Indian J. Pure Appl. Math., **50**(1): 121-135, March 2019

DOI: 10.1007/s13226-019-0311-1

PRIMITIVE PERMUTATION GROUPS WITH A SOLVABLE SUBCONSTITUENT OF DEGREE 7

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(*Received* 15 *May* 2017; *after final revision* 28 *December* 2017;
accepted 15 *March* 2018)

In this paper all primitive permutation groups which have a solvable subconstituent of degree 7 are determined.

Key words : Primitive group; subconstituent; irreducible character; Galois conjugacy class.

1. Introduction

Let G be a primitive permutation group acting on the finite set Ω. Denote by G_α the stabilizer of $\alpha \in \Omega$ and $\Delta(\alpha)$ an orbit of G_α on Ω which is often called a suborbit of G. Consider the action induced by G_α on $\Delta(\alpha)$. Its transitive constituent $G_\alpha^{\Delta(\alpha)}$ is a homomorphic image of G_α which is called a subconstituent of G. The subconstituent of G is said to be faithful if the kernel of G_α on $\Delta(\alpha)$ is trivial.

Lots of work has been done for primitive permutation groups whose subconstituents satisfy certain properties. For example, Praeger [1] studied primitive groups with a doubly transitive subconstituent.

Wang [2, 3] classified primitive groups with a sharply 2-transitive and solvable 2-transitive subconstituent, respectively. Here we mainly focus on the primitive permutation groups with a suborbit of small length. It is well known that the primitive group with a suborbit of length 2 is a Frobenius group of prime degree [4, Theorem 18.7]. However, the structure of G would be more complicated when G_α has an orbit of length more than 2. The primitive permutation groups with a suborbit of length 3 were determined by Wong in [5]. The classification of primitive groups with a suborbit of length 4 were given by Quirin, Sims and Wang in [6, 7, 8]. Furthermore, Quirin, Wang, Fawcett, Giudici, *et al.*, researched the primitive groups with a suborbit of length 5 in [6, 9-12]. Their results have completed the classification of all finite primitive permutation groups having a suborbit of length 5.

So it is natural to consider the problem of classifying the primitive groups which have a suborbit of length 7. Then the transitive constituent $G_\alpha^{\Delta(\alpha)}$ of G_α acts on the suborbit $\Delta(\alpha)$ of length 7 is either solvable, or isomorphism to $PSL(2,7), A_7, S_7$. If the situation comes to the latter one, it is difficult to determine G_α when $G_\alpha^{\Delta(\alpha)}$ is unfaithful, and it is more difficult to determine G by G_α. Therefore, in this paper, our main goal is to study the case that the subconstituent $G_\alpha^{\Delta(\alpha)}$ is solvable and we have the following result.

***Theorem* 1.1** — *Let G be a primitive permutation group acting on a finite set Ω with a suborbit $\Delta(\alpha)$ of length 7. Assume that the subconstituent $G_\alpha^{\Delta(\alpha)}$ is solvable. Then one of the following holds:*

(1) *If $G_\alpha^{\Delta(\alpha)} \cong \mathbb{Z}_7$, then G is solvable and $G \cong G(7,p)$ which is listed in subsection 3.1;*

(2) *If $G_\alpha^{\Delta(\alpha)} \cong \mathrm{D}_{14}$ is faithful, then either G is solvable and $G \cong G(14,p)$ which is listed in subsection 3.2, or $G \cong \mathrm{PSL}(2,8)$, $\mathrm{PSL}(2,13)$ or ${}^2B_2(8)$;*

(3) *If $G_\alpha^{\Delta(\alpha)} \cong \mathrm{D}_{14}$ is unfaithful, then $G \cong \mathrm{PSL}(2,27)$, $\mathrm{PSL}(2,29)$, $\mathrm{PGL}(2,27)$ or $\mathrm{PGL}(2,29)$;*

(4) *If $G_\alpha^{\Delta(\alpha)} \cong \mathbb{Z}_7 \rtimes \mathbb{Z}_3$, then either $G \cong G(21,p)$ which is listed in subsection 3.3, or $G \cong \mathrm{PSL}(3,2)$, $\mathrm{PGL}(3,4)$ or $\mathrm{PGU}(3,5)$;*

(5) *If $G_\alpha^{\Delta(\alpha)} \cong \mathbb{Z}_7 \rtimes \mathbb{Z}_6$, then either $G \cong G(42,p)$ which is listed in subsection 3.4, or $G \cong \mathrm{S}_7$, J_1, $\mathrm{P\Sigma L}(2,27)$, $\mathrm{PSL}(3,4).\mathbb{Z}_6$, $\mathrm{PGL}(3,4)$ or ${}^2B_2(8).\mathbb{Z}_3$.*

The notation and terminology used in this paper are standard (see [4, 13-17]). For two groups K and H, $K.H$ is an arbitrary extension of K by H, while $K \rtimes H$ stands for a split one. We use $\mathbb{Z}_m$ and D_{2m} to denote the cyclic group of order m and the dihedral group of order $2m$, respectively. For a prime p and an integer $k > 1$, the notation $\mathbb{Z}_p^k$ stands for the elementary abelian group of order p^k, and $GF(p^k)$ for the finite field of p^k elements. We denote by $V(k,p)$ a k-dimensional vector space over the field $GF(p)$. The definition of the group $G(i,p)$ for $i = 7, 14, 21, 42$ appeared in Theorem 1.1 will be given in Section 3.

The organization of the paper is as follows. In Section 2, we give some preliminary results. In Section 3, we determine the faithful irreducible representations of $\mathbb{Z}_7$, D_{14}, $\mathbb{Z}_7 \rtimes \mathbb{Z}_3$ and $\mathbb{Z}_7 \rtimes \mathbb{Z}_6$ over the field $F := GF(p)$ for prime $p \neq 7$. For each case, using the computer algebra system GAP [18] we construct primitive groups G with a suborbit of length 7. In the last section, we complete the proof of Theorem 1.1.

