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ON MULTIPLICITY OF QUADRILATERALS IN COMPLETE GRAPHS[1]

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In this paper, we determine a lower bound for the multiplicity of quadrilaterals (cycles on four vertices) in complete graph K_n for any positive integer n.

Key words : Complete graphs; edge coloring; monochromatic; multiplicity; quadrilaterals.

1. INTRODUCTION

A graph with colored edges is said to be monochromatic if all its edges have the same color. If G and F are graphs, define $M(G, F)$ to be the minimum number of monochromatic copies of G that occur in any 2-coloring of the edges of F. $M(G, F)$ is called the multiplicity of G in F.

Consider the complete graph K_n on n vertices, say, $v_1, v_2, ..., v_n$. Color the edges of K_n with two colors (say red and blue) arbitrarily. We call this a '2-coloring of K_n'. Without loss of generality, we may assume that the number of red edges is always less than or equal to the number of blue edges in any 2-coloring of K_n. In a 2-coloring C of K_n, let r_i and b_i denote the red and blue degrees of the vertex v_i and let $\langle r\rangle = (r_1, r_2, ..., r_n)$ and $\langle b\rangle = (b_1, b_2, ..., b_n)$ denote the red and blue degree sequences of C.

In 1959, Goodman [3] determined the multiplicity of triangles in the complete graph K_n for any positive integer n. This result has been re-proved and extended in various ways by Sauve [8], Lorden [5] and Schwenk [9]. In 1962, Moon and Moser [6] gave a lower bound for $M(C_4, K_{n,n})$. The leading term in their bound agreed with the expected value of this quantity. In 2015, Rukmani and

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Vijayalakshmi [7] gave an upper bound for $M(C_4, K_{n,n})$. In this paper, we give a lower bound for $M(C_4, K_n)$. The mean and variance of the number of monochromatic copies of C_4 in K_n are $\frac{3}{8}\binom{n}{4}$ and $\frac{3}{64}\binom{n}{4}(4n-7)$, respectively.

For any simple connected graph G, the Ramsey number $r(G)$ is defined as the smallest positive integer N such that every 2-coloring of the edges of the complete graph K_N must contain a monochromatic copy of G. The Ramsey multiplicity $R(G)$ is defined as the minimum number of monochromatic copies of G in any 2-coloring of the edges of $K_{r(G)}$. In 1972, Chvatal and Harary [1] determined that $r(C_4) = 6$ and in 1974 Harary and Prins [4] found $R(C_4) = 2$. Hence, $M(C_4, K_n) = 0$ for $n < 6$ and $M(C_4, K_6) = 2$. In this paper, we determine a lower bound for $M(C_4, K_n)$, $n \geq 7$.

2. A Lower Bound for the Multiplicity of Quadrilaterals

Let the vertices of K_n be $v_1, v_2, ..., v_n$. Corresponding to any 2-coloring C of K_n, we associate an $n \times n$ symmetric matrix $A = [a_{ij}]$ whose entries are given by

$$a_{ij} = \begin{cases} 1, & \text{if } v_iv_j \text{ is a red edge;} \\ 0, & \text{if } v_iv_j \text{ is a blue edge;} \\ \infty, & \text{if } v_i = v_j. \end{cases}$$

There are r_i 1's and $n-1-r_i$ 0's in the i^{th} row of A. Let the $\binom{n}{2}$ pairs of columns of A be numbered $1, 2, 3, ..., N = \binom{n}{2}$. If v denotes the pair (k, l), let t_v denote the number of rows i in which $a_{ik} = a_{il} = 1$ and h_v denote the number of rows i in which $a_{ik} = a_{il} = 0$ $(1 \leq i \leq n)$. It can be observed that

$$\sum_{v=1}^{N} t_v = \sum_{i=1}^{n} \binom{r_i}{2} = R \text{ (say) and}$$
$$\sum_{v=1}^{N} h_v = \sum_{i=1}^{n} \binom{n-1-r_i}{2} = B \text{ (say)}.$$

A monochromatic quadrilateral in C corresponds to a 2×2 submatrix in A whose elements are all 1's or all 0's. If P and Q denote the number of red and blue quadrilaterals in a given 2-coloring C, then

$$P + Q = \frac{1}{2}\left[\sum_{v=1}^{N} \binom{t_v}{2} + \sum_{v=1}^{N} \binom{h_v}{2}\right],$$

since each pair of rows included in the rows counted by $\binom{t_v}{2}$ and $\binom{h_v}{2}$ determines a monochromatic quadrilateral and each such quadrilateral is counted twice, once for each pair of diagonally opposite vertices it contains.

