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A FREE BOUNDARY PROBLEM FOR A REACTION-DIFFUSION EQUATION APPEARING IN BIOLOGY

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This works studies a quasilinear, reaction-diffusion type parabolic free boundary problem modelling population dynamics. Hölder norm á priori estimates are established for the free boundary and the solution. Uniqueness of the solution is shown and qualitative properties of the solution are investigated.

Key words : Free boundary problem, quasilinear equations, a priori estimates.

1. Introduction

Nonlinear dissipative systems has been an active area of research in the last decade. Complex interaction of the components of the system together with advection process result in non-trivial spacio-temporal structures. Study of such systems, often referred to as active systems, is of great practical importance in many sciences. Model of population dynamics is one example, and it has almost become the ”mathematical foundation” of the entire mathematical biology. Moreover, the basic population dynamics models have served as building blocks for more sophisticated models in cell biology, microbiology, epidomology, mathematical genetics and other branches of the discipline, cf. [1-3].

Models described by nonlinear, parabolic reaction-diffusion type systems also serve as fundamental models of synergetics. Many articles from this field have been printed [4-6], which studied the appearance of inhomogeneity due to the stability loss of spatially homogenous state under the process of diffusion (e.g., diffusion of genes or morphogenesis).

To date, large number of mathematical models describing various physical, chemical and biological systems, have been proposed. These models allow us to identity some common principles that are specific to active systems.

We view that the introduction of free boundary problems into the theory of nonlinear dissipative structures - is perfectly acceptable and even natural. These models more adequately describe the processes. In these settings, the free boundary represents spreading front of the concentration, while in multicomponent systems, they are interfaces between the components.

To our knowledge, one of the first works to propose modelling population dynamics as free boundary diffusion-reaction problem was [7]. [7] along with [8-16] are among the fundamental works in the area, and have attracted the interests of more researchers since then.

2. Statement of the Problem

In this manuscript we consider a free boundary problem related to the one proposed in [7].

Problem 1 : Find $(s(t), u(t,x))$, satisfying

$$k(u)u_t - du_{xx} - cu_x = u\,(a - bu) \text{ in } D = \{(t,x) : t > 0,\ 0 < x < s(t)\}, \tag{1}$$

$$u\,(0,x) = u_0(x), \quad 0 \leqslant x \leqslant s(0) = s_0, \tag{2}$$

$$u_x\,(t,0) = 0, u\,(t, s(t)) = 0, \quad t \geqslant 0, \tag{3}$$

$$s'(t) = -\mu\, u_x\,(t, s(t))\,, \quad t \geqslant 0, \tag{4}$$

where $x = s(t)$ is a moving (unknown) boundary, which determines the front of spreading of the species, $u\,(t,x)$ is the concentration (intensity) of the species at point (t,x), $k(u)$ is the carrying capacity of the medium, d is diffusion coefficient, a represents the intrinsic growth rate of the species, b measures its intraspecific competition, cu_x is said to be an advection term, which means that the spreading of a species is affected by advection, $u_0(x)$ is initial concentration of the pollutant. In particular, we assume that a, b, c, μ are positive constants, $u_0 \in C^2([0, s_0]), u'{}_0(0) = u_0(s_0) = 0, u_0(x) > 0$ in $[0, s_0), k(u) \geq k_0 > 0$ for any $u > 0$.

The free boundary condition (4) is firstly established by Lin [11] from an ecological point by using the Fick's first law. This condition coincides with the well-known one-phase Stefan condition arising from the investigation of the melting of ice in contact with water, and has been applied in many other application field, for example, the modeling of wound healing [12], tumor growth [13], spreading of disease [14-16] and spreading of species [7, 11].

Our goal is to prove the solvability of the system, to give precise description of the behaviour of $s(t)$ with respect to t, and to investigate the qualitative properties of $s(t)$, $u(t,x)$.

For functional spaces and norms, we will employ the notations of [7], and we will also make use of its results.

The manuscript is organized as follows. First, we establish two-sided bounds for $u\,(t,x)$ and $s'(t)$, and then bounds for $|u|_{1+\alpha}$, $|u|_{2+\alpha}$ for appropriate $0<\alpha<1$. Afterwards, using those auxiliary results, we prove global existence and uniqueness theorems, and we also investigate some qualitative properties of the solution.

The system (1)-(4) was studied in [7] for $c=0$ and $k(u)=1$ case. The techniques we employ in this study are different then the ones used in [7], and moreover, we prove a theorem on the uniqueness of the solution, which is of interest in its own right.

3. Some a Priori Estimates and the Behaviour of the Free Boundary

In this section we derive some a priori estimate on the solution, and describe the behavior of the free boundary.

