ON 3-REGULAR PARTITIONS IN 3-COLORS

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We consider \( p_{\{3,3\}}(n) \), the number of 3-regular partitions in 3-colors. We find the generating functions for \( p_{\{3,3\}}(n) \) and deduce congruences modulo large powers of 3. We also find the generating functions and congruences for linear combination of \( p_3(n) \) (the number of partitions of \( n \) in 3-colors) by finding the relation connecting \( p_3(n) \) and \( p_{\{3,3\}}(n) \). As an application, we find finite discrete convolution of \( p_{\{3,1\}}(n) \) and \( p_{\{3,2\}}(n) \).

Key words: Partitions; 3-colors; 3-regular partitions; congruences.

1. INTRODUCTION

We start by defining the generating functions by setting \( E(q) = \prod_{n \geq 1} (1 - q^n) \).

- Let \( p(n) \) denotes the number of partitions of \( n \), given by
  \[
  \sum_{n \geq 0} p(n)q^n = \frac{1}{E(q)}. \tag{1.1}
  \]

- Let \( p_3(n) \) denotes the number of partitions of \( n \) in three colors, given by
  \[
  \sum_{n \geq 0} p_3(n)q^n = \frac{1}{E(q)^3}. \tag{1.2}
  \]

- Let \( p_{\{3,k\}}(n) \) denotes the number of 3-regular partitions of \( n \) in \( k \)-colors, given by
  \[
  \sum_{n \geq 0} p_{\{3,k\}}(n)q^n = \frac{E(q^3)^k}{E(q)^k}. \tag{1.3}
  \]
Recently, Hirschhorn [2] established infinite family of congruences modulo powers of 3 for $p_3(n)$. For example, for all $\alpha, n \geq 0$, he proved that

$$p_3 \left( 3^{2\alpha+1} n + \frac{5 \times 3^{2\alpha+1} + 1}{8} \right) \equiv 0 \pmod{3^{3\alpha+2}}.$$

In this paper, we will show that for each $\alpha \geq 0$,

$$\sum_{n \geq 0} p_{\{3,3\}} \left( 3^{2\alpha} n + \frac{3^{2\alpha} - 1}{4} \right) q^n = \sum_{i=1}^{(3^{2\alpha+3}/4)} x_{2\alpha,i} q^{i-1} \frac{E(q^3)^{3(4i-3)}}{E(q)^{3(4i-3)}},$$

and

$$\sum_{n \geq 0} p_{\{3,3\}} \left( 3^{2\alpha+1} n + \frac{3^{2\alpha+2} - 1}{4} \right) q^n = \sum_{i=1}^{(3^{2\alpha+1+1}/4)} x_{2\alpha+1,i} q^{i-1} \frac{E(q^3)^{3(4i-1)}}{E(q)^{3(4i-1)}},$$

where the coefficient vectors $x_0 = (x_{0,1}, x_{0,2}, \ldots)$ are given by

$$x_0 = (x_{0,1}, x_{0,2}, x_{0,3}, \ldots) = (1, 0, 0, \ldots),$$

and for $\alpha \geq 0$,

$$x_{\alpha+1} = x_\alpha A \text{ if } \alpha \text{ is even},$$

$$x_{\alpha+1} = x_\alpha B \text{ if } \alpha \text{ is odd},$$

where $A = (a_{i,j})_{i,j \geq 1}$ and $B = (b_{i,j})_{i,j \geq 1}$ are defined by

$$a_{i,j} = m_{4i-3,i+j-1}, \quad b_{i,j} = m_{4i-1,i+j-1},$$

where $M = (m_{i,j})_{i,j \geq 1}$ is defined as follows. The first three rows of $M$ are

$$M = \begin{pmatrix}
9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
6 & 243 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
1 & 243 & 6561 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\end{pmatrix}$$

and

for $i \geq 4$, $m_{i,1} = 0$, and for $j \geq 2$, $m_{i,j} = 27m_{i-1,j-1} + 9m_{i-2,j-1} + m_{i-3,j-1}$.