2. Preliminaries

We first give some lemmas about the structure of G_α, $G_\alpha^{\Delta(\alpha)}$ and G.

Lemma 2.1 — Let G be a primitive permutation group with a suborbit $\Delta(\alpha)$ of prime length p.

(1) [6, Proposition 3.2]. If $G_\alpha^{\Delta(\alpha)} \cong \mathbb{Z}_p$, then $G_\alpha \cong \mathbb{Z}_p$ and G is solvable;

(2) [6, Theorem 3.5]. If $G_\alpha^{\Delta(\alpha)} \cong \mathrm{D}_{2p}$ is not faithful, then G is isomorphic to $\mathrm{PSL}(2, 4p \pm 1)$ or $\mathrm{PGL}(2, 2p \pm 1)$.

Lemma 2.2 — [9, Theorem 1.2]. Let G be a primitive permutation group of degree n. Suppose that G has a suborbit $\Delta(\alpha)$ of length p $(p \geq 5)$ and a faithful subconstituent $G_\alpha^{\Delta(\alpha)} \cong \mathrm{D}_{2p}$. Then one of the following holds:

(1) G is solvable, n is a prime power and $p \nmid n$;

(2) $G = \mathrm{A}_5, p = 5, n = 6$;

(3) $G = \mathrm{PSL}(2,q), p = \frac{q-\epsilon}{(2,\, q-1)}, \epsilon = \pm 1, (\epsilon,\, q) \neq (1, 11)$ and $(-1, 9)$, $n = \frac{1}{2}q(q+\epsilon)$;

(4) $G = {}^2B_2(q), p = q - 1, n = \frac{1}{2}q^2(q^2+1)$.

Lemma 2.3 — [6, Theorem 2.2]. Suppose G is a primitive permutation group with an suborbit $\Delta(\alpha)$ of length p^a, where p is a prime. If

(1) $G_\alpha^{\Delta(\alpha)}$ is primitive and

(2) $|G_\alpha^{\Delta(\alpha)}| = |G_\alpha^{\Delta'(\alpha)}| = p^a r^b$, where Δ' is an orbital of G paired with Δ, $a \neq 0, b \neq 0$, and $r \neq p$ is a prime such that

(i) $r \neq 2$ if $p = 3$,

(ii) $a \leq 2$ if $p = 2$ and $r = 3$.

Then (a) p^{a+1} does not divide $|G_\alpha|$,

(b) $G_{\alpha\cup\Delta(\alpha)} = G_{\alpha\cup\Delta'(\alpha)} = O_q(G_\alpha)$,

(c) $O_p(G_\alpha)$ is a Sylow p-subgroup of G_α and

(d) $|G_\alpha| \mid p^a r^{2b}$.

Lemma 2.4 — [7, Lemma 9]. Let G be a primitive permutation group on the finite set Ω. Suppose that G contains a regular normal subgroup. Then every non-trivial transitive constituent of G_α is faithful.

The following two lemmas will be used to determine the faithful irreducible representations of the subconstituent $G_\alpha^{\Delta(\alpha)}$. We denote $\deg Y$ be degree of an irreducible representation Y and $F(\chi)$ be the field generated over F by the value of χ.

Lemma 2.5 — [15, Corollary 9.23]. Let $F \subseteq E$ be fields of prime characteristic, X be an irreducible E-representation of H which affords the character χ and Y be an irreducible F-representation such that X is a constituent of Y^E. Then $\deg Y = |F(\chi) : F| \deg X$. In particular, X is similar to Y^E if $F(\chi){=}F$.

Lemma 2.6 — [13, Theorem 26.2]. Let V be an irreducible FG-module, E be a finite extension of F and $\Gamma = Gal(E/F)$. Then

(1) $V^E = \bigoplus_{a\in A} W^a$ for some irreducible EG-module W and any set A of coset representatives for $N_\Gamma(W)$ in Γ, where $N_\Gamma(W) = \{\gamma \in \Gamma : W^\gamma \cong W\}$.

(2) Let U be an irreducible EG-module. Then V is an FG-submodule of U precisely when U is EG-isomorphic to W^σ for some $\sigma \in \Gamma$.

Lemma 2.7 — [13, Theorem 5.15.1]. Let H be a finite group of exponent m and F be a finite field with $char(F) = p$ such that $p \nmid |H|$. Let E be a splitting field of H, where $E = F(\zeta)$ and ζ is a primitive m-th root of unit. If (φ, U) is an irreducible F-representation of H, then there exists an irreducible E-representation (χ, W) of H such that

$$U^E \cong \ell\Big(\bigoplus_{W^\sigma\in \mathrm{Orb}(W)} W^\sigma\Big), \quad \varphi^E \cong \ell\Big(\bigoplus_{\chi^\sigma\in \mathrm{Orb}(\chi)} \chi^\sigma\Big)$$

where $\mathrm{Orb}(W)$ denotes an orbit containing W which $Gal(E/F)$ acts on all of the irreducible E-modules of H and $\mathrm{Orb}(\chi)$ denotes an orbit containing χ which $Gal(E/F)$ acts on all of the irreducible E-characters of H, and ℓ is an integer.

3. Faithful Irreducible Representations of $\mathbb{Z}_7$, D_{14}, $\mathbb{Z}_7 \rtimes \mathbb{Z}_3$ and $\mathbb{Z}_7 \rtimes \mathbb{Z}_6$

In this section, suppose that H is an affine primitive group of degree 7. Then H is isomorphic to one of the following groups: $\mathbb{Z}_7$, $\mathrm{D}_{14} \cong \mathbb{Z}_7 \rtimes \mathbb{Z}_2$, $\mathbb{Z}_7 \rtimes \mathbb{Z}_3$ or $\mathrm{AGL}(1,7) \cong \mathbb{Z}_7 \rtimes \mathbb{Z}_6$. We will determine the faithful irreducible representations of H and all primitive groups G with H as the subconstituent in four subsections. In the following we denote by ε a primitive 7-th root of unity and ω a primitive 6-th root of unity.