Theorem 1 — *Assume that $(s(t),u(t,x))$ is a solution of (1)-(4) in the region $D_T=\{(t,x):\ 0\le t\le T,\ 0<x<s(t)\}$, and assume there exists a constant N satisfying*

$$N\geq \max\left\{\frac{u_0(x)}{s_0-x},\frac{a^2}{bc}\right\},\ 0<u_0(x)\leq \frac{a}{b},\ \text{for } 0<x<s_0. \tag{5}$$

Then there exist positive constants M_1, M_2, independent of T, for which

$$0<u(t,x)\leqslant M_1=\frac{a}{b},\quad 0<s'(t)\leqslant M_2=\mu N\quad \text{in } D_T. \tag{6}$$

Proof : We will employ the result of the Theorem 5.1 [18, Chapter 2]. If we pick $\hat{\rho}=0,\tilde{\rho}=\frac{a}{b}$, then $f(t,x,u)=u(a-bu)$ satisfies all the assumptions of the theorem. Consequently,

$$0\leq u(t,x)\leq \frac{a}{b}=M_1 \text{ in } D_T, \tag{7}$$

and, by the strong maximum principle [19],

$$u(t,x)>0 \text{ in } D_T. \tag{8}$$

Therefore,

$$u_x(t,s(t))<0, t>0. \tag{9}$$

Using (3), we obtain that $s'(t)>0, 0<t\leq T$.

To derive an upper bound for $s'(t)$, we introduce $v(x,t)$ as

$$v(t,x)=u(t,x)+N(x-s(t)), \tag{10}$$

where N is appropriate positive constant, satisfying (5). We find that

$$k(v)v_t - dv_{xx} - cv_x \leqslant M_1 a - cN \leqslant 0, \quad \text{in } D_T,$$

$$v_x\,(t,0) = N > 0, \quad 0 \leqslant t \leqslant T,$$

$$v\,(0,x) = u_0(x) + N\,(x - s_0) \leqslant 0, \quad 0 \leqslant x \leqslant s_0,$$

$$v\,(t, s(t)) = 0, \quad 0 \leqslant t \leqslant T.$$

Invoking the maximum principle one more time, we obtain

$$v(t,x) \leqslant 0, \quad (t,x) \in \bar{D}_T.$$

Then, (10) also implies that

$$u(t,x) \leqslant N\,(s(t) - x)\,, 0 \leqslant x \leqslant s(t).$$

Therefore,

$$v_x(t, s(t)) = u_x(t, s(t)) + N \geqslant 0$$

or

$$u_x(t, s(t)) \geq -N.$$

The last inequality combined with (4) gives

$$s'(t) \leq \mu N = M_2,$$

which completes the proof. □

Corollary 1 — Under the assumptions of the Theorem 1 and $k'(u) \geq 0, u \geq 0, u_0'(x) < 0$, we have $u_x(t,x) \leq 0$ for all D_T.

PROOF : The proof immediately follows by differentiating (1) with respect to x, considering the free boundary problem for $v(t,x) = u_x(t,x)$ in D_T and invoking the maximum principle. □

4. THE UNIQUENESS RESULT

First we derive an integral expression for the free boundary. To this end, we rewrite (1) as

$$(q(u))_t - du_{xx} - cu_x = u(a - bu), \tag{11}$$

where

$$q(u) = \int\limits_0^u k(\eta) d\eta.$$

Integrating (11) over D_t, we obtain

$$\int\limits_0^t d\eta \int\limits_0^{s(\eta)} \Big[(d\, u_\xi - c\, u)_\xi - q(u)_\eta \Big] \, d\xi + \int\limits_0^t d\eta \int\limits_0^{s(\eta)} u\,(a - b\, u)\, d\xi = 0.$$

We get

$$\frac{d}{\mu} s\,(t) = \frac{d}{\mu} s_0 + c \int\limits_0^t u\,(\eta, 0)\, d\eta - \int\limits_0^{s(t)} q(u) d\xi + \int\limits_0^{s_0} q(u_0) d\xi + \iint\limits_{D_t} u\,(a - b\, u)\, d\xi d\eta\,. \tag{12}$$

***Theorem* 2** — *Under the assumptions of Theorem 1, (1)-(4) has a unique solution.*

PROOF : We first establish the result for smaller values of t, and then extend the proof to the general case of $0 < t < \infty$.

Assume that $s_1(t)$, $u_1\,(t, x)$ and $s_2(t)$, $u_2\,(t, x)$ are the solutions of the problem (1)-(4) and let $y(t) = \min\,(s_1(t),\, s_2(t))$, $h(t) = \max\,(s_1(t),\, s_2(t))$. Then each pair satisfies the identity (12). Subtracting, we obtain that

$$\begin{aligned} \frac{d}{\mu}\,|s_1(t) - s_2(t)| \leqslant c \int\limits_0^t |u_1\,(\eta, 0) - u_2\,(\eta, 0)|\, d\eta \\ + \int\limits_0^{y(t)} |q(u_1\,(t, \xi)) - q(u_2\,(t, \xi))|\, d\xi + \int\limits_{y(t)}^{h(t)} |q(u_i\,(t, \xi))|\, d\xi \\ + \int\limits_0^t d\eta \int\limits_0^{y(\eta)} |u_1\,(a - b\, u_1) - u_2\,(a - b\, u_2)|\, d\xi + \int\limits_0^t d\eta \int\limits_{y(\eta)}^{h(\eta)} |u_i\,(a - b\, u_i)|\, d\xi, \end{aligned} \tag{13}$$