Definition 2.1 — A sequence of non-negative integers $(a_1, a_2, ..., a_n)$ is said to be balanced if $|a_i - a_j| \leq 1$ for all i, j.

Lemma 2.2 — Let $w_1, w_2, ..., w_s$ be a sequence of $s(\geq 2)$ non-negative integers and let $W = \sum_{i=1}^{s} w_i$. If $W = a_W s + b_W$, where $0 \leq b_W < s$, then

$$\sum_{i=1}^{s} \binom{w_i}{2} \geq (s - b_W)\binom{a_W}{2} + b_W\binom{a_W + 1}{2} = s\binom{a_W}{2} + a_W b_W \qquad (\#)$$

with equality holding iff the sequence $\langle w \rangle$ is balanced.

PROOF : Suppose that $\langle w \rangle$ is not balanced and that $w_i - w_j \geq 2$ for some $i \neq j$. Let $\langle w' \rangle$ denote the sequence that differs from $\langle w \rangle$ only in that $w_i' = w_i - 1$ and $w_j' = w_j + 1$. The sums of the elements of the two sequences both equal W. But

$$\binom{w_i}{2} + \binom{w_j}{2} - \binom{w_i - 1}{2} - \binom{w_j + 1}{2} = w_i - w_j - 1 \geq 1.$$

So the minimum value of the sum in the left-hand side of $(\#)$, subject to the given constraints, can occur only for a balanced sequence. But equality certainly holds when $\langle w \rangle$ is balanced. This implies the required result.

Notation : When working with the sequences $\langle t \rangle$ and $\langle h \rangle$, where W equals R or B and $s = N$, we will usually let $f(R)$ and $f(B)$ denote the expressions on the right-hand side of inequality $(\#)$. We usually assume that $b_W < N$ in these cases; but it will not occasionally be convenient to allow for the possibility that $b_W = N$. This does not affect the value of $f(W)$ since $N\binom{a_W}{2} + Na_W = N\binom{a_W+1}{2}$.

Lemma 2.3 — Let C be a 2-coloring of K_n. Let R, B, P and Q be defined as above and $N = \binom{n}{2}$. Then $P + Q \geq \frac{1}{2}[f(R) + f(B)]$.

PROOF : Let $R = \sum_{v=1}^{N} t_v = a_R N + b_R$ where $a_R \geq 0$, $0 \leq b_R \leq N - 1$ and $B = \sum_{v=1}^{N} h_v = a_B N + b_B$ where $a_B \geq 0$, $0 \leq b_B \leq N - 1$. Then

$$P = \frac{1}{2}\sum_{v=1}^{N} \binom{t_v}{2} \geq \frac{1}{2} f(R) \quad \text{by Lemma 2.2}$$

$$Q = \frac{1}{2}\sum_{v=1}^{N} \binom{h_v}{2} \geq \frac{1}{2} f(B) \quad \text{by Lemma 2.2}$$

Therefore, $P + Q \geq \frac{1}{2}[f(R) + f(B)]$.

Lemma 2.4 — Let C and C' be two 2-colorings of K_n, $n \geq 7$, with red degree sequences $\langle r \rangle = (r_1, r_2, ..., r_n)$ and $\langle r' \rangle = (r_1', r_2', ..., r_n')$ such that $\sum_{i=1}^{n} r_i = \sum_{i=1}^{n} r_i'$. If $\langle r' \rangle$ is a balanced sequence but $\langle r \rangle$ is not, then $f(R) + f(B) > f(R') + f(B')$.