3.1 *Faithful irreducible representations of* $\mathbb{Z}_7$

Let $H = \mathbb{Z}_7$. Then H has a faithful irreducible representation over $GF(p)$ for prime $p \neq 7$. This representation is unique up to an automorphism of H. The representation has degree 1 if $p \equiv 1 \pmod 7$, degree 2 if $p \equiv -1 \pmod 7$, degree 3 if $p \equiv 2 \pmod 7$ or $p \equiv -3 \pmod 7$, and degree 6 if $p \equiv -2 \pmod 7$ or $p \equiv 3 \pmod 7$. So we can define the semidirect product:

$$G(7,p) = \begin{cases} \mathbb{Z}_p \rtimes \mathbb{Z}_7 & \text{if } p \equiv 1 \pmod 7, \\ \mathbb{Z}_p^2 \rtimes \mathbb{Z}_7 & \text{if } p \equiv -1 \pmod 7, \\ \mathbb{Z}_p^3 \rtimes \mathbb{Z}_7 & \text{if } p \equiv 2 \pmod 7 \text{ or } p \equiv -3 \pmod 7, \\ \mathbb{Z}_p^6 \rtimes \mathbb{Z}_7 & \text{if } p \equiv -2 \pmod 7 \text{ or } p \equiv 3 \pmod 7. \end{cases}$$

Thus $G = G(7,p)$ is a primitive group of degree p, p^2, p^3 or p^6, respectively. The stabilizer G_α has an orbit of length 7 on $\Omega \setminus \{\alpha\}$.

3.2 *Faithful irreducible representations of* D_{14}

Let $H = \langle a, b \,|\, a^7 = b^2 = 1, bab = a^{-1}\rangle \cong \mathrm{D}_{14}$. Since H has five conjugacy classes, namely $\{1\}, \{a, a^{-1}\}, \{a^2, a^{-2}\}, \{a^3, a^{-3}\}$ and $\{b, ab, \cdots, a^6 b\}$, one has that H has five irreducible characters over the complex field $\mathbb{C}$. The irreducible character table of H over $\mathbb{C}$ is listed in Table 1. It

Table 1: The character table of D_{14}

	1	a	a^2	a^3	b
χ_1	1	1	1	1	1
χ_2	1	1	1	1	-1
χ_3	2	$\varepsilon + \varepsilon^{-1}$	$\varepsilon^2 + \varepsilon^{-2}$	$\varepsilon^3 + \varepsilon^{-3}$	0
χ_4	2	$\varepsilon^2 + \varepsilon^{-2}$	$\varepsilon^3 + \varepsilon^{-3}$	$\varepsilon + \varepsilon^{-1}$	0
χ_5	2	$\varepsilon^3 + \varepsilon^{-3}$	$\varepsilon + \varepsilon^{-1}$	$\varepsilon^2 + \varepsilon^{-2}$	0

follows from [15, Theorem 9.10] that the splitting field E for H is

$$E = \begin{cases} GF(p) \text{ if } p & \equiv 1 \pmod 7, \\ GF(p^2) \text{ if } p & \equiv -1 \pmod 7, \\ GF(p^3) \text{ if } p & \equiv 2 \pmod 7, \\ GF(p^6) \text{ if } p & \equiv -2 \pmod 7 \text{ or } p \equiv \pm 3 \pmod 7. \end{cases}$$

Assume $p \equiv 1 (\bmod 7)$. Then F is a splitting field for H. Hence H has exactly three faithful irreducible representations X_i which affords the irreducible character χ_i over F where $i = 3$, 4 and 5, all of which have degree 2. Then X_i can be represented by

$$X_3(a) = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix}, \ X_3(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};$$

$$X_4(a) = \begin{pmatrix} \varepsilon^2 & 0 \\ 0 & \varepsilon^{-2} \end{pmatrix}, \ X_4(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};$$

$$X_5(a) = \begin{pmatrix} \varepsilon^3 & 0 \\ 0 & \varepsilon^{-3} \end{pmatrix}, \ X_5(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

From the facts $\langle X_i(a), X_i(b)\rangle \cong D_{14}\ (i{=}3,4,5)$ and $X_3(H) = X_4(H) = X_5(H)$ we know that there exists a unique split extension of $\mathbb{Z}_p^2$ by H. Define

$$G(14,p) = \mathbb{Z}_p^2 \rtimes \mathrm{D}_{14} \text{ if } p \equiv 1 \pmod 7.$$

Then $G = G(14,p)$ is a primitive group of degree p^2 with $\Omega = V(2,p) \cong \mathbb{Z}_p^2$. Let $\alpha{=}(0,0) \in \Omega$. Then $G_\alpha \cong H$ has an orbit $\Delta{=}\{ (1,1), (\varepsilon,\varepsilon^{-1}), (\varepsilon^2,\varepsilon^{-2}), (\varepsilon^3,\varepsilon^{-3}), (\varepsilon^{-3},\varepsilon^{3}), (\varepsilon^{-2},\varepsilon^{2}), (\varepsilon^{-1},\varepsilon)\}$ of length 7.