where $u_i\,(t, x)$ is the solution between $y(t)$ and $h(t)$, i.e.,

$$u_i\,(t, x) = \begin{cases} u_1\,(t, x)\,, & \text{if} \quad s_2\,(t) < s_1\,(t)\,, \\ u_2\,(t, x)\,, & \text{if} \quad s_2\,(t) > s_1\,(t)\,, \end{cases} \tag{14}$$

From Theorem 1, we have that

$$|u_i\,(t,x)| \leqslant N\,(y\,(t)-x)\,,$$
$$|u_1\,(t,y\,(t)) - u_2\,(t,y\,(t))| \leqslant N\,|s_1\,(t)-s_2\,(t)|\,.$$

Considering the difference $w(t,x) = u_1(t,x) - u_2(t,x)$, we obtain an equation with bounded coefficients and the problem

$$dw_{xx} + cw_x - b_1(t,x)w_t - b_2(t,x)w +$$
$$+\left[a - b\left(u_1+u_2\right)\right]w = 0, \quad 0 < t < T, \quad 0 < x < y(t),$$
$$w(0,x) = 0, \quad 0 \le x \le s_0,$$
$$w_x\,(t,0) = 0, \quad t \ge 0,$$
$$|w\,(t,y\,(t))| \leqslant N \max_{0\leqslant\eta\leqslant t} |s_1\,(\eta) - s_2\,(\eta)|\,,$$

where $b_1(t,x) = a(u_1(t,x)), b_2(t,x) = u_{2t}(t,x)b(\bar{u})$.

From this problem, invoking the maximum principle, we conclude that

$$|w(t,x)| \leqslant N \max_{0\leqslant\eta\leqslant t} |s_1\,(\eta) - s_2\,(\eta)|\,.$$

Let

$$A(t_0) = \max_{0\le t\le t_0} |s_1(t) - s_2(t)| > 0.$$

We have that

$$\frac{\partial(s_1-s_2)}{\partial t} = \mu[u_{2x}(\cdot) - u_{1x}(\cdot)] \le -\mu u_{1x}(t, s_1(t)) \le \mu N = M_2.$$

Integrating from 0 to $t \le t_0$, we get

$$s_1(t) - s_2(t) \le M_2 t,$$

which implies

$$\max_{0\le t\le t_0} |s_1(t) - s_2(t)| \le M_2 t_0 \text{ or } A(t_0) \le M_2 t_0.$$

Bounding the terms in (13), we obtain

$$I_1 = c\int_0^t |u_1(\eta,0) - u_2(\eta,0)|\, d\eta \le c t_0 A(t_0),$$

$$I_2 = \int_0^{y(t)} |q(u_1(t,\xi)) - q(u_2(t,\xi))|\, d\xi \le M_{11}\int_0^{y(t)} |u_1(t,\xi) - u_2(t,\xi)| d\xi,$$

$$I_3 = \int_{y(t)}^{h(t)} |q(u_i(t,\xi))|\, d\xi \le M_{12} A^2(t_0),$$

$$I_4 = \int_0^t d\eta \int_0^{y(\eta)} |u_1(a - b\,u_1) - u_2(a - b\,u_2)|\, d\xi \le a\int_0^t d\eta\int_0^{y(\eta)} |u_1(\eta,\xi) - u_2(\eta,\xi)| d\xi,$$

$$I_5 = \int_0^t d\eta \int_{y(\eta)}^{h(\eta)} |u_i(a - b\,u_i)|\, d\xi \le \frac{a^2}{4b} t_0 N A(t_0).$$

Therefore,

$$\begin{aligned}\frac{d}{\mu}A(t_0) &\le M_{10}t_0A(t_0) + M_{11}\int_0^{y(t)} |u_1(t,\xi) - u_2(t,\xi)| d\xi + M_{12}A^2(t_0)\\ &+ a\int_0^t d\eta\int_0^{y\eta} |u_1(\eta,\xi) - u_2(\eta,\xi)| d\xi + \frac{a^2}{4b}t_0NA(t_0).\end{aligned} \tag{15}$$

Dividing (15) by $A(t_0)$, we end up with

$$\begin{aligned}\frac{d}{\mu} &\le M_{10}t_0 + M_{11}\int_0^{y(t)} \frac{|u_1(t,\xi) - u_2(t,\xi)|}{A(t_0)} d\xi + M_{12}A^2(t_0)\\ &+ a\int_0^t d\eta\int_0^{y\eta} \frac{|u_1(\eta,\xi) - u_2(\eta,\xi)|}{A(t_0)} d\xi + \frac{a^2}{4b}t_0N.\end{aligned} \tag{16}$$