PROOF : Let $R = a_R N + b_R$ and $R^{'} = a_{R'} N + b_{R'}$ where $a_R \geq 0, a_{R'} \geq 0$ and $0 \leq b_R, b_{R'} \leq N - 1$. When Lemma 2.2 is applied to $\langle r \rangle$ with $s = n$, we get $R > R^{'}$, so $a_R \geq a_{R'}$. Now

$$\begin{aligned} f(R) - f(R^{'}) &= N\left[\binom{a_R}{2} - \binom{a_{R'}}{2}\right] + b_R a_R - b_{R'} a_{R'} \\ &= \frac{1}{2} N (a_R - a_{R'})(a_R + a_{R'} - 1) + b_R a_R - b_{R'} a_{R'}. \end{aligned}$$

Case i : $a_R = a_{R'}$. This implies that $b_R > b_{R'}$, so

$$f(R) - f(R^{'}) = (b_R - b_{R'}) a_R \geq 0$$

with equality holding only when $a_R = a_{R'} = 0$. That is, $f(R) = f(R^{'}) = 0$. We shall pursue this further shortly.

Case ii : $a_R \geq a_{R'} + 1$. This implies that

$$\begin{aligned} f(R) - f(R^{'}) &\geq N a_{R'} - b_{R'} a_{R'} + b_R a_R \\ &\geq a_{R'} + b_R \geq 0 \end{aligned}$$

with equality holding only when $a_{R'} = b_R = 0$. If $a_{R'} = 0$, then $f(R^{'}) = 0$. And if $b_R = 0$, then $R = a_R N$, where $a_R \geq 1$. If $a_R \geq 2$, then $f(R) - f(R^{'}) = N\binom{a_R}{2} > 0$. If $a_R = 1$, then $f(R) = f(R^{'}) = 0$. So $f(R) - f(R^{'}) > 0$ unless $f(R) = f(R^{'}) = 0$.

Similarly, we find that $f(B) > f(B^{'})$ unless $f(B) = f(B^{'}) = 0$; consequently, $f(R) + f(B) > f(R^{'}) + f(B^{'})$ unless $f(R) = f(B) = f(R^{'}) = f(B^{'}) = 0$. It remains to show that this cannot happen when $n \geq 7$. For, if $f(R) = f(B) = 0$, then $R \leq N$ and $B \leq N$, so $R + B \leq 2N$. But

$$\begin{aligned} R + B - 2N &= \sum_{i=1}^{n} \binom{r_i}{2} + \sum_{i=1}^{n} \binom{n-1-r_i}{2} - 2N \\ &= \sum_{i=1}^{n} \left[\binom{n-1}{2} - r_i(n-1-r_i)\right] - n(n-1) \\ &\geq \frac{n(n-1)(n-4)}{2} - \frac{n(n-1)^2}{4} \\ &= \frac{n(n-1)(n-7)}{4} > 0 \ \text{ when } n > 7. \end{aligned}$$

Equality could hold only when $n = 7$ and $r_i = 3$ for $1 \leq i \leq 7$. But this cannot happen since $\langle r \rangle$ is not balanced. This suffices to complete the proof of the Lemma.

Remark : In view of Lemma 2.4 we may restrict our attention to balanced sequences $\langle r \rangle$ from now on.

Lemma 2.5 — Let C and $C^{'}$ be two 2-colorings of $K_n, n \geq 7$, with balanced red degree sequences $\langle r \rangle = (r_1, r_2, ..., r_n)$ and $\langle r^{'} \rangle = (r^{'}_1, r^{'}_2, ..., r^{'}_n)$ respectively such that

$$r_i = \begin{cases} p, & \text{for } k \text{ values of } i; \\ p+1, & \text{for } n-k \text{ values of } i; \end{cases} \quad \text{and } r^{'}_i = \begin{cases} p, & \text{for } k-1 \text{ values of } i; \\ p+1, & \text{for } n-k+1 \text{ values of } i; \end{cases}$$

where $1 \leq k \leq n$ and $0 \leq p \leq \begin{cases} \frac{n-4}{2}, & \text{if } n \text{ is even;} \\ \frac{n-5}{2}, & \text{if } n \text{ is odd.} \end{cases}$

Then $f(R) + f(B) > f(R^{'}) + f(B^{'})$.