If $p \equiv -1 (\bmod 7)$. Then H has exactly three faithful irreducible E-representations X_i where $i = 3$, 4, 5, all of which have degree 2. Let $\sigma : x \mapsto x^p$ be a Frobenius automorphism of E. It follows that $\sigma \in Gal(E/F)$ and σ fixes $\varepsilon + \varepsilon^{-1}$, $\varepsilon^2 + \varepsilon^{-2}$ and $\varepsilon^3 + \varepsilon^{-3}$. It implies that $\varepsilon + \varepsilon^{-1}$, $\varepsilon^2 + \varepsilon^{-2}$ and $\varepsilon^3 + \varepsilon^{-3}$ are all in F. Hence $F(\chi_i) = F$ for each irreducible character χ_i, where $1 \leq i \leq 5$ and $F(\chi_i)$ is the field over F generated by the values of χ_i. Let Y be an irreducible representation of H over F and X be an irreducible constituent of Y^E. Then Y^E is similar to X by Lemma 2.5. Therefore H has exactly three faithful irreducible F-representations $Y_i\ (i{=}3,4,5)$ of degree 2 and Y_i

has matrix representation:

$$Y_3(a) = \begin{pmatrix} \varepsilon+\varepsilon^{-1} & -1 \\ 1 & 0 \end{pmatrix}, \qquad Y_3(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};$$

$$Y_4(a) = \begin{pmatrix} \varepsilon^2+\varepsilon^{-2}+1 & -(\varepsilon+\varepsilon^{-1}) \\ \varepsilon+\varepsilon^{-1} & -1 \end{pmatrix}, \qquad Y_4(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};$$

$$Y_5(a) = \begin{pmatrix} -(1+\varepsilon^2+\varepsilon^{-2}) & -(1+\varepsilon^2+\varepsilon^{-2}) \\ 1+\varepsilon^2+\varepsilon^{-2} & -(\varepsilon+\varepsilon^{-1}) \end{pmatrix}, \quad Y_5(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

As $\langle Y_i(a), Y_i(b)\rangle \cong D_{14}(i=3,4,5)$ and $Y_3(H)=Y_4(H)=Y_5(H)$ we have that there exists a unique split extension of $\mathbb{Z}_p^2$ by H. Define

$$G(14,p)=\mathbb{Z}_p^2 \rtimes \mathrm{D}_{14} \quad \text{if } p \equiv -1 \pmod 7.$$

Hence $G=G(14,p)$ is a primitive permutation group of degree p^2 with $\Omega=V(2,p)\cong\mathbb{Z}_p^2$. Let α=$(0,0)\in\Omega$. Then $G_\alpha$$\cong$$H$ has an orbit Δ=$\{(1,1),\ (1+\varepsilon+\varepsilon^{-1},\ -1),\ (-\varepsilon^3-\varepsilon^{-3},\ -1-\varepsilon-\varepsilon^{-1}),\ (0,\varepsilon^3+\varepsilon^{-3}),\ (\varepsilon^3+\varepsilon^{-3},0),(-1-\varepsilon-\varepsilon^{-1},-\varepsilon^3-\varepsilon^{-3}),\ \ (-1,1+\varepsilon+\varepsilon^{-1})\}$ of length 7.

Assume $p\equiv d \pmod 7$ where $d\in\{\pm2,\pm3\}$ and $p\neq 2$. Then H has exactly three faithful irreducible E-representations which all have degree 2. Let $\sigma: x\mapsto x^p$ be a Frobenius automorphism of E. Then σ belongs to $Gal(E/F)$ and it cyclicly permutes $\varepsilon+\varepsilon^{-1}$, $\varepsilon^2+\varepsilon^{-2}$ and $\varepsilon^3+\varepsilon^{-3}$. It follows that χ_3, χ_4 and χ_5 comprise a Galois conjugacy class over the field F (see [15, Chapter 9]). Furthermore, there is an irreducible representation Y of H over F, which affords an irreducible character μ by Lemma 2.6 such that Y^E is similar to $X_3\oplus X_4\oplus X_5$ and $\mu=\chi_3+\chi_4+\chi_5$. Thus Y is an irreducible representation of degree 6.

If $p=2$. Then $E=GF(8)$ is a splitting field for H. Let $K=\mathbf{Q}(\omega)$ where ω is a primitive 7th complex root of 1 and $\mathbf{Q}$ is a rational number field. Then K is also a splitting field for H. Let $R=\mathbf{Z}[\omega]$ (where $\mathbf{Z}$ is a complex number field) and $P=2R$ be a maximal ideal in R. Choose a nonarchimedean $P-$adic valuation on K. It is well known that the valuation ring R_P has a unique maximal ideal $P\cdot R_P$ and $R_P/P\cdot R_P\cong R/P=\mathbf{Z}[\omega]\cong\mathbf{E}$. Then (K,R,E) is a $2-$modular system (cf.$[16]$, $\S4$ and $\S$ 16). By the corollary of Fong-Swan-Rukolaine Theorem $([16], 22.5)$, the degree of each irreducible $E-$representation of H coincides with the degree of some irreducible $K-$representation, because H is $2-$solvable. It follows that a faithful irreducible $E-$representation of H must have degree 2. By using the results in $([17],\S86)$, we get the conclusion that H has exactly 4 irreducible $E-$representation; Let $X_i\,(1\le i\le 4)$ be irreducible representations over E which affords the Brauer characters $\psi_i (1 \le i \le 4)$. It is easy to determine the Brauer character table of H over E which is given in Table 2.

Table 2: Brauer character table of D_{14} for $p = 2$

	1	a	a^2	a^3
ψ_1	1	1	1	1
ψ_2	2	$\varepsilon+\varepsilon^{-1}$	$\varepsilon^2+\varepsilon^{-2}$	$\varepsilon^3+\varepsilon^{-3}$
ψ_3	2	$\varepsilon^2+\varepsilon^{-2}$	$\varepsilon^3+\varepsilon^{-3}$	$\varepsilon+\varepsilon^{-1}$
ψ_4	2	$\varepsilon^3+\varepsilon^{-3}$	$\varepsilon+\varepsilon^{-1}$	$\varepsilon^2+\varepsilon^{-2}$

Let $\sigma : x \mapsto x^2$ be a Frobenius automorphism of E. Then σ belongs to $Gal(E/F)$ and it cyclicly permutes $\varepsilon+\varepsilon^{-1}$, $\varepsilon^2+\varepsilon^{-2}$ and $\varepsilon^3+\varepsilon^{-3}$. Hence ψ_2, ψ_3 and ψ_4 comprise a Galois conjugacy class over the field F. Furthermore, there is an irreducible representation Y of H over F which affords an irreducible character μ by Lemma 2.6 such that Y^E is similar to $X_2 \oplus X_3 \oplus X_4$ and $\mu = \psi_2+\psi_3+\psi_4$. Therefore Y is an irreducible representation of degree 6.