Thus,

$$|w(t,x)| = |u_1(t,x) - u_2(t,x)| \le NA(t_0) \le M_2 t_0. \tag{17}$$

Now we bound the following integrals from (16) as

$$\int_{y(0)}^{y(t)} \frac{|w(t,x)|}{A(t_0)} dx \leq N(y(t) - y(0)) \leq M_{13} t_0, \tag{18}$$

$$\int_0^t d\eta \int_{y(0)}^{y(t)} \frac{|w(t,x)|}{A(t_0)} d\xi \leq N \int_0^t (y(\eta) - y(0)) d\eta \leq M_{14} t_0^2. \tag{19}$$

Above, $M_{i0}, (i = \overline{0,4})$ are some known constants. Since $A(t_0) \to 0$ as $t_0 \to 0$, in the light of (17)-(19), the inequality (16) can not hold, except for possibly small enough values of t_0, unless $A(t_0) = 0$.

But then, $s_1(t) = s_2(t)$ for all $t \leq \lambda$ (λ small enough) and, consequently, $u_1(t,x) = u_2(t,x)$ also holds for $t \leq \lambda$. □

Having proved uniqueness for $0 \leq t \leq \lambda$, we can be extended to all $t > 0$.

5. The Existence Result

Before proving the existence result, we will establish Hölder norm bounds $|u|_{1+\alpha}$ and $|u|_{2+\alpha}$ in $\overline{D}_T$.

Let

$$Q = \{(t,x) : 0 \leqslant x \leqslant s_0, \, 0 \leqslant t \leqslant T\}, \tag{20}$$

***Theorem* 3** — *Let* $M_1 = \max\limits_{Q} |u|$ *and assume that* $u(t,x), u_x(t,x)$ *are continuous in* Q *and* u *satisfies (1) in* Q *together with (2)-(4). Then*

$$|u_x(t,x)| \leqslant C_1(M_1), \quad (t,x) \in Q. \tag{21}$$

Moreover, if the weak second derivatives u_{xx}, u_{tx} *are in* $L^2(Q)$*, then there exists* $\alpha = \alpha(M, \delta)$*, such that*

$$|u|_{1+\alpha,Q} \leqslant C_2(M_1, C_1). \tag{22}$$

Additionally, assume that, $u(t,x)$ *satisfying (1) in* Q*, is continuous with its derivatives* u_t*,* u_x*,* u_{xx} *and* $|u|_{2+\alpha,Q} < \infty$.

Then

$$|u|_{2+\alpha,Q} \leqslant C_3(M_1, C_1, C_2). \tag{23}$$

PROOF : The derivative $u_x\,(t,x)$ satisfies a homogenous parabolic equation with bounded coefficients and is bounded at the boundary points $x = 0$, $x = s(t)$, and at $t = 0$. According to the maximum principle, $u_x\,(t,x)$ is bounded in $\overline{D_T}$. Therefore, (21) holds. The estimates (22) and (23) for $(t,x) \in Q$ are immediate consequences of the results of [17].

In order to bound the solution near the free boundary, we will make a change of variables by setting $\tau = t$, $y = \frac{x}{s(t)}$. As a result, the region D_T is mapped into $\Omega = \{(\tau, y) :\ 0 < \tau < T,\ 0 < y < 1\}$, and $v\,(\tau, y) = u\,(\tau,\ y\,s(\tau))$, which is bounded, solves

$$
\begin{aligned}
k(v)v_\tau &= \frac{d}{s^2(\tau)} v_{yy} + f\,(\tau, y, \theta)\,, (\tau, y, \theta) \in \Omega \\
v\,(0, y) &= u_0(y\,s_0), \quad 0 \leqslant y \leqslant 1, \\
v_y(\tau, 0) &= 0, 0 \leqslant \tau \leqslant T, \\
v\,(\tau, 1) &= 0, \quad 0 \leqslant \tau \leqslant T,
\end{aligned}
\tag{24}
$$

where

$$
f\,(\tau, y,\, v) = \frac{c}{s(\tau)} v_y - k(v) \frac{y\mu}{s^2(\tau)} v_y\,(\tau,\, y)\ v_y\,(\tau,\, 1) + a\,v - b\,v^2,
$$

$$
h(\tau) = \frac{s_0}{s(\tau)} > 0.
$$

The function $s(t)$ and the right hand side of (24) satisfy the Hölder conditions and theorems. Applying the method of even continuation through the right boundary [17], we obtain (21), (22). The remaining bounds for the higher order derivatives are established using the results for the linear equations [20]. □

In determining the maximal existence time for the Stephan's problem, three factors are taken into account:

1) domain nondegeneracy;

2) existence of the a priori bounds for norms in appropriate space;

3) upper and lower boundedness of the gradient of the solution on the free boundary.