PROOF : Let $R = a_R N + b_R$ and $B = a_B N + b_B$ where $a_R \geq 0$, $a_B \geq 0$ and $0 \leq b_R, b_B \leq N-1$. The assumptions on $r_i, r^{'}_i$ and p imply that

$$\begin{aligned} R &= k\binom{p}{2} + (n-k)\binom{p+1}{2} = n\binom{p+1}{2} - kp \text{ and} \\ B &= k\binom{n-1-p}{2} + (n-k)\binom{n-2-p}{2} = n\binom{n-2-p}{2} + k(n-2-p). \end{aligned}$$

$$\begin{aligned} \Rightarrow B - R &= n\left[\binom{n-2-p}{2} - \binom{p+1}{2}\right] + k(n-2) \\ &= \frac{n}{2}(n-2)(n-3-2p) + k(n-2) \\ &\geq \frac{n(n-2)}{2} + (n-2) = N + \frac{n-4}{2}. \end{aligned} \quad (1)$$

$$\begin{aligned} \text{Thus,} \quad (a_B - a_R)N &= B - R + b_R - b_B \geq N + \frac{n-4}{2} + 1 - N \\ &= \frac{n-2}{2} > 0, \text{ since } n \geq 7. \\ \Rightarrow a_B - a_R &\geq 1. \end{aligned}$$

Also, $R^{'} = R + p = a_R N + b_R + p$ and $B^{'} = B - (n-p-2) = a_B N + b_B - (n-p-2)$.

Case A : $b_R + p < N$.

$$f(R) - f(R^{'}) = N\binom{a_R}{2} + b_R a_R - N\binom{a_R}{2} - (b_R + p)a_R = -pa_R.$$

Case B : $b_R + p \geq N$.

$$\begin{aligned} f(R) - f(R^{'}) &= N\binom{a_R}{2} + b_R a_R - N\binom{a_R+1}{2} - (b_R + p - N)(a_R + 1) \\ &= -pa_R - (b_R + p - N). \end{aligned}$$

Case C : $b_B - (n-p-2) \geq 0$.

$$\begin{aligned} f(B) - f(B^{'}) &= N\binom{a_B}{2} + b_B a_B - N\binom{a_B}{2} - (b_B - (n-p-2))a_B \\ &= (n-p-2)a_B. \end{aligned}$$

Case D : $b_B - (n-p-2) < 0$.

$$\begin{aligned} f(B) - f(B^{'}) &= N\binom{a_B}{2} + b_B a_B - N\binom{a_B - 1}{2} \\ &\qquad - (N + b_B - (n-p-2))(a_B - 1) \\ &= (n-p-2)(a_B - 1) + b_B. \end{aligned}$$

Now we want to prove that $f(R) + f(B) - f(R^{'}) - f(B^{'}) > 0$. Here we consider four cases.

Case 1 : $b_R + p < N$ and $b_B - (n-p-2) \geq 0$.

$$\begin{aligned} f(R) + f(B) - f(R^{'}) - f(B^{'}) &= -pa_R + (n-p-2)a_B \\ &= (n-2p-2)a_B + p(a_B - a_R) \\ &\geq 2a_B + p \geq 2. \end{aligned}$$

Case 2 : $b_R + p < N$ and $b_B - (n-p-2) < 0$.

$$\begin{aligned} f(R) + f(B) - f(R^{'}) - f(B^{'}) &= -pa_R + (n-p-2)(a_B - 1) + b_B \\ &\geq (n-2p-2)(a_B - 1) + b_B \\ &\geq 2(a_B - 1) + b_B \\ &> 0. \ (\text{since } B > N \text{ for } n \geq 7) \end{aligned}$$

Case 3 : $b_R + p \geq N$ and $b_B - (n-p-2) \geq 0$.

$$f(R) + f(B) - f(R^{'}) - f(B^{'})$$

$$\begin{aligned} &= -pa_R - (b_R + p - N) + (n-p-2)a_B \\ &\geq -p(a_B - 1) - (p-1) + (n-p-2)a_B \\ &= (n-2p-2)a_B + 1 \\ &\geq 2a_B + 1 \\ &\geq 3. \end{aligned}$$

Case 4 : $b_R + p \geq N$ and $b_B - (n-p-2) < 0$.