For these two cases, Y can be represented by

$$Y(a) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & -1 \end{pmatrix}, \; Y(b) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

It is easy to see that $\langle Y(a), Y(b)\rangle \cong \mathrm{D}_{14}$. Therefore, there is a unique split extension of $\mathbb{Z}_p^6$ by H. Define

$$G(14,p) = \mathbb{Z}_p^6 \rtimes \mathrm{D}_{14} \;\text{ if } p \equiv d \pmod 7, d \in \{\pm 2, \pm 3\}.$$

Then $G = G(14,p)$ is a primitive group of degree p^6 with $\Omega = V(6,p) \cong \mathbb{Z}_p^6$. Let $\alpha = (0,0,0,0,0,0) \in \Omega$. Then H has an orbit $\Delta = \{(1,1,0,0,0,0)$, $(-1,0,0,0,0,0)$, $(0,0,0,0,0,-1)$, $(0,0,0,0,1,1)$, $(0,0,0,-1,-1,0)$, $(0,0,1,1,0,0)$, $(0,-1,-1,0,0,0)\}$ of length 7.

Table 3: The character table of $\mathbb{Z}_7 \rtimes \mathbb{Z}_3$

	1	a	a^3	b	b^2
χ_1	1	1	1	1	1
χ_2	1	1	1	ω^2	ω^4
χ_3	1	1	1	ω^4	ω^2
χ_4	3	$\varepsilon+\varepsilon^2+\varepsilon^4$	$\varepsilon^3+\varepsilon^5+\varepsilon^6$	0	0
χ_5	3	$\varepsilon^3+\varepsilon^5+\varepsilon^6$	$\varepsilon+\varepsilon^2+\varepsilon^4$	0	0

Then Y has the form:

$$Y(a)=\begin{pmatrix} 0&0&0&0&0&1\\ 1&0&0&0&0&1\\ 0&1&0&0&0&1\\ 0&0&1&0&0&1\\ 0&0&0&1&0&1\\ 0&0&0&0&1&1 \end{pmatrix},\ Y(b)=\begin{pmatrix} 1&0&0&0&0&0\\ 1&0&0&0&0&1\\ 1&0&0&0&1&0\\ 1&0&0&1&0&0\\ 1&0&1&0&0&0\\ 1&1&0&0&0&0 \end{pmatrix}.$$

Therefore, there is a unique split extension of $\mathbb{Z}_2^6$ by H. Define $G(14,2)=\mathbb{Z}_2^6 \rtimes \mathrm{D}_{14}$. Then $G=G(14,2)$ is a primitive group of degree p^6 with $\Omega=V(6,2)\cong\mathbb{Z}_2^6$. Let $\alpha=(0,0,0,0,0,0)\in\Omega$, then $G_\alpha \cong H$ has an orbit $\Delta=\{(1,1,0,0,0,0), (1,0,0,0,0,0), (0,0,0,0,0,1), (0,0,0,0,1,1), (0,0,0,1,1,0), (0,0,1,1,0,0), (0,1,1,0,0,0)\}$ of length 7.

3.3 *Faithful irreducible representations of* $\mathbb{Z}_7 \rtimes \mathbb{Z}_3$

Let $H=\langle a,b \mid a^7=b^3=1, b^{-1}ab=a^2\rangle \cong \mathbb{Z}_7 \rtimes \mathbb{Z}_3$. Since H has five conjugacy classes, namely $\{1\}$, $\{a,a^2,a^4\}$, $\{a^3,a^5,a^6\}$, $\{a^ib \mid 0\le i\le 6\}$, $\{a^ib^2 \mid 0\le i\le 6\}$, it follows that H has five irreducible characters over $\mathbb{C}$. By [19] the irreducible character table of H over $\mathbb{C}$ is listed in Table 3.

It is clear that the splitting field E for H is as follows:

$$E=\begin{cases} GF(3^6) & \text{if } p=3,\\ GF(p) & \text{if } p\equiv 1 \pmod{21},\\ GF(p^2) & \text{if } p\equiv d \pmod{21}, d\in\{-1,\pm 8\},\\ GF(p^3) & \text{if } p\equiv 4 \pmod{21},\\ GF(p^6) & \text{if } p\equiv d \pmod{21}, d\in\{\pm 2,-4,\pm 5,\pm 10\}. \end{cases}$$

Assume $p \equiv 1 \pmod{21}$. Then F is a splitting field for H. Hence H has only two faithful irreducible representations X_i which affords the irreducible character χ_i over F for $i = 4, 5$, all of which have degree 3. Then X_i can be given by the following matrices:

$$X_4(a) = \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon^2 & 0 \\ 0 & 0 & \varepsilon^4 \end{pmatrix}, \ X_4(b) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix};$$

$$X_5(a) = \begin{pmatrix} \varepsilon^3 & 0 & 0 \\ 0 & \varepsilon^6 & 0 \\ 0 & 0 & \varepsilon^5 \end{pmatrix}, \ X_5(b) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

It is easy to verify that $X_4(H) = X_5(H)$. Therefore, there is a unique split extension of $\mathbb{Z}_p^3$ by H. Define $G(21, p) = \mathbb{Z}_p^3 \rtimes (\mathbb{Z}_7 \rtimes \mathbb{Z}_3)$ when $p \equiv 1 \pmod{21}$. Then $G = G(21, p)$ is a primitive group of degree p^3 with $\Omega = V(3, p) \cong \mathbb{Z}_p^3$. Let $\alpha = (0, 0, 0) \in \Omega$. Then $G_\alpha \cong H$ has an orbit $\Delta = \{(1, 1, 1), (\varepsilon, \varepsilon^2, \varepsilon^4), (\varepsilon^2, \varepsilon^4, \varepsilon), (\varepsilon^3, \varepsilon^6, \varepsilon^5), (\varepsilon^4, \varepsilon, \varepsilon^2), (\varepsilon^5, \varepsilon^3, \varepsilon^6), (\varepsilon^6, \varepsilon^5, \varepsilon^3)\}$ of length 7.