Theorem 4 — *Under the assumptions of Theorems 1 and 3, there exists a solution* $u(t, x) \in C^{2+\alpha}\,(D)$, $s(t) \in C^{1+\alpha}\,[0, T]$ *of (1)-(4).*

PROOF : Note that we have established the nessasary Schauder type bounds in Hölder norm, and the free boundary $x = s(t)$ is increasing function of time t. We also showed the Hölder regularity of

$s'(t)$ and obtained bounds in $C^{2+\alpha}$ norm. Therefore, we can conclude the proof of the theorem by a standard argument.

In order to show the solvability of the nonlinear problem, we can make use of the Leray-Schauder theorem, keeping in mind that we already have uniqueness of the classical solution. Moreover, we also make use of the established bounds for all possible solutions in $|\cdot|_{1+\alpha}^{Q}$ norm and the existence results for linear problems in space of Hölder functions. We can view the equation as a linear one with respect to $u(t,x)$ with Hölder continuous coefficients.

We will consider the system (1)-(4) together with a one parameter family of problems of the same type. The linear equation determines a transformation $\omega = F(\omega, k)$, $0 \le k \le 1$, for which we can apply the Leray-Schauder principle. The operator is nonlinear and depends on k. Its fixed point at $k = 1$ is a solution of the problem.

Let us denote by $H^{1+\alpha}, \alpha \in (0,1)$ the Banach space of functions in $\bar{Q}$ with norm $||\cdot|| = |\cdot|_{1+\alpha}^{Q}$, which additionally satisfy the initial and boundary conditions of the problem. For any $u \in H^{1+\alpha}$ and arbitrary $k \in [0,1]$, we denote by u^k the solution of the linear problem, which possesses a unique solution with $u^k \in C^{2+\alpha}, s(t) \in C^{1+\alpha}$.

We note that the same transformation as in Theorem 2 is applied, which transforms the problem in D_T to the one in a fixed domain with Hölder continuous coefficients.

Uniform continuity and absolute continuity of the transformation operator F with respect to k, uniform bounds with respect to k for the solutions and solvability of the linear problems follow from the established a priori bounds in Hölder norms.

Similar proofs are demonstrated in detail, for example, in [19, Chapter 7] and [20, Chapter 6]. □

6. Some Properties of the Solution

Lemma 1 —

$$\lim_{t \to +\infty} u(t,x) = \frac{a}{b}$$

uniformly $x \in (0, \infty)$.

PROOF : We will construct a spatially homogenous solution of (1)-(4). We will look for the u as

$$u(t,x) = y(t).$$

From (1), we find that

$$y'(t) = \alpha(t)y(t) - \beta(t)y^2(t), \tag{25}$$
$$y(0) = y_0 = ||u_0||_\infty,$$

where $\alpha(t) = \frac{a}{k(y(t))}, \beta(t) = \frac{b}{k(y(t))}$.

Rewrite (25) as

$$-\frac{d}{dt}\left(\frac{1}{y(t)}\right) = \frac{\alpha(t)}{y(t)} - \beta(t).$$

Denoting $m(t) = \frac{1}{y(t)}$, we have that

$$\frac{d}{dt}\left(m(t)e^{\int_0^t \alpha(\eta)d\eta}\right) = \beta(t)e^{\int_0^t \alpha(\eta)d\eta}.$$

It then follows that

$$y(t) = \frac{e^{\int_0^t \alpha(\eta)d\eta}}{\frac{1}{y_0} + \int_0^t \beta(u)\left(e^{\int_0^\eta \alpha(\xi)d\xi}\right)d\eta} = \frac{f_1(t)}{f_2(t)}. \tag{26}$$

By L'Hospital's rule, we get that

$$\lim_{t\to\infty} y(t) = \frac{a}{b}.\square \tag{27}$$

We will mention the following comparison theorem without a proof.

***Theorem* 5** — *Let* $D_T = \{(t,x) : 0 < t \le T, 0 < x < \bar{s}(t)\}$. *Assume* $\bar{u} \in C(\bar{D}_T) \cap C^{1,2}(D_T)$, $\bar{s}(t) \in C^1[0,T]$,

$$k(\bar{u})\bar{u}_t - d\bar{u}_{xx} - c\bar{u}_x \ge \bar{u}(a - b\bar{u}),\ 0 < t < T,\ 0 < x < \bar{s}(t),$$
$$\bar{u}(t,\bar{s}(t)) = 0,\ \bar{s}'(t) \ge -\mu\bar{u}_x(t,\bar{s}(t)),$$
$$\bar{u}_x(t,0) \le 0,$$
$$\bar{s}(0) \ge s(0),\ \bar{u}(0,x) \ge u(0,x),\ 0 \le x \le s(0).$$

Then

$$s(t) \le \bar{s}(t) \text{ and } u(t,x) \le \bar{u}(t,x).$$

Above we proved that $u(t,x) \leq \frac{a}{b}$ for $0 \leq t \leq T, 0 \leq x \leq s(t)$.

