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EULER RELATED BINOMIAL SUMS

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We develop new closed form representations of sums of reciprocal binomial coefficients. We also identify new integral and hypergeometric representation for the binomial-harmonic number sums.

Key words : Binomial coefficients; harmonic numbers; combinatorial series identities; summation formulas; partial fraction approach.

1. Introduction and Preliminaries

In the interesting paper [6], Nimbran considers the representation of

$$S(k)=\sum_{n=1}^{\infty}\frac{(nk-k)!}{(nk)!}, \tag{1.1}$$

for $k\in\mathbb{N}\setminus\{1\}$, in closed form and evaluates $S(k)$ for $k=\{2,3,4,5,6,8,10,12\}$. In particular $S(2)=\ln 2$ is listed in [4], $S(3)=\frac{\sqrt{3}\pi}{12}-\frac{1}{4}\ln 3$ and $S(4)=\frac{1}{4}\ln 2-\frac{\pi}{24}$ are listed in [5]. Nimbran's search of the literature yields no other evaluation of $S(k)$ for $k\geq 5$ and then sets out to evaluate $S(k)$ for $k=\{5,6,8,10,12\}$. Nimbran claims $S(10)$ is difficult to evaluate and finds it impossible to evaluate $S(k)$ for any other values of k. Nimbran's method of evaluating $S(k)$ is indeed ingenious and relies on the representation

$$\ln p=\sum_{m\geq 1}\left(\sum_{r=1}^{p-1}\left(\frac{1}{mp+r-m}-\frac{1}{mp}\right)\right)$$

which is a generalization of an identity given by Euler in 1734, [3]. As a by-product of Nimbran's investigations, he also obtains some rather interesting representations of π including

$$\pi=\frac{22}{7}-\sum_{n=1}^{\infty}\frac{60}{(4n^2-1)(16n^2-1)(16n^2-9)}.$$

In this paper we shall investigate (1.1) and give a general identity for $S(k)$ for every $k \in \mathbb{N}\backslash\{1\}$. Furthermore we shall extend our investigation of (1.1) and evaluate representations for harmonic number sums of the form $H_{kn}S(k)$. First we recall some definitions of some special functions that will be useful throughout this paper. The Gamma function, for $z \in \mathbb{C}$, as given by Euler in integral form is

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, \; \Re(z) > 0,$$

the special case for $z \in \mathbb{N}$ reduces to, from the recurrence relation, $\Gamma(n+1) = n\Gamma(n) = n!$. The Pochhammer, or shifted factorial is defined by $(\lambda)_\nu = \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)}$. The Beta function, or Euler integral of the first kind is

$$\begin{aligned} B(z,w) &= \int_0^1 t^{z-1}(1-t)^{w-1}\,dt \\ &= \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}, \; \Re(z) > 0, \Re(w) > 0. \end{aligned}$$

Let

$$H_n = \sum_{r=1}^n \frac{1}{r} = \int_0^1 \frac{1-t^n}{1-t}\,dt = \gamma + \psi(n+1) = \sum_{j=1}^\infty \frac{n}{j(j+n)}, \qquad H_0 := 0$$

be the nth harmonic number, where γ denotes the Euler-Mascheroni constant, $H_n^{(m)} = \sum_{r=1}^n \frac{1}{r^m}$ is the m^{th} order harmonic number and $\psi(z)$ is the Digamma (or Psi) function defined by

$$\psi(z) := \frac{d}{dz}\{\log \Gamma(z)\} = \frac{\Gamma'(z)}{\Gamma(z)} \text{ and } \psi(1+z) = \psi(z) + \frac{1}{z},$$

moreover

$$\psi(z) = -\gamma + \sum_{n=0}^\infty \left(\frac{1}{n+1} - \frac{1}{n+z} \right).$$