Setting $\alpha = 0$ in (1.5) and $\alpha = 1$ in (1.4), we obtain

$$\sum_{n \geq 0} p_{\{3,3\}} (3n + 2) q^n = 9 \frac{E(q^3)^9}{E(q)^9}.$$
and
\[ \sum_{n \geq 0} p_{\{3,3\}}(9n + 2)q^n = 9 \frac{E(q)^3}{E(q)^3} + 2187q \frac{E(q)^{15}}{E(q)^{15}} + 59049q^2 \frac{E(q)^{27}}{E(q)^{27}}. \] (1.13)

These are analogous to Ramanujan’s most beautiful identities [3, p. 239 and p. 243]
\[ \sum_{n \geq 0} p(5n + 4)q^n = 5 \frac{E(q)^5}{E(q)^6} \] (1.14)
and
\[ \sum_{n \geq 0} p(7n + 5)q^n = 7 \frac{E(q)^7}{E(q)^8}. \] (1.15)

From the result (1.5), we prove that for \( \alpha \geq 0 \) and \( n \geq 0 \),
\[ p_{\{3,3\}} \left( \frac{3^{2\alpha+1} n + 3^{2\alpha+2} - 1}{4} \right) \equiv 0 \pmod{3^{2\alpha+2}}. \] (1.16)

The result (1.16) is analogous to Ramanujan’s congruences modulo powers of 5 [1], for \( \alpha \geq 0 \) and for \( n \geq 0 \),
\[ p \left( \frac{5^{2\alpha+1} n + 19 \times 5^{2\alpha+1} + 1}{24} \right) \equiv 0 \pmod{5^{2\alpha+1}} \] (1.17)
and
\[ p \left( \frac{5^{2\alpha+2} n + 23 \times 5^{2\alpha+2} + 1}{24} \right) \equiv 0 \pmod{5^{2\alpha+2}}. \] (1.18)

Using (1.4) and (1.5), we find the generating functions for linear combination of \( p_3(n) \), for each \( \alpha \geq 0 \),
\[
\sum \sum (-1)^k (2k + 1) p_3 \left( 3^{2\alpha} n + \frac{3^{2\alpha} - 1}{4} - \frac{3k(k + 1)}{2} \right) q^n \\
= \sum_{i=1}^{(3^{2\alpha+3})/4} x_{2\alpha,i} q^{i-1} E(q)^{3(4i-3)} \frac{E(q)^{3(4i-3)}}{E(q)} \] (1.19)
and
\[
\sum \sum (-1)^k (2k + 1) p_3 \left( 3^{2\alpha+1} n + \frac{3^{2\alpha+1} - 1}{4} - \frac{3k(k + 1)}{2} \right) q^n \\
= \sum_{i=1}^{(3^{2\alpha+1+1})/4} x_{2\alpha+1,i} q^{i-1} E(q)^{3(4i-1)} \frac{E(q)^{3(4i-1)}}{E(q)}. \] (1.20)

where here and in the sequel, we set for \( t < 0 \), \( p_3(t) = 0 \).
Using (1.20), we find the congruences for linear combination of $p_3(n)$, for each $\alpha \geq 0$ and $n \geq 0$,

\[
\sum_{k \geq 0} (-1)^k (2k + 1)p_3 \left(3^{2\alpha+1}n + \frac{3^{2\alpha+2} - 1}{4} - \frac{3k(k + 1)}{2}\right) \equiv 0 \pmod{3^{2\alpha+2}}. \tag{1.21}
\]

**Example**: Setting $\alpha = 1$ and $n = 0$ in (1.21),

\[
-7p_3(2) + 5p_3(11) - 3p_3(17) + p_3(20) = -7(9) + 5(4599) - 3(90882) + 341649 = 1135 \cdot 3^4 \equiv 0 \pmod{3^4}.
\]

As an application in Section 5, we find finite discrete convolution of $p_{\{3,1\}}(n)$ and $p_{\{3,2\}}(n)$.