Taking into account (27), we obtain that $\lim_{t\to\infty} \sup u(t,x) \leq \frac{a}{b}$ uniformly for $x \in [0,\infty)$.

We showed that, the curve $x = s\,(t) -$ is increasing function of t and $\lim_{t\to\infty} s\,(t) = s_\infty \in (0,+\infty]$ holds.

***Theorem* 6** — *If $s_\infty < \infty$, then*

$$s_\infty \leqslant l_0 = \frac{\pi}{2}\left(a - \frac{c^2}{2d}\right)^{-\frac{1}{2}} \sqrt{d}$$

and

$$\lim_{t\to+\infty} \|u(t,x)\|_{C[0,s(t)]} = 0.$$

PROOF : First, we will show that $s_\infty \leqslant l_0$. Suppose that $s_\infty > l_0$. Then there exists T, such that $l = s\,(T) > l_0$. For this fixed T, we consider the following function $v\,(x) = u\,(T,x)$. The system (1)-(4) implies that

$$\begin{aligned} -dv'' - cv' = v\,(a - bv)\,, 0 < x < l, \\ v'\,(0) = 0, v\,(l) = 0. \end{aligned} \tag{28}$$

We have proved that $u_x\,(T,x) = v'\,(x) < 0$, $0 < x \leqslant l$ and $v\,(x) < \frac{a}{b}$. If we extend $v\,(x)$ about $x = 0$ evenly, then we arrive the following problem

$$\begin{aligned} -dv'' - cv' = v\,(a - bv)\,, -l < x < l, \\ v\,(-l) = v\,(l) = 0, \end{aligned} \tag{29}$$

with $v\,(x) < \frac{a}{b}$, $-l < x < l$.

In order to make use of the result of [18, Theorem 4, pg. 111], we define the following spectral problem

$$\begin{aligned} -dy'' - cy' = \lambda y, -l < x < l, \\ y\,(-l) = y\,(l) = 0. \end{aligned} \tag{30}$$

The equation (29) has a unique positive solution, if $a > \lambda_1$, where λ_1 is the first eigenvalue of the problem (30).

We can construct λ_1 explicitly:

$$\lambda_1 = \frac{c^2}{2d} + d\Big(\frac{\pi}{2l}\Big)^2.$$

If $a > \lambda_1$, then $l \geqslant \frac{\pi}{2}\Big(a - \frac{c^2}{2d}\Big)^{-\frac{1}{2}}\sqrt{d} = l_0$.

Since $s_\infty < +\infty$, then $s'(t) \to 0$ as $t \to +\infty$. Then we can find $T_0 > T$, such that $s'(t) < c\frac{l}{s_\infty}$, $t \geqslant T_0$. Therefore, as $t \geqslant T_0$, $x \in [0, s(t)]$, $\frac{x\,s'(t)}{l} \leqslant c$ and for

$$w(t,x) := v\left(\frac{l}{s(t)}\,x\right)$$

we have

$$\begin{aligned} kw_t - dw_{xx} - cw_x &= -\frac{xlk}{s^2(t)}s(t)v' - \frac{dl^2}{s^2(t)}v'' - \frac{cl}{s(t)}v' \\ &\leq -\frac{c}{2}v' + \frac{l^2}{s^2(t)}\left(-dv'' - \frac{x\dot{s}(t)k}{l}v'\right) \\ &\leq \frac{l^2}{s^2(t)}\left(-dv'' - \frac{c}{2}v'\right) - \frac{c}{2}v' \\ &\leq (ab - bv^2). \end{aligned}$$

Since $0 \leq v \leq \frac{a}{b}$, then $av - bv^2 \geq 0$ and $\frac{l}{s(t)} \leq 1$.

Consequently, we obtain the following problem.

$$\begin{cases} kw_t - dw_{xx} - cw_x \leq av - bv^2 = aw - bw^2 \text{ for } t \geq T_0,\ x \in [0, s(t)], \\ w_x(t,0) = 0, w(t, s(t)) = 0, w(T_0, x) = v\left(\frac{l}{s(T_0)}x\right). \end{cases}$$

Choosing $\delta \in (0,1)$ such that $\delta\, w(T_0, x) \leqslant u(T_0, x)$. Then $\underline{u}(t,x) = \delta\, w(t,x)$ satisfies

$$\begin{cases} k\underline{u}_t - d\underline{u}_{xx} - c\underline{u}_x \leq \underline{u}(a - b\underline{u}), & t \geq T_0, \quad x \in [0, s(t)], \\ \underline{u}_x(t,0) = 0, \quad \underline{u}(t, s(t)) = 0, & t \geq T_0, \\ \underline{u}(T_0, x) \leq u(T_0, x), & 0 \leq x \leq s_0. \end{cases}$$

According to the comparison principle,

$$v(t,x) = \underline{u}(t,x) - u(t,x) \leq 0,\ t \geq T_0,\ x \in [0, s(t)]$$

and

$$v(t, s(t)) = 0, v_x(t, s(t)) \geq 0.$$

It follows that

$$u_x(t,s(t)) \leq \underline{u}_x(t,s(t)) = \delta\frac{l}{s(t)}v'(l) \to \delta\frac{l}{s_\infty}v'(l) < 0.$$

On the other hand,

$$u_x(t,s(t)) = -\frac{1}{\mu}\dot{s}(t) \to 0 \text{ as } t \to +\infty.$$

The contradiction shows that $s_\infty \leq l_0$.