If $p \equiv d \pmod{21}$ for $d \in \{2, 4, 8, -5, -10\}$. Then H has only two faithful irreducible E-representations. Let $\sigma : x \mapsto x^p$ be a Frobenius automorphism of E. Then $\sigma \in Gal(E/F)$ and it fixes $\varepsilon + \varepsilon^2 + \varepsilon^4$ and $\varepsilon^3 + \varepsilon^5 + \varepsilon^6$. It implies that both $\varepsilon + \varepsilon^2 + \varepsilon^4$ and $\varepsilon^3 + \varepsilon^5 + \varepsilon^6$ are in F. Hence $F(\chi_i) = F$ for each irreducible character χ_i $(1 \leq i \leq 5)$. Let Y be an irreducible representation of H over F and X be an irreducible constituent of Y^E. Then Y^E is similar to X. Therefore H has exactly two irreducible F-representations Y_4 and Y_5 of degree 3 which are listed as follows:

Assume that $G \cong \mathbb{Z}_p^3 \rtimes (\mathbb{Z}_7 \rtimes \mathbb{Z}_3)$. Since the average value of the character χ_4 on a subgroup $\mathbb{Z}_3$ of $\mathbb{Z}_7 \rtimes \mathbb{Z}_3$ is 1, so that $\mathbb{Z}_3$ fixes a non-trivial element of $V(3, p)$, then $\mathbb{Z}_7 \rtimes \mathbb{Z}_3$ has an orbit of length 7 on $V(3, p)$, namely $G_\alpha \cong (\mathbb{Z}_7 \rtimes \mathbb{Z}_3)$ has an orbit of length 7 on Ω.

$$Y_4(a) = \begin{pmatrix} \varepsilon + \varepsilon^2 + \varepsilon^4 & 0 & 1 \\ 1 & 0 & 0 \\ -\varepsilon^3 - \varepsilon^5 - \varepsilon^6 & 1 & 0 \end{pmatrix}, \qquad Y_4(b) = \begin{pmatrix} 1 & 0 & \varepsilon^3 + \varepsilon^5 + \varepsilon^6 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix};$$

$$Y_5(a) = \begin{pmatrix} \varepsilon^3 + \varepsilon^5 + \varepsilon^6 & \varepsilon + \varepsilon^2 + \varepsilon^4 & -1 \\ -1 & 1 & \varepsilon + \varepsilon^2 + \varepsilon^4 \\ -1 & -\varepsilon^3 - \varepsilon^5 - \varepsilon^6 & -1 \end{pmatrix}, \quad Y_5(b) = \begin{pmatrix} 1 & 0 & \varepsilon^3 + \varepsilon^5 + \varepsilon^6 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}.$$

As $\langle Y_i(a), Y_i(b) \rangle \cong \mathbb{Z}_7 \rtimes \mathbb{Z}_3$ and $Y_4(H) = Y_5(H)$ we have that there exists a unique split extension of $\mathbb{Z}_p^2$ by H. Therefore, there is a unique split extension of $\mathbb{Z}_p^3$ by H. Define

$G(21, p) = \mathbb{Z}_p^3 \rtimes (\mathbb{Z}_7 \rtimes \mathbb{Z}_3)$ if $p \equiv d \pmod{21}$ and $d \in \{2, 4, 8, -5, -10\}$.

Then $G = G(21, p)$ is a primitive group of degree p^3 on $\Omega = V(3, p) \cong \mathbb{Z}_p^3$. Let $\alpha = (0, 0, 0) \in \Omega$. Then $G_\alpha \cong H$ has an orbit $\Delta = \{(\varepsilon+\varepsilon^2+\varepsilon^4,\, \varepsilon^3+\varepsilon^5+\varepsilon^6,\, 3),\, (\varepsilon+\varepsilon^2+\varepsilon^4,\, 3,\, \varepsilon+\varepsilon^2+\varepsilon^4),\, (\varepsilon^3+\varepsilon^5+\varepsilon^6,\, \varepsilon+\varepsilon^2+\varepsilon^4,\, \varepsilon+\varepsilon^2+\varepsilon^4),\, (\varepsilon+\varepsilon^2+\varepsilon^4,\, \varepsilon+\varepsilon^2+\varepsilon^4,\, \varepsilon^3+\varepsilon^5+\varepsilon^6),\, (\varepsilon^3+\varepsilon^5+\varepsilon^6,\, \varepsilon+\varepsilon^2+\varepsilon^4,\, \varepsilon^3+\varepsilon^5+\varepsilon^6),\, (\varepsilon^3+\varepsilon^5+\varepsilon^6,\, \varepsilon^3+\varepsilon^5+\varepsilon^6,\, \varepsilon+\varepsilon^2+\varepsilon^4),\, (3,\, \varepsilon^3+\varepsilon^5+\varepsilon^6,\, \varepsilon^3+\varepsilon^5+\varepsilon^6)\}$ of length 7.

If $p \equiv d \pmod{21}$ for $d \in \{-1, -2, -4, 5, -8, 10\}$. Then H has only two faithful irreducible E-representations. Let $\sigma : x \mapsto x^p$ be a Frobenius automorphism of E. Then $\sigma \in Gal(E/F)$ and it interchanges $\varepsilon+\varepsilon^2+\varepsilon^4$ and $\varepsilon^3+\varepsilon^5+\varepsilon^6$. So χ_4 and χ_5 comprise a Galois conjugacy class over F. Furthermore, there is an irreducible representation Y of G over F which affords an irreducible character μ by Lemma 2.6 such that Y^E is similar to $X_4 \oplus X_5$ and $\mu = \chi_4 + \chi_5$. Therefore Y is an irreducible F-representation of degree 6.