We observe that in this case

$$\begin{aligned}(a_B - a_R)N &= B - R + b_R - b_B \\ &> N + \frac{n-4}{2} + N - n + 2 \quad \text{(by (1))} \\ &= 2N - \frac{n}{2} > N. \\ \Rightarrow a_B - a_R &\geq 2.\end{aligned}$$

$$f(R) + f(B) - f(R^{'}) - f(B^{'})$$

$$\begin{aligned}&= -pa_R - (b_R + p - N) + (n-p-2)(a_B - 1) + b_B \\ &\geq -p(a_B - 2) - (p-1) + (n-p-2)(a_B - 1) + 0 \\ &= (n-2p-2)(a_B - 1) + 1 \\ &\geq 2(a_B - 1) + 1 \geq 3.\end{aligned}$$

From all the above four cases, we have $f(R) + f(B) > f(R^{'}) + f(B^{'})$.

Corollary 2.6 — Let C be a 2-coloring of K_n with balanced red degree sequence $\langle r \rangle = (r_1, r_2, ..., r_n)$ where $r_i = \begin{cases} p, & \text{for } k \text{ values of } i; \\ p+1, & \text{for } n-k \text{ values of } i; \end{cases}$ $1 \leq k \leq n$ and $0 \leq p \leq \begin{cases} \frac{n-4}{2}, & \text{if } n \text{ is even;} \\ \frac{n-5}{2}, & \text{if } n \text{ is odd.} \end{cases}$

Let $C^{'}$ be an another 2-coloring of K_n with balanced red degree sequence $\langle r^{'} \rangle = (r_1^{'}, r_2^{'}, ..., r_n^{'})$ where $r_i^{'} = \begin{cases} \frac{n-2}{2}, & \text{if } n \text{ is even;} \\ \frac{n-3}{2}, & \text{if } n \text{ is odd.} \end{cases}$

Then $f(R) + f(B) > f(R^{'}) + f(B^{'})$.

PROOF : By repeated application of Lemma 2.5 for the red degree sequence $\langle r \rangle$ of C starting with $p = 0$ and $k = n$, we get the result.

***Theorem* 2.7** — *For any integer* $n \geq 7$,

$$M(C_4, K_n) \geq \begin{cases} u(u-1)(4u^2 - 7u + 2), & if\ n = 4u; \\ u(4u+1)(u-1)^2, & if\ n = 4u+1; \\ \frac{u}{2}(u-1)(4u-1)(2u+1), & if\ n = 4u+2; \\ \frac{u^2}{2}(8u^2 + 2u - 7), & if\ n = 4u+3. \end{cases}$$

PROOF : From Lemma 2.3, we have $P + Q \geq \frac{1}{2}[f(R) + f(B)] \geq \frac{1}{2}(\min_{R,B}[f(R) + f(B)])$. By Lemma 2.4, $f(R) + f(B)$ is minimum only when $\langle r \rangle$ and $\langle b \rangle$ are balanced. By the assumption that the number of red edges is less than or equal to the number of blue edges and by Corollary 2.6, it is enough to find the $\min_{R,B}[f(R)+f(B)]$ among the colorings whose red degree sequences are given by $\langle r\rangle = (r_1, r_2, ..., r_n)$ where

$$r_i = \begin{cases} p, & \text{for } k \text{ values of } i; \\ p+1, & \text{for } n-k \text{ values of } i; \end{cases}$$

where p equals $\frac{n-2}{2}$ or $\frac{n-3}{2}$ according as n is even or odd, respectively. If n is even, $\frac{n}{2} \le k \le n$ and if n is odd, $0 \le k \le n$. Since the number of vertices of odd degree must be even, we find by inspection that k must be even except when $n = 4u+3$ where it must be odd.

In what follows we shall be giving expressions for $R = n\binom{p+1}{2} - kp$ of the form $R = a_R N + b_R$ where $a_R \ge 0$ and $0 \le b_R \le N$. It can happen that a given expression for b_R does not satisfy the required inequalities for all values in the range of the variable k. In such cases, we give different expressions for a_R and b_R for different subintervals of the range of k. For each such expression, it is easy to verify that $0 \le b_R \le N$ in the subinterval indicated. So, we omit the details of these verifications.