We now establish that $\|u(t,x)\|_{C[0,s(t)]} \to 0$ as $t \to +\infty$. Let $\overline{u}$ be the unique solution of

$$\begin{aligned} k\overline{u}_t - d\overline{u}_{xx} - c\overline{u}_x \leqslant \overline{u}\left(a - b\overline{u}\right), \quad t>0, \quad 0<x<s_\infty,\\ \overline{u}_x(t,0)=0,\\ \overline{u}(t,s_\infty)=0,\\ \overline{u}\left(0,x\right)=\overline{u}_0\left(x\right), 0\leqslant x\leqslant s_\infty, \end{aligned} \tag{31}$$

where

$$\overline{u}_0\left(x\right) = \begin{cases} u_0(x), & 0\leq x\leq s_0,\\ 0, & x\geq s_0. \end{cases}$$

In $\Omega = \{(t,x) : t>0, 0<x<s(t)\}$ we consider the function

$$v\left(t,x\right) = \overline{u}(t,x) - u(t,x).$$

We then have

$$\begin{cases} kv_t - dv_{xx} - cv_x + (\tilde{k} - a + b(u+\bar{u})v = 0,\\ v_x\left(t,0\right)=0, \quad v\left(t,s(t)\right) = \overline{u}\left(t,s(t)\right) \geqslant 0, \qquad t>0,\\ v\left(0,x\right)=0, 0\leqslant x\leqslant s_0. \end{cases}$$

According to the maximum principle

$$v\left(t,x\right)\geqslant 0 \quad \text{or} \quad \overline{u}\left(t,x\right)\geqslant u(t,x)\geqslant 0.$$

Since

$$s_\infty \leqslant l_0 = \frac{\pi}{2}\left(a - \frac{c^2}{2d}\right)^{-\frac{1}{2}}\sqrt{d},$$

it then implies that

$$a \leqslant \frac{c^2}{2d} + d\left(\frac{\pi}{2s_\infty}\right)^2 = \lambda_1.$$

By the property of the solution of (31) (cf. [18]), we have that $\overline{u}(t,x) \to 0$ uniformly in $x \in [0, s(t)]$ as $t \to +\infty$.

Therefore, $\lim\limits_{t\to+\infty} \|u(t,x)\|_{C[0,s(t)]} = 0$. □

The following Theorem 7 is proved similar to the arguments of [7].

***Theorem* 7** — *If* $s_\infty = +\infty$, *then* $\lim\limits_{t\to+\infty} u(t,x) = \frac{a}{b}$ *uniformly in any bounded subset of* $[0,\infty)$.

PROOF : We have already shown that $u(t,x) \leqslant \overline{u}(t) = y(t)$ for $t > 0$, $x \in [0, s(t)]$, where $y(t)$ is given in (26).

As $\lim\limits_{t\to+\infty} y(t) = \frac{a}{b}$, we then get that $\lim\limits_{t\to+\infty} \sup u(t,x) \leqslant \frac{a}{b}$ uniformly for $x \in [0, s(t)]$.

On the other hand, for $l > \max\{s_0, l_0\}$ there exists t_l, such that $s(t_l) = l$. By comparison principle $u(t,x) \geqslant \underline{u}_l(t,x)$ in $(t_l,\infty)\times(0,l)$, where $\underline{u}_l$ is the solution of the following problem with fixed boundaries

$$\begin{cases} k(\underline{u}_l)_t - d(\underline{u}_l)_{xx} - c(\underline{u}_l)_x = (\underline{u}_l)(a - b\underline{u}_l), & t > t_l, \quad 0 < x < l, \\ 0 = (\underline{u}_l)_x(t,0) = (\underline{u}_l)(t,l), & t > t_l, \\ (\underline{u}_l)(t_l,x) = (u_l)(t_l,x), & 0 \le x \le l. \end{cases}$$

Since $a > \lambda_1$, we can conclude that $u_l(t,x) \to u_l^*(x)$ as $t \to +\infty$ uniformly in any compact subset of $[0,l]$, where u_l^* is the unique positive solution of

$$\begin{aligned} -d(u_l^*)_{xx} - c(u_l^*)_x &= (u_l^*)\left(a - b\left(u_l^*\right)\right), & -l < x < l, \\ u_l^*(-l) &= u_l^*(l) = 0. \end{aligned}$$

Therefore, $\lim\limits_{t\to+\infty} \inf u(t,x) \geqslant u_l^*(x)$ uniformly in compact subset of $[0,l]$.