## 2. Preliminaries

Due to Jacobi, we have

\[
E(q)^3 = \sum_{n \geq 0} (-1)^n (2n + 1)q^{n(n+1)/2}. \tag{2.1}
\]

Which is equivalent to

\[
E(q)^3 = 1 - 3q + 5q^3 - 7q^6 + 9q^{10} - 11q^{15} + 13q^{21} - 15q^{28} + 17q^{36} - 19q^{45} + 21q^{55} - \cdots
\]

\[
= (1 + 5q^3 - 7q^6 - 11q^{15} + 13q^{21} + 17q^{36} - 19q^{45} + \cdots)
- 3q(1 - 3q^9 + 5q^{27} - 7q^{54} + \cdots)
= P(q^3) - 3qE(q^9)^3,
\]

where

\[
P(q) = \sum_{-\infty}^{\infty} (-1)^n (6n + 1)q^{n(3n+1)/2}.
\]

Let $\omega = e^{2\pi i/3}$. Replacing $q$ by $q$, $\omega q$, and $\omega^2 q$ in (2.2), and multiplying the three results, we find that

\[
P(q^3)^3 - 27q^3E(q^9)^9
= E(q)^3E(\omega q)^3E(\omega^2 q)^3
= \prod_{n \geq 1} \left(1 - q^n\right)^3\left(1 - \omega^n q^n\right)^3\left(1 - \omega^2 q^n\right)^3
\]

\[
= \prod_{3 \nmid n} \left(1 - q^n\right)^3\left(1 - \omega^n q^n\right)^3\left(1 - \omega^2 q^n\right)^3 \times \prod_{3 \nmid n} \left(1 - q^n\right)^3\left(1 - \omega^n q^n\right)^3\left(1 - \omega^2 q^n\right)^3
\]
\[
\prod_{n \geq 1} (1 - q^{3n})^9 \times \prod_{n \geq 1} (1 - q^{3(3n-1)})^3 (1 - q^{3(3n-2)})^3
\]
\[
= \prod_{n \geq 1} (1 - q^{3n})^9 \times \prod_{n \geq 1} (1 - q^{3n})^3 / \prod_{n \geq 1} (1 - q^{9n})^3
\]
\[
= \frac{E(q)^{12}}{E(q^{9})^{12}},
\]
that is
\[
P(q)^3 = \frac{E(q)^{12}}{E(q^{3})^{3}} + 27qE(q^{3})^9.
\]

Now let
\[
\zeta = \frac{E(q)^{3}}{qE(q^{9})^{3}},\quad \rho = \frac{P(q)^{3}}{qE(q^{9})^{3}},\quad T = \frac{E(q)^{12}}{q^{3}E(q^{9})^{12}}.
\]

Then, from (2.2) and (2.3),
\[
\zeta = \frac{E(q)^{3}}{qE(q^{9})^{3}} = \frac{P(q)^{3} - 3qE(q^{9})^{3}}{qE(q^{9})^{3}} = \frac{P(q)^{3}}{qE(q^{9})^{3}} - 3 = \rho - 3,
\]
\[
\zeta^2 = 9 - 6\rho + \rho^2
\]
and from (2.3)-(2.5),
\[
\zeta^3 = -27 + 27\rho - 9\rho^2 + \rho^3
\]
\[
= -27 + 27\rho - 9\rho^2 + \frac{P(q)^{3}}{qE(q^{9})^{9}}
\]
\[
= -27 + 27\rho - 9\rho^2 + \frac{1}{q^{3}E(q^{9})^{9}} \left( \frac{E(q)^{12}}{E(q^{9})^{3}} + 27q^{3}E(q^{9})^{9} \right)
\]
\[
= -27 + 27\rho - 9\rho^2 + (T + 27)
\]
\[
= 27\rho - 9\rho^2 + T.
\]

It follows from (2.5)-(2.7) that
\[
\zeta^3 + 9\zeta^2 + 27\zeta = T.
\]

We can write (2.8)
\[
\frac{1}{\zeta} = \frac{1}{T} (27 + 9\zeta + \zeta^2),
\]
so
\[
\frac{1}{\zeta^i} = \frac{1}{T} \left( \frac{27}{\zeta^{i-1}} + \frac{9}{\zeta^{i-2}} + \frac{1}{\zeta^{i-3}} \right).
\]