Case 1 : $n = 4u, u \ge 2, N = 2u(4u-1), p = 2u-1, 2u \le k \le 4u$ and k is even.

In this case

$$\begin{aligned} R &= 4u\binom{2u}{2} - k(2u-1) = Nu - 2u^2 - k(2u-1) \\ &= \begin{cases} N(u-1) + N - 2u^2 - k(2u-1), & \text{if } 2u \le k \le 3u; \\ N(u-2) + 2N - 2u^2 - k(2u-1), & \text{if } 3u+1 \le k \le 4u. \end{cases} \end{aligned}$$

Hence,

$$f(R) = \begin{cases} N\binom{u-1}{2} + (N - 2u^2 - k(2u-1))(u-1), & \text{if } 2u \le k \le 3u; \\ N\binom{u-1}{2} + (N - 2u^2 - k(2u-1))(u-2), & \text{if } 3u+1 \le k \le 4u. \end{cases}$$

Note that we have used the relation $\binom{u-2}{2} + (u-2) = \binom{u-1}{2}$ in re-writing the last expression.

Similarly,

$$\begin{aligned} B &= 4u\binom{2u-1}{2} + k(2u-1) \\ &= N(u-1) - 2u(u-1) + k(2u-1) \ \text{ if } 2u \le k \le 4u. \end{aligned}$$

Hence,

$$f(B) = N\binom{u-1}{2} + (k(2u-1) - 2u^2 + 2u)(u-1) \ \text{ if } 2u \le k \le 4u.$$

Therefore,

$$\begin{aligned} f(R)+f(B) &= 2N\binom{u-1}{2}+4u^2(u-1) \text{ if } 2u\le k\le 3u \text{ and} \\ f(R)+f(B) &= 2N\binom{u-1}{2}+2u(2u^2-5u+1)+k(2u-1) \\ &\ge 2N\binom{u-1}{2}+4u^3-4u^2+u-1 \text{ if } 3u+1\le k\le 4u. \end{aligned}$$

So the minimum value occurs when $2u\le k\le 3u$ and

$$M(C_4,K_n)\ge \frac{1}{2}\left[2N\binom{u-1}{2}+4u^2(u-1)\right]=u(u-1)(4u^2-7u+2).$$

Case 2 : $n=4u+1, u\ge 2, N=2u(4u+1), p=2u-1, 0\le k\le 4u+1$ and k is even.

In this case

$$\begin{aligned} R &= (4u+1)\binom{2u}{2}-k(2u-1)=Nu-4u^2-u-k(2u-1) \\ &= \begin{cases} N(u-1)+N-4u^2-u-k(2u-1), & \text{if } 0\le k\le 2u; \\ N(u-2)+2N-4u^2-u-k(2u-1), & \text{if } 2u+2\le k\le 4u. \end{cases} \end{aligned}$$

Hence,

$$f(R)=\begin{cases} N\binom{u-1}{2}+(N-4u^2-u-k(2u-1))(u-1), & \text{if } 0\le k\le 2u; \\ N\binom{u-1}{2}+(N-4u^2-u-k(2u-1))(u-2), & \text{if } 2u+2\le k\le 4u. \end{cases}$$

Similarly,

$$\begin{aligned} B &= (4u+1)\binom{2u}{2}+k(2u) \\ &= \begin{cases} N(u-1)+N-4u^2-u+2ku, & \text{if } 0\le k\le 2u; \\ Nu+2ku-4u^2-u, & \text{if } 2u+2\le k\le 4u. \end{cases} \end{aligned}$$

Hence,

$$f(B)=\begin{cases} N\binom{u}{2}+(2ku-4u^2-u)(u-1), & \text{if } 0\le k\le 2u; \\ N\binom{u}{2}+(2ku-4u^2-u)u, & \text{if } 2u+2\le k\le 4u. \end{cases}$$

Therefore,

$$\begin{aligned} f(R)+f(B) &= 2N\binom{u-1}{2}+N(u-1)+k(u-1) \\ &\ge 2N\binom{u-1}{2}+N(u-1) \text{ if } 0\le k\le 2u \text{ and} \\ f(R)+f(B) &= 2N\binom{u-1}{2}+N(u-1)+k(5u-2)-8u^2-2u \\ &\ge 2N\binom{u-1}{2}+N(u-1)+2u^2+4u-4 \text{ if } 2u+2\le k\le 4u. \end{aligned}$$

So the minimum value occurs when $k = 0$ and

$$M(C_4, K_n) \geq \frac{1}{2}\left[2N\binom{u-1}{2} + N(u-1)\right] = u(4u+1)(u-1)^2.$$

Case 3 : $n = 4u+2, u \geq 2, N = (2u+1)(4u+1), p = 2u, 2u+1 \leq k \leq 4u+2$ and k is even.

