leq j \leq n)$ and ϕ_B is an isometry of $\mathbb{C}P^n$, $d\phi_B(E_j(\gamma(s))) = -E_j(\gamma(s)) \ \forall \ s$. Hence ϕ_B reverses any normal geodesic along γ. □

Now we prove the main theorem.

Consider any two pairs (p, v) and (q, w) where

$$p, q \in \mathbb{C}P^n, \ v \in T_p\mathbb{C}P^n \ \& \ w \in T_q\mathbb{C}P^n.$$

Then there exists an isometry f of $\mathbb{C}P^n$ such that $f(p) = q$ and $Df_p(v) = w$. Also the Laplace-Beltrami operator Δ of $\mathbb{C}P^n$ is invariant under isometries of $\mathbb{C}P^n$. Hence it suffices to study the first Dirichlet eigenvalue $\lambda_1(\Omega)$ only on domains

$$\Omega_s := B(p, r_1) \setminus \overline{B(\gamma(s), r_0)}; 0 \leq s < r_1 - r_0.$$

Define $\lambda_1(s) : (r_0 - r_1, r_1 - r_0) \longrightarrow \mathbb{R}$ by $\lambda_1(s) = \lambda_1(\Omega(\gamma_s))$. Fix s_0 such that $0 \leq s_0 < r_1 - r_0$. Now put $B_0 = B(\gamma(s_0), r_0)$ and $B_1 = B(p, r_1)$. Also fix r_2 such that $r_0 < r_2 < r_1 - r_0$.

Consider a smooth function $\rho : \mathbb{C}P^n \longrightarrow \mathbb{R}$ satisfying $\rho = 1$ on $\overline{B(\gamma(s_0), r_2)}$ and $\rho = 0$ on ∂B_1. Define a vector field W on $\mathbb{C}P^n$ by

$$W(q) = \rho(q)V(q) \; \forall \; q \in \mathbb{C}P^n \quad \text{(V is defined in §3).}$$

Let $\{\phi_s\}_{s\in\mathbb{R}}$ be the one-parameter family of diffeomorphisms of $\mathbb{C}P^n$ associated with W. Then for all $|s|$ sufficiently close to 0, $\phi_s|_{B(p,r_1)} = \psi_s|_{B(p,r_1)}$ and $\phi_s(\Omega_{s_0}) = \Omega_{s_0+s}$. Hence $\lambda_1(s_0 + s) = \lambda_1(\phi_s(\Omega_{s_0}))$. By Proposition 3.1 of [2], λ_1 is differentiable at $s = s_0$. As $\lambda_1(\phi_s(\Omega)) = \lambda_1(\phi_{-s}(\Omega))$, λ_1 is an even function which is differentiable at 0. Thus $\lambda_1'(0) = 0$. Put $\Omega = \Omega_{s_0}$. Let n be the outward unit normal field on $\partial\Omega$. Then by Hadamard formula stated in Proposition 3.4 of [2] we get,

$$\begin{aligned} \lambda_1'(s_0) &= -\int_{\partial\Omega} \left(\frac{\partial y_1}{\partial n}\right)^2 \langle W, n\rangle(q)dS \\ &= -\int_{\partial B_0} \left(\frac{\partial y_1}{\partial n}\right)^2 \langle W, n\rangle(q)dS \qquad (\because W = 0 \text{ on } \partial B_1). \end{aligned}$$

Since $W = V$ on $\overline{B(\gamma(s_0), r_2)}$ we have

$$\lambda_1'(s_0) = -\int_{\partial B_0} \left(\frac{\partial y_1}{\partial n}\right)^2 \langle V, n\rangle(q)dS. \tag{4.12}$$

Let $\mathcal{I}_{\gamma(s_0)}$ be the isometry such that $D\mathcal{I}_{\gamma(s_0)} : T_{\gamma(s_0)}\mathbb{C}P^n \longrightarrow T_{\gamma(s_0)}\mathbb{C}P^n$ is the antipodal map $-I$. Hence for any geodesic σ such that $\sigma(0) = \gamma(s_0)$, we have $\mathcal{I}_{\gamma(s_0)} o\sigma(s) = \sigma(-s)$. For $z \in \mathcal{O}$ define $z' := \mathcal{I}_{\gamma(s_0)}(z)$. Hence $z' := \mathcal{I}_{\gamma(s_0)}(z) = \mathcal{I}_{\gamma(s_0)}(\mu_z(r)) = \mu_z(-r)$.

$$\mathcal{I}_{\gamma(s_0)}(\partial B_0) = \partial B_0 \tag{4.13}$$

$d(z, p) < r_1$ for any $z \in \mathcal{O}$. Hence by Lemma 4.3

$$d(z', p) < d(z, p) < r_1.$$

Thus we have

$$\mathcal{I}_{\gamma(s_0)}(\mathcal{O}) \subset B(p, r_1). \tag{4.14}$$

Let $q \in \partial B_0^+$. Then by Lemma 4.1

$$\langle V(q), n(q)\rangle < 0. \tag{4.15}$$

And from Lemma 4.4 we have

$$\langle V(q'), n(q')\rangle = -\langle V(q), n(q)\rangle \qquad \forall q \in \partial B_0. \tag{4.16}$$

Hence from 4.12

$$\lambda_1'(s_0) = -\int_{\partial B_0 \cap \partial \mathcal{O}} \left\{ \left(\frac{\partial y_1}{\partial n}(q)\right)^2 - \left(\frac{\partial y_1}{\partial n}(q')\right)^2 \right\} \langle V, n\rangle(q) dS. \tag{4.17}$$