Now let \( H \) be the “huffing” operator modulo 3, that is,
\[
H \left( \sum a_nq^n \right) = \sum a_{3n}q^{3n}.
\]
If we apply \( H \) to (2.10), we find
\[
H \left( \frac{1}{\zeta^i} \right) = \frac{1}{T} \left( 27H \left( \frac{1}{\zeta^{i-1}} \right) + 9H \left( \frac{1}{\zeta^{i-2}} \right) + H \left( \frac{1}{\zeta^{i-3}} \right) \right). \tag{2.12}
\]

Now,
\[
H \left( \zeta^2 \right) = H \left( 9 - 6\rho + \rho^2 \right) = 9, \tag{2.13}
\]
\[
H \left( \zeta \right) = H \left( -3 + \rho \right) = -3, \tag{2.14}
\]
\[
H \left( 1 \right) = 1. \tag{2.15}
\]

From (2.12)-(2.15), we find
\[
H \left( \frac{1}{\zeta^i} \right) = \frac{3^2}{T}, \tag{2.16}
\]
\[
H \left( \frac{1}{\zeta^3} \right) = \frac{2 \cdot 3 + 3^5}{T^2}, \tag{2.17}
\]
\[
H \left( \frac{1}{\zeta^5} \right) = \frac{1 + 3^5}{T^2} + \frac{3^8}{T^3}, \tag{2.18}
\]
and so on.

Indeed, for \( i \geq 1 \) we can write
\[
H \left( \frac{1}{\zeta^i} \right) = \sum_{j=1}^{i} \frac{m_{i,j}}{T^j}, \tag{2.19}
\]
where the \( m_{i,j} \) are defined in the following matrix.

The \( m_{i,j} \) form a matrix \( M \), the first nine rows of which are
\[
M = \begin{pmatrix}
3^2 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
2 \cdot 3 & 3^5 & 0 & 0 & 0 & 0 & 0 & \cdots \\
1 & 3^5 & 3^8 & 0 & 0 & 0 & 0 & \cdots \\
0 & 2 \cdot 3^2 \cdot 5 & 2 \cdot 3^7 & 3^{11} & 0 & 0 & 0 & \cdots \\
0 & 3 \cdot 5 & 2^2 \cdot 3^5 \cdot 5 & 3^{10} \cdot 5 & 3^{14} & 0 & 0 & \cdots \\
0 & 1 & 2 \cdot 3^6 & 3^9 \cdot 11 & 2 \cdot 3^{14} & 3^{17} & 0 & \cdots \\
0 & 0 & 2^2 \cdot 3^2 \cdot 7 & 2 \cdot 3^8 \cdot 7 & 3^{11} \cdot 7^2 & 3^{16} \cdot 7 & 3^{20} & \cdots \\
0 & 0 & 2^3 \cdot 3 \cdot 7 & 2 \cdot 3^6 \cdot 17 & 2^4 \cdot 3^{10} \cdot 5 & 2^2 \cdot 3^{14} \cdot 17 & 2^3 \cdot 3^{19} & \cdots \\
0 & 0 & 1 & 2 \cdot 3^7 & 3^{10} \cdot 29 & 3^{16} \cdot 5 & 2 \cdot 3^{19} \cdot 5 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{pmatrix} \tag{2.20}
\]
It is clear that if \( i > 3j \) then \( m_{i,j} = 0 \). So \( m_{4i-3,j} = 0 \) if \( j \leq i - 1 \).

It follows that we can write

\[
H \left( \frac{1}{\zeta^{4i-3}} \right) = \sum_{j=i}^{4i-3} \frac{m_{4i-3,j}}{T^j} = \sum_{j=1}^{3i-2} \frac{m_{4i-3,j+1}}{T^{i+j-1}} = \sum_{j=1}^{3i-2} \frac{a_{i,j}}{T^{i+j-1}} \tag{2.21}
\]

where the \( a_{i,j} \) are defined in (1.9).

Similarly, \( m_{4i-1,j} = 0 \) if \( j \leq i - 1 \), so we can write

\[
H \left( \frac{1}{\zeta^{4i-1}} \right) = \sum_{j=i}^{4i-1} \frac{m_{4i-1,j}}{T^j} = \sum_{j=1}^{3i} \frac{m_{4i-1,j+1}}{T^{i+j-1}} = \sum_{j=1}^{3i} \frac{b_{i,j}}{T^{i+j-1}} \tag{2.22}
\]

where the \( b_{i,j} \) are defined in (1.9).