If $p = 3$. Then H has three irreducible Brauer characters $\psi_i\,(1 \le i \le 3)$ over E since H has three 3-regular classes. Let $Y_i\,(1 \le i \le 3)$ be irreducible representations over E which affords the Brauer characters $\psi_i(1 \le i \le 3)$. It is easy to determine the Brauer character table of H over E, see Table 4 below. Let $\sigma : x \mapsto x^3$ be a Frobenius automorphism of E. Then $\sigma \in Gal(E/F)$ and it interchanges $\varepsilon+\varepsilon^2+\varepsilon^4$ and $\varepsilon^3+\varepsilon^5+\varepsilon^6$. So ψ_2 and ψ_3 comprise a Galois conjugacy class over F. Furthermore, there is an irreducible representation Y of H over F which affords an irreducible character μ by Lemma 2.6 such that Y^E is similar to $Y_2 \oplus Y_3$ and $\mu = \psi_2 + \psi_3$. We also have that Y is an irreducible F-representation of degree 6.

Table 4: Brauer character table of $\mathbb{Z}_7 \rtimes \mathbb{Z}_3$ for $p = 3$

	1	a	a^3
ψ_1	1	1	1
ψ_2	3	$\varepsilon+\varepsilon^2+\varepsilon^4$	$\varepsilon^3+\varepsilon^5+\varepsilon^6$
ψ_3	3	$\varepsilon^3+\varepsilon^5+\varepsilon^6$	$\varepsilon+\varepsilon^2+\varepsilon^4$

For these two cases, Y can be represented by

$$Y(a) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & -1 \end{pmatrix},\ Y(b) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 & -1 \end{pmatrix}.$$

It is easy to verify that $\langle Y(a), Y(b)\rangle \cong \mathbb{Z}_7 \rtimes \mathbb{Z}_3$. Therefore, there is a unique split extension of $\mathbb{Z}_p^6$ by H. Define

$G(21,p) = \mathbb{Z}_p^6 \rtimes (\mathbb{Z}_7 \rtimes \mathbb{Z}_3)$ if $p = 3$ or $p \equiv d \pmod{21}$ for $d \in \{-1,-2,-4,5,-8,10\}$.

Then $G = G(21,p)$ is a primitive group of degree p^6 on $\Omega = V(6,p) \cong \mathbb{Z}_p^6$. Let $\alpha = (0,0,0,0,0,0) \in \Omega$. Then $G_\alpha \cong H$ has an orbit $\Delta = \{(1,1,0,0,0,0), (-1,0,0,0,0,0), (0,0,0,0,0,-1), (0,0,0,0,1,1), (0,0,0,-1,-1,0), (0,0,1,1,0,0), (0,-1,-1,0,0,0)\}$ of length 7.

3.4 *Faithful irreducible representations of* $\mathbb{Z}_7 \rtimes \mathbb{Z}_6$

Let $H = \langle a, b \mid a^7 = b^6 = 1, b^{-1}ab = a^5\rangle \cong \mathbb{Z}_7 \rtimes \mathbb{Z}_6$. Since H has seven conjugacy classes, namely $\{1\}$, $\{a^i \mid 1 \le i \le 6\}$, $\{a^ib^j \mid 0 \le i \le 6, 1 \le j \le 5\}$, it follows that H has seven irreducible characters over $\mathbb{C}$ which are listed in Table 5 (also see [19]).

Table 5: The character table of $\mathbb{Z}_7 \rtimes \mathbb{Z}_6$

	1	b	b^2	b^3	b^4	b^5	a
χ_1	1	1	1	1	1	1	1
χ_2	1	ω	ω^2	ω^3	ω^4	ω^5	1
χ_3	1	ω^2	ω^4	1	ω^2	ω^4	1
χ_4	1	ω^3	1	ω^3	1	ω^3	1
χ_5	1	ω^4	ω^2	1	ω^4	ω^2	1
χ_6	1	ω^5	ω^4	ω^3	ω^2	ω	1
χ_7	6	0	0	0	0	0	-1

Since H has only one nonlinear character, one gets that H has only one faithful irreducible representations X_7 which affords the irreducible character χ_7 over F of degree 6. Then X_7 has the form:

$$Y(a) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & -1 \end{pmatrix}, \; Y(b) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & -1 & -1 & -1 & -1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & -1 & -1 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

It is easy to verify that $\langle Y(a), Y(b)\rangle \cong \mathbb{Z}_7 \rtimes \mathbb{Z}_6$. Thus there is a unique split extension of

$\mathbb{Z}_p^6$ by H. Define $G(42,p) = \mathbb{Z}_p^6 \rtimes (\mathbb{Z}_7 \rtimes \mathbb{Z}_6)$. Then $G = G(42,p)$ is a primitive group of degree p^6 on $\Omega = V(6,p) \cong \mathbb{Z}_p^6$. Let $\alpha = (0,0,0,0,0,0) \in \Omega$. Thus $G_\alpha \cong H$ has an orbit Δ=$\{(1,1,0,0,0,0),\ (-1,0,0,0,0,0), (0,0,0,0,0,-1),\ (0,0,0,0,1,1),\ (0,0,0,-1,-1,0), (0,0,1,1,0,0),\ (0,-1,-1,0,0,0)\}$ of length 7.

4. Proof of Theorem 1.1

In this section we will use the results in Section 3 to provide the proof of Theorem 1.1.