Let $\mathcal{P} := \{z \in B^+(p, r_1) \mid \langle \mu_z'(0), \gamma'(0)\rangle = 0\}$. Thus

$$\mathcal{I}_{\gamma(s_0)}(\mathcal{P}) = \mathcal{P}. \tag{4.18}$$

Also

$$D\mathcal{I}_{\gamma(s_0)}(n(q)) = n(\mathcal{I}_{\gamma(s_0)}(q)) \qquad \forall\, q \in \partial B_0. \tag{4.19}$$

The Laplace-Beltrami operator Δ of $\mathbb{C}P^n$ is uniformly elliptic. Since $y_1 > 0$ on Ω, by Hopf-maximum principle

$$\frac{\partial y_1}{\partial n}(q) < 0 \qquad \forall\, q \in \partial\Omega \tag{4.20}$$

Put $\tilde{y_1}(z) = (y_1 o \mathcal{I}_{\gamma(s_0)})(z) \qquad \forall\, z \in \mathcal{O}$. Then from 4.19 we have

$$\frac{\partial \tilde{y_1}}{\partial n}(q) = \frac{\partial y_1}{\partial n}(q') \qquad \forall\, q \in \partial\Omega \cap \partial\mathcal{O}. \tag{4.21}$$

Define $\omega(z) := y_1(z) - \tilde{y_1}(z)\ \forall\, z \in \mathcal{O}$. Since Δ commutes with the isometry $\mathcal{I}_{\gamma(s_0)}$ and using 4.18, ω satisfies the following:

$$\left.\begin{aligned} \Delta\omega + \lambda_1\omega = 0 &\qquad \text{in } \mathcal{O} \\ \omega = 0 &\qquad \text{on } \partial\mathcal{O} \cap \partial B_0 \\ \omega = 0 &\qquad \text{on } \partial\mathcal{O} \cap \mathcal{P} \text{ (by lemma } 4.5) \\ \omega < 0 &\qquad \text{on } \partial\mathcal{O} \cap \partial B_1. \end{aligned}\right\} \tag{4.22}$$

Define $\omega^+ = \max\{\omega, 0\}$. Since $\omega \leq 0$ on $\partial\mathcal{O}$,$\omega^+ \in H_0^1(\mathcal{O})$. From problem 1.1 and 4.22, we get

$$\begin{aligned} 0 &= \int_{\mathcal{O}} (-\Delta\omega)\omega^+ dx - \lambda_1(\Omega) \int_{\mathcal{O}} \omega\omega^+ dx \\ &= \int_{\mathcal{O}\cap\{\omega\geq 0\}} (-\Delta\omega)\omega^+ dx - \lambda_1(\Omega) \int_{\mathcal{O}\cap\{\omega\geq 0\}} \omega\omega^+ dx \\ &= \int_{\mathcal{O}} (\parallel \nabla\omega^+ \parallel^2 dx - \lambda_1(\Omega) \int_{\mathcal{O}} (\omega^+)^2 dx \\ &> \int_{\mathcal{O}} (\parallel \nabla\omega^+ \parallel^2 dx - \lambda_1(\mathcal{O}) \int_{\mathcal{O}} (\omega^+)^2 dx \quad (\because \lambda_1(\Omega) < \lambda_1(\mathcal{O})). \end{aligned}$$

Thus $\lambda_1(\mathcal{O}) > \frac{\int_{\mathcal{O}}(\|\nabla\omega^+\|)^2 dx}{\int_{\mathcal{O}}(\omega^+)^2 dx}$ which is a contradiction. Hence $\omega^+ \equiv 0$. So $\omega \leq 0$ in $\mathcal{O}$. It follows that $\Delta\omega + \lambda_1\omega \geq 0$ on $\mathcal{O}$. By generalized maximum principle $\omega < 0$ in $\mathcal{O}$. Then by applying Hopf-maximum principle $\frac{\partial\omega}{\partial n} > 0$ a.e.on $\partial\mathcal{O} \cap \partial B_0$. Consequently we get

$$|\frac{\partial y_1}{\partial n}(q)| < |\frac{\partial y_1}{\partial n}(q')| \quad (\text{from } 4.20 \text{ and } 4.21).$$

Hence from 4.17, $\lambda_1'(s_0) < 0$.

ACKNOWLEDGEMENT

The second author is grateful to M.K. Vemuri for useful discussions on this research topic.

REFERENCES

1. S. Kesavan, On two functionals connected to the Laplacian in a class of doubly connected domains, *Proc. R. Soc. Edinburgh*, **133**A (2003), 617-624.
2. M. H. C. Anisa and A. R. Aithal, On two functionals connected to the Laplacian in a class of doubly connected domains in space-forms, *Proc. Indian Acad. Sci.*, **115**(1) (2005), 93-102.
3. M. H. Chorwadwala Anisa and M. K. Vemuri, On two functionals connected to the Laplacian in a class of doubly connected domains of rank one symmetric non-compact type, *Geom Dedicata*, **167** (2013), 11-21.
4. T. Aubin, *Non-linear analysis on manifolds-Monge-Ampere equations*, Springer-Verlag, (1982).
5. M. Protter and H. Weinberger, *Maximum principles in differential equations*, Prentice-Hall Inc., (1967).
6. J. Cheeger and D. G. Ebin, *Comparisons in Riemannian geometry*, North Holland Publishing Company, (1975).