We can write (2.21)

\[
H \left( \frac{qE(q^3)^{4i-3}}{E(q)^3} \right) = \sum_{j=1}^{3i-2} a_{i,j} \left( \frac{q^3 E(q^9)^{12}}{E(q^3)^{12}} \right)^{i+j-1}, \tag{2.23}
\]

and this can be rearranged to

\[
H \left( q^{i-3} E(q^3)^{3(4i-3)} \frac{E(q)^{3(4i-3)}}{E(q)^3} \right) = \sum_{j=1}^{3i-2} a_{i,j} q^{3j-3} E(q^9)^{3(4j-1)} E(q)^3 \left( \frac{q^{3(4i-3)}}{E(q)^3(4j-3)} \right)^{i+j-1}. \tag{2.24}
\]

Similarly, (2.22) is

\[
H \left( \frac{qE(q^3)^{4i-1}}{E(q)^3} \right) = \sum_{j=1}^{3i} b_{i,j} \left( \frac{q^3 E(q^9)^{12}}{E(q^3)^{12}} \right)^{i+j-1}, \tag{2.25}
\]

and this can be rearranged to

\[
H \left( q^{i-1} E(q^3)^{3(4i-1)} \frac{E(q)^{3(4i-1)}}{E(q)^3} \right) = \sum_{j=1}^{3i} b_{i,j} q^{3j-3} E(q^9)^{3(4j-3)} E(q)^3 \left( \frac{q^{3(4i-1)}}{E(q)^3(4j-3)} \right)^{i+j-1}. \tag{2.26}
\]

3. PROOF OF GENERATING FUNCTIONS

The identity (1.3) is the \( \alpha = 0 \) case of (1.4).

Suppose (1.4) holds for some \( \alpha \geq 0 \). Then

\[
\sum_{n \geq 0} p_{\{3,3\} \ (3^{2\alpha} n + \frac{3^{2\alpha} - 1}{4} )} q^n = \sum_{i=1}^{(3^{2\alpha}+3)/4} x_{2\alpha,i} q^{i-1} E(q^3)^{3(4i-3)} E(q)^{3(4i-3)} \tag{3.1}
\]
which is equivalent to
\[ \sum_{n \geq 0} p_{3,3} \left( 3^{2\alpha} n + \frac{3^{2\alpha} - 1}{4} \right) q^{n-2} = \sum_{i=1}^{(3^{2\alpha}+3)/4} x_{2\alpha,i} q^{i-3} \frac{E(q^3)3^{4i-3}}{E(q)3^{4i-3}}. \] (3.2)

Applying the operator \( H \) to (3.2), we find that
\[ \sum_{n \geq 0} p_{3,3} \left( 3^{2\alpha} (3n + 2) + \frac{3^{2\alpha} - 1}{4} \right) q^{3n} = \sum_{i=1}^{(3^{2\alpha}+3)/4} x_{2\alpha,i} H \left( q^{i-3} \frac{E(q^3)3^{4i-3}}{E(q)3^{4i-3}} \right) \]
\[ = \sum_{i=1}^{(3^{2\alpha}+3)/4} x_{2\alpha,i} \sum_{j=1}^{3i-2} a_{i,j} q^{3j-3} \frac{E(q^9)3^{4j-1}}{E(q)3^{4j-1}} \]
\[ = \sum_{j=1}^{(3^{2\alpha}+1+1)/4} \left( \sum_{i=1}^{(3^{2\alpha}+3)/4} x_{2\alpha,i} a_{i,j} \right) q^{3j-3-1} \frac{E(q^9)3^{4j-1}}{E(q)3^{4j-1}} \]
\[ = \sum_{j=1}^{(3^{2\alpha}+1+1)/4} x_{2\alpha+1,j} q^{3j-3} \frac{E(q^9)3^{4j-1}}{E(q)3^{4j-1}}. \]

Which implies that
\[ \sum_{n \geq 0} p_{3,3} \left( 3^{2\alpha+1} n + \frac{3^{2\alpha+2} - 1}{4} \right) q^n = \sum_{j=1}^{(3^{2\alpha+1}+1)/4} x_{2\alpha+1,j} q^{j-1} \frac{E(q^3)3^{4j-1}}{E(q)3^{4j-1}}, \] (3.3)
which is (1.5).