We also have $u_l^*(x) \to \frac{a}{b}$ as $t \to +\infty$, cf. [18, Lemma 2.2].

Consequently, $\lim\limits_{t\to+\infty} \inf u(t,x) \geqslant \frac{a}{b}$ uniformly in x.

Combining the results of $\lim\limits_{t\to+\infty} \sup u(t,x)$ and $\lim\limits_{t\to+\infty} \inf u(t,x)$ we arrive at the desired result.□

***Theorem* 8** — *If* $s_0 < \frac{\pi}{2}\left(a - \frac{c^2}{2d}\right)^{-\frac{1}{2}}\sqrt{d}$ *and*

$$\mu \ge \mu_0 = \max\left\{1, \frac{b}{a}||u_0||_\infty\right\} d(s_\infty - s_0)\left(\int_0^{s_0} q(u_0(x))dx\right)^{-1},$$

then $s_\infty = +\infty$.

PROOF : First, we consider the $||u_0|| \le \frac{a}{b}$ case. Under this assumption, $y(t) \le \frac{a}{b}, t > 0$, which implies that $u(t,x) < y(t) \le \frac{a}{b}, t > 0, x \in [0, s(t)]$.

We now rewrite the equation (11) as

$$\frac{d}{dt}\int\limits_0^{s(t)} q(u)dx = \int\limits_0^{s(t)} (q(u))_t dx + s'(t)u(t,s(t))$$
$$= d\int\limits_0^{s(t)} u_{xx}dx + c\int\limits_0^{s(t)} u_x dx + \int\limits_0^{s(t)} u(a-bu)dx. \quad (32)$$

Integrating (32) with respect to t from 0 to t, we obtain that

$$\int\limits_0^{s(t)} q(u(t,x))dx = \int\limits_0^{s_0} q(u_0(x))dx + d\int\limits_0^t d\eta \int\limits_0^{s(\eta)} u_{xx}(\eta,x)dx$$
$$+ c\int\limits_0^t d\eta \int\limits_0^{s(\eta)} u_x(\eta,x)dx + \int\limits_0^t d\eta \int\limits_0^{s(\eta)} (au - bu^2)dx. \quad (33)$$

Consider the second and third terms in (33).

$$J_1 = d\int\limits_0^t d\eta \int\limits_0^{s(\eta)} u_{xx}(\eta,x)dx = d\int\limits_0^t u_x\left[(\eta, s(\eta)) - u_x(\eta,0)\right]d\eta = \frac{d}{\mu}(s_0 - s(t)),$$
$$J_2 = c\int\limits_0^t (u(\eta,s(\eta)) - u(\eta,0))d\eta = -c\int\limits_0^t u(\eta,0)d\eta.$$

Using $0 < u(t,x) < \frac{a}{b}, t > 0, x \in [0, s(t))$ in the integral $\int\limits_0^t d\eta \int\limits_0^{s(\eta)} (au - bu^2)dx$ gives

$$\int\limits_0^t d\eta \int\limits_0^{s(\eta)} (au - bu^2)dx \ge \int\limits_0^1 d\eta \int\limits_0^{s(\eta)} (au - bu^2)dx > 0$$

for all $t \ge 1$.

Applying the last inequality in (33), we obtain

$$\int_0^{s(t)} q(u(t,x))dx = \int_0^{s_0} q(u_0(x))dx - \frac{d}{\mu}(s(t)-s_0) - c\int_0^t u(\eta,0)d\eta$$
$$+\int_0^t d\eta \int_0^{s(\eta)} (au-bu^2)dx. \qquad (34)$$

We know that if $s_\infty \neq \infty$, then $s_\infty \leq \frac{\pi}{2}\left(a-\frac{c^2}{2d}\right)^{-\frac{1}{2}}\sqrt(d) = l_0$ and $\lim\limits_{t\to\infty} ||u(t,\cdot)||_\infty = 0$.

Assuming $t \to +\infty$ in (34), we have

$$\int_0^{s_0} q(u_0(x))dx < \frac{d}{\mu}(s_\infty - s_0)$$

or

$$\mu < d(s_\infty - s_0)\left(\int_0^{s_0} q(u_0(x))dx\right)^{-1},$$

which contradicts $\mu \geq \mu_0$.

In the case of $||u_0||_\infty > \frac{a}{b}$, we take $\underline{u}_0 = \frac{a}{b||u_0||_\infty}u_0(x)$.

The solution of (1)-(4) $(\underline{u},\underline{s})$ with initial conditions $\underline{u}_0$ is a subsolution and therefore $s(t) \geq \underline{s}(t), t > 0$. But from the first case proven above, based on $||\underline{u}_0|| = \frac{a}{b}$ and our assumption about μ, we obtain that $\lim\limits_{t\to\infty} \underline{s}(t) = \infty$. Then $s_\infty = \infty$, which completes the proof of the Theorem. □

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