In this case,

$$\begin{aligned} R &= (4u+2)\binom{2u+1}{2} - k(2u) = Nu + 2u^2 + u - 2ku \\ &= N(u-1) + N + 2u^2 + u - 2ku \text{ if } 2u+2 \leq k \leq 4u+2. \end{aligned}$$

Hence,

$$\Rightarrow f(R) = N\binom{u-1}{2} + (N + 2u^2 + u - 2ku)(u-1) \text{ if } 2u+2 \leq k \leq 4u+2.$$

Similarly,

$$\begin{aligned} B &= (4u+2)\binom{2u}{2} + k(2u) \\ &= \begin{cases} N(u-1) + N - 6u^2 - 3u + 2ku, & \text{if } 2u+2 \leq k \leq 3u+1; \\ Nu - 6u^2 - 3u + 2ku, & \text{if } 3u+2 \leq k \leq 4u+2. \end{cases} \end{aligned}$$

Hence,

$$f(B) = \begin{cases} N\binom{u}{2} + (2ku - 6u^2 - 3u)(u-1), & \text{if } 2u+2 \leq k \leq 3u+1; \\ N\binom{u}{2} + (2ku - 6u^2 - 3u)u, & \text{if } 3u+2 \leq k \leq 4u+2. \end{cases}$$

Therefore,

$$\begin{aligned} f(R) + f(B) &= 2N\binom{u-1}{2} + N(u-1) + (4u^2 + 4u + 1)(u-1) \\ &= 2N\binom{u-1}{2} + N(u-1) + 4u^3 - 3u - 1 \text{ if } 2u+2 \leq k \leq 3u+1 \text{ and} \\ f(R) + f(B) &= 2N\binom{u-1}{2} + N(u-1) + 4u^3 - 6u^2 - 6u + 2ku - 1 \\ &\geq 2N\binom{u-1}{2} + N(u-1) + 4u^3 - 2u - 1 \text{ if } 3u+2 \leq k \leq 4u+2. \end{aligned}$$

So the minimum value occurs when $2u + 2 \leq k \leq 3u + 1$ and

$$M(C_4, K_n) \geq \frac{1}{2}\left[2N\binom{u-1}{2} + N(u-1) + 4u^3 - 3u - 1\right] = \frac{u}{2}(u-1)(4u-1)(2u+1).$$

Case 4 : $n = 4u + 3, u \geq 1, N = (4u + 3)(2u + 1), p = 2u, 0 \leq k \leq n$ and k is odd.

In this case

$$\begin{aligned} R &= (4u+3)\binom{2u+1}{2} - 2ku = Nu - 2ku \\ &= N(u-1) + N - 2ku \ \text{ if } 1 \leq k \leq 4u+3. \end{aligned}$$

Hence,

$$f(R) = N\binom{u}{2} - 2ku(u-1) \ \text{ if } 1 \leq k \leq 4u+3.$$

Similarly,

$$B = (4u+3)\binom{2u+1}{2} + k(2u+1) = Nu + k(2u+1) \ \text{ if } 1 \leq k \leq 4u+3.$$

Hence,

$$f(B) = N\binom{u}{2} + k(2u+1)u \ \text{ if } 1 \leq k \leq 4u+3.$$

Therefore,

$$f(R) + f(B) = 2N\binom{u}{2} + 3ku \ \text{ if } 1 \leq k \leq 4u+3.$$

So the minimum value occurs when $k = 1$ and

$$M(C_4, K_n) \geq \frac{1}{2}\left[2N\binom{u}{2} + 3u\right] = \frac{u^2}{2}(8u^2 + 2u - 7).$$

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