Now suppose (1.5) holds for each \( \alpha \geq 0 \). Then
\[ \sum_{n \geq 0} p_{3,3} \left( 3^{2\alpha+1} n + \frac{3^{2\alpha+2} - 1}{4} \right) q^n = \sum_{i=1}^{(3^{2\alpha+1}+1)/4} x_{2\alpha+1,i} q^{i-1} \frac{E(q^3)3^{4i-1}}{E(q)3^{4i-1}}. \] (3.4)

Applying the operator \( H \) to (3.4), we find that
\[ \sum_{n \geq 0} p_{3,3} \left( 3^{2\alpha+1}(3n) + \frac{3^{2\alpha+2} - 1}{4} \right) q^{3n} = \sum_{i=1}^{(3^{2\alpha+1}+1)/4} x_{2\alpha+1,i} H \left( q^{i-1} \frac{E(q^3)3^{4i-1}}{E(q)3^{4i-1}} \right) \]
\[
\sum_{n \geq 0} p_{\{3,3\}} \left( 3^{2\alpha + 2} n + \frac{3^{2\alpha + 2} - 1}{4} \right) q^n = \sum_{j = 1}^{(3^{2\alpha + 2} + 3)/4} x_{2\alpha + 2,j} q^{3j - 3} \frac{E(q^3)^{3(4j - 3)}}{E(q^3)^{3(4j - 3)}},
\]

which is \( \alpha + 1 \) case of (1.4). This completes the proof of (1.4) and (1.5) by induction.

From (1.2) and (1.3), we find that

\[
\sum_{n \geq 0} p_{\{3,3\}}(n)q^n = \sum_{n \geq 0} p_3(n)q^n \times E(q^3)^3
\]

\[
= \sum_{n \geq 0} p_3(n)q^n \sum_{k \geq 0} (-1)^k (2k + 1) q^{3k(k+1)/2}
\]

\[
= \sum_{n \geq 0} \sum_{k \geq 0} (-1)^k (2k + 1) p_3 \left( n - \frac{3k(k + 1)}{2} \right) q^n,
\]

where we use Jacobi identity and Cauchy product of two power series.

Using (1.4) and (1.5) in (3.6), we arrive at (1.19) and (1.20).

4. PROOF OF THE CONGRUENCES

Let \( \nu(N) \) be the largest power of 3 that divides \( N \).

It follows from (1.10) and (1.11) that

\[
\nu(m_{i,j}) \geq (3j - 1) - \frac{3}{2}(i - j) - \frac{1}{2} = \frac{9j - 3i - 3}{2},
\]

and then follows from (1.9) and (4.1) that

\[
\nu(a_{i,j}) \geq \frac{9(i + j - 1) - 3(4i - 3) - 3}{2} = \frac{9j - 3i - 3}{2},
\]

and

\[
\nu(b_{i,j}) \geq \frac{9(i + j - 1) - 3(4i - 1) - 3}{2} = \frac{9j - 3i - 9}{2}.
\]
It not hard to show that
\[ \nu(x_{2\alpha,j}) \geq 2\alpha + \frac{9j - 12}{2} \] (4.4)
and
\[ \nu(x_{2\alpha+1,j}) \geq 2\alpha + 2 + \frac{9j - 10}{2}. \] (4.5)

The identity (4.4) is true for \( \alpha = 0 \), by (1.6).

Suppose (4.4) is true for some \( \alpha \geq 0 \). Then
\[ \nu(x_{2\alpha+1,j}) \geq \min_{i \geq 1} (\nu(x_{2\alpha,i}) + \nu(a_{i,j})) = \nu(x_{2\alpha,1}) + \nu(a_{1,j}) \geq 2\alpha + \frac{9j - 6}{2} \geq 2\alpha + 2 + \frac{9j - 10}{2}, \]
which is (4.5).