Proof : Since $G_\alpha^{\Delta(\alpha)}$ is solvable and $|\Delta(\alpha)| = 7$, it follows that $G_\alpha^{\Delta(\alpha)}$ is an affine primitive permutation group of degree 7. Thus it is isomorphic to one of the following groups: $\mathbb{Z}_7, \mathrm{D}_{14} \cong \mathbb{Z}_7 \rtimes \mathbb{Z}_2, \mathbb{Z}_7 \rtimes \mathbb{Z}_3$ or $\mathrm{AGL}(1,7) \cong \mathbb{Z}_7 \rtimes \mathbb{Z}_6$.

Case (1) : Assume $G_\alpha^{\Delta(\alpha)} \cong Z_7$. Then $G_\alpha^{\Delta(\alpha)} \cong G_\alpha \cong \mathbb{Z}_7$ and G is solvable by Lemma 2.1. It follows that $G \cong G(7,p)$ by the argument in Subsection 3.2.

Case (2) : Assume $G_\alpha^{\Delta(\alpha)} \cong \mathrm{D}_{14}$. If G_α acts faithfully on $\Delta(\alpha)$, we have that either G is solvable or G is isomorphic to $\mathrm{PSL}(2,8)$, $\mathrm{PSL}(2,13)$ or ${}^2B_2(8)$ by Lemma 2.2. For the former case, we have that $G \cong G(14,p)$ by the argument in Subsection 3.3. If G_α acts unfaithfully on $\Delta(\alpha)$. Then G is isomorphic to $\mathrm{PSL}(2,27)$, $\mathrm{PSL}(2,29)$, $\mathrm{PGL}(2,27)$ or $\mathrm{PGL}(2,29)$ by Lemma 2.1.

Case (3) : Assume $G_\alpha^{\Delta(\alpha)} \cong \mathbb{Z}_7 \rtimes \mathbb{Z}_3$. One has that G_α is solvable and $|G_\alpha| \mid 7 \cdot 3^2$ by Lemma 2.3. By [20] we get that G must be one of the following three types: (1) an affine type primitive group, (2) an almost simple primitive group, (3) a product action type primitive group.

If G is an affine type primitive group. Then $G_\alpha^{\Delta(\alpha)} \cong G_\alpha \cong \mathbb{Z}_7 \rtimes \mathbb{Z}_3$ by Lemma 2.4. Combining it and the results of Subsection 3.3 together, we have $G \cong G(21,p)$.

If G is an almost simple group. Since $|G_\alpha| \mid 7 \cdot 3^2$, we have that the pair (G, G_α) must be one of the following: $(\mathrm{PSL}(3,2), \mathbb{Z}_7 \rtimes \mathbb{Z}_3)$, $(\mathrm{PGL}(3,4), \mathbb{Z}_7 \rtimes (\mathbb{Z}_3 \times \mathbb{Z}_3))$ or $(\mathrm{PGU}(3,5), \mathbb{Z}_7 \rtimes (\mathbb{Z}_3 \times \mathbb{Z}_3))$ by checking [20, Tables 14-20].

Assume that G is a product action type primitive group, i.e., G satisfies $T^m \trianglelefteq G \leq G_1 \wr S_m (m \geq 2)$ where G_1 is an almost simple primitive group acting on Δ with socle T. Let $\alpha = (\delta, \delta, \cdots, \delta)$ where $\delta \in \Delta$. Then $(T_\delta)^m = (T^m)_\alpha \leq G_\alpha$. It follows that $m = 2$ and $T_\delta \cong \mathbb{Z}_3$. However, the pair (T, T_δ) must belong to one of Tables 14-20 [20] which is contradicts $T_\delta \cong \mathbb{Z}_3$ by [20, Theorem 1.1].

Case (4) : Assume $G_\alpha^{\Delta(\alpha)} \cong \mathbb{Z}_7 \rtimes \mathbb{Z}_6$. Then $G_\alpha^{\Delta(\alpha)}$ is sharply 2-transitive on $\Delta(\alpha)$. By [2, Main Theorem] we get that one of the following holds: (1) $Soc(G)$ is elementary abelian; (2) G is the Monster M or the Baby Monster B; (3) G is almost simple except for (2).

If G is an affine type primitive group. Then by Lemma 2.4 we have $G_\alpha^{\Delta(\alpha)} \cong G_\alpha \cong \mathbb{Z}_7 \rtimes \mathbb{Z}_6$. It follows that $G \cong G(42, p)$ by the results of Subsection 3.4.

If $G = B$ or M. Then by [20, Table 15] we have $7^2 \mid |G_\alpha|$ and $7 \nmid |G_\alpha|$, respectively. However, this contradicts $7 \| |G_\alpha|$.

If G is an almost simple group except for the Monster M and the Baby Monster B. Then by [2, Main Theorem], we have that (G, G_α) must be one of the following: $(\mathrm{S}_7, \mathbb{Z}_7 \rtimes \mathbb{Z}_6)$, $(\mathrm{J}_1, \mathbb{Z}_7 \rtimes \mathbb{Z}_6)$, $(\mathrm{P\Sigma L}(2, 27), \mathbb{Z}_{14} \rtimes \mathbb{Z}_6)$, $(\mathrm{PSL}(3, 4).\mathbb{Z}_6, \mathbb{Z}_7 \rtimes (\mathbb{Z}_6 \times \mathbb{Z}_3))$ or $({}^2B_2(8).\mathbb{Z}_3, \mathbb{Z}_7 \rtimes \mathbb{Z}_6)$.

This completes the proof of Theorem 1.1. □

ACKNOWLEDGEMENT

We are very grateful to Prof. Shenglin Zhou for helping us to modify and discuss this paper. The first author was supported by the Basic Ability Improvement Project for Young and Middle Aged Teachers in Guangxi (NO. 2017KY1302). The second author was supported by Guangxi Natural Science Foundation Program (2013GXNSFAA019018). The third author was supported in part by Shandong Provincial Natural Science Foundation, China (ZR2017MA022) and NSFC (11201401, 11561021 and 11761079).

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