Now suppose (4.5) is true for all \( \alpha \geq 0 \). Then
\[ \nu(x_{2\alpha+2,j}) \geq \min_{i \geq 1} (\nu(x_{2\alpha+1,i}) + \nu(b_{i,j})) = \nu(x_{2\alpha+1,1}) + \nu(b_{1,j}) \geq 2\alpha + 2 + \frac{9j - 12}{2}, \]
which is \( \alpha + 1 \) case of (4.4). This completes the proof of (4.4) and (4.5) by induction.

The congruence (1.16) follows from (1.5) together with (4.5).

It follows from (1.4) and (4.4) that for \( \alpha \geq 0 \) and \( n \geq 0 \),
\[ p_{\{3,3\}} \left( 3^{2\alpha}n + \frac{3^{2\alpha} - 1}{4} \right) \equiv 0 \pmod{3^{2\alpha}}, \] (4.6)
which is the special case of (1.16).

Equating coefficients of \( q^n \) on both sides of (3.6), we find that for each \( n \geq 0 \),
\[ p_{\{3,3\}}(n) = \sum_{k \geq 0} (-1)^k(2k + 1)p_3 \left( n - \frac{3k(k + 1)}{2} \right). \] (4.7)

Invoking (1.16) in (4.7), we arrive at (1.21).
5. Finite Discrete Convolution of \( p_{\{3,1\}}(n) \) and \( p_{\{3,2\}}(n) \)

It follows from (1.3) that

\[
\sum_{n \geq 0} p_{\{3,3\}}(n)q^n = \frac{E(q^3)^3}{E(q)^3} = \sum_{n \geq 0} p_{\{3,1\}}(n)q^n \sum_{n \geq 0} p_{\{3,2\}}(n)q^n
\]

\[
= \sum_{n \geq 0} \sum_{k=0}^{n} p_{\{3,1\}}(k)p_{\{3,2\}}(n-k)q^n,
\]

where we use Cauchy product of two power series.

Equating coefficients of \( q^n \) on both sides of (5.1), we obtain

\[
p_{\{3,3\}}(n) = \sum_{k=0}^{n} p_{\{3,1\}}(k)p_{\{3,2\}}(n-k). \quad (5.2)
\]

**Corollary 5.1** — For each \( \alpha \geq 0 \),

\[
\sum_{n \geq 0} 3^{2\alpha n + (3^{2\alpha} - 1)/4} \sum_{k=0}^{n} p_{\{3,1\}}(k)p_{\{3,2\}} \left( 3^{2\alpha n} + \frac{3^{2\alpha} - 1}{4} - k \right) q^n
\]

\[
= \sum_{i=1} x_{2\alpha,i} q^{1-1} E(q^3)^{3(4i-3)} \frac{E(q)^3(4i-3)}{E(q)^{3(4i-3)}} \quad (5.3)
\]

and

\[
\sum_{n \geq 0} 3^{2\alpha+1 n + (3^{2\alpha+2} - 1)/4} \sum_{k=0}^{n} p_{\{3,1\}}(k)p_{\{3,2\}} \left( 3^{2\alpha+1 n} + \frac{3^{2\alpha+2} - 1}{4} - k \right) q^n
\]

\[
= \sum_{i=1} x_{2\alpha+1,i} q^{1-1} E(q^3)^{3(4i-1)} \frac{E(q)^3(4i-1)}{E(q)^{3(4i-1)}} \quad (5.4)
\]

**Proof:** Substituting (1.4) into (5.1), we arrive at (5.3). Similarly, we obtain (5.4) from (1.5) and (5.1).

**Corollary 5.2** — For each \( \alpha \geq 0 \) and \( n \geq 0 \),

\[
3^{2\alpha+1 n + (3^{2\alpha+2} - 1)/4} \sum_{k=0}^{n} p_{\{3,1\}}(k)p_{\{3,2\}} \left( 3^{2\alpha+1 n} + \frac{3^{2\alpha+2} - 1}{4} - k \right) \equiv 0 \pmod{3^{2\alpha+2}}. \quad (5.5)
\]

**Proof:** Invoking (1.16) and (5.2), we obtain (5.5).
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REFERENCES