A generalized hypergeometric function is defined by

$$\begin{aligned} {}_pF_q[z] &= {}_pF_q\left[\begin{matrix} a_1, a_2, ..., a_p \\ b_1, b_2, ..., b_q \end{matrix} \middle| z \right] = {}_pF_q[(a_p);(b_q) \mid z] \\ &= \sum_{n\geq 0} \frac{(a_1)_n ... (a_p)_n}{(b_1)_n ... (b_q)_n} \frac{z^n}{n!} = \sum_{n \geq 0} \frac{\prod_{j=1}^p (a_j)_n}{\prod_{j=1}^q (b_j)_n} \frac{z^n}{n!} \end{aligned} \tag{1.2}$$

for b_j non-negative integers or zero. When $p \le q$; ${}_pF_q[z]$ converges for all complex values of z, ${}_pF_q[z]$ is an entire function. When $p > q+1$; ${}_pF_q[z]$ converge for $z=0$, unless it terminates, which it does when one of the parameters a_j is a negative integer, hence ${}_pF_q[z]$ is a polynomial in z. When $p = q+1$ the series converges in the unit disc $|z| < 1$, and also for $|z| = 1$ provided that $\Re\left(\sum\limits_{j=1}^{q} b_j - \sum\limits_{j=1}^{p} a_j\right) > 0$. When $p = 2, q = 1$ we have the familiar Gauss hypergeometric function

$$ {}_2F_1\left[\left.\begin{array}{c} a,b \\ c \end{array}\right| z\right] = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int\limits_0^1 \frac{t^{b-1}(1-t)^{c-b-1}}{(1-zt)^a} dt, $$

where $|z| < 1$, $\Re(c-b) > 0$ and $\Re(b) > 0$. The following Lemma will be useful in the development of the main Theorem.

Lemma 1 — Let $p(n)$ and $q(n)$ be polynomials in n where all the roots of $q(n)$ are simple. No root of $q(n)$ is in $\mathbb{N}$ and let the $\deg(p(n)) \le \deg(q(n) - 2)$. Let $v_n = \frac{p(n)}{q(n)}$. Then

$$ \sum_{n=0}^{\infty} v_n = -\sum_{r=1}^{k} \alpha_r \psi(\beta_r) \tag{1.3} $$

where

$$ v_n = \frac{p(n)}{q(n)} = \sum_{r=1}^{k} \frac{\alpha_r}{n+\beta_r}. \tag{1.4} $$

PROOF : From $v_n = \frac{p(n)}{q(n)}$ we have $\sum\limits_{n=0}^{\infty} v_n = \sum\limits_{n=0}^{\infty} \frac{p(n)}{q(n)}$. By partial fraction expansion $v_n = \sum_{r=1}^{k} \frac{\alpha_r}{n+\beta_r}$ since all the roots of $q(n)$ are simple. For the series $\sum\limits_{n=0}^{\infty} v_n$ to converge it suffices to have $\lim\limits_{n\to\infty} n v_n = 0$, in which case $\sum_{r=1}^{k} \alpha_r = 0$. Now

$$ \begin{aligned} \sum_{n=0}^{\infty} v_n &= \sum_{n=0}^{\infty}\sum_{r=1}^{k} \frac{\alpha_r}{n+\beta_r} \\ &= \sum_{n=0}^{\infty}\sum_{r=1}^{k} \alpha_r \left(\frac{1}{n+\beta_r} - \frac{1}{n+1}\right) \end{aligned} $$

$$= \sum_{r=1}^{k} \alpha_r \sum_{n=0}^{\infty} \left(\frac{1}{n+\beta_r} - \frac{1}{n+1} \right)$$

$$= -\sum_{r=1}^{k} \alpha_r \left(\gamma + \psi(\beta_r) \right)$$

$$= -\sum_{r=1}^{k} \alpha_r \psi(\beta_r)$$

and the Lemma is proved. □

2. Closed form Summation

We now prove the following theorem.

***Theorem* 1** — *Let $k \in \mathbb{N} \backslash \{1\}$ and $j \in \mathbb{R}^+$ then we have the novel representation*

$$T(j,k) = \sum_{n=1}^{\infty} \frac{1}{\binom{nk+j}{k}} = \sum_{r=1}^{k} (-1)^r \binom{k-1}{r-1} \psi(\frac{r+j}{k}). \tag{2.1}$$

The case $j = 0$ reduces to

$$T(0,k) = \sum_{n=1}^{\infty} \frac{1}{\binom{nk}{k}} = \sum_{r=1}^{k} (-1)^r \binom{k-1}{r-1} \psi(\frac{r}{k}). \tag{2.2}$$

PROOF : Consider the expansion

$$T(j,k) = \sum_{n=1}^{\infty} \frac{1}{\binom{nk+j}{k}} = \sum_{n=1}^{\infty} \frac{k!\ (nk-k+j)!}{(nk+j)!}$$

$$= k! \sum_{n=1}^{\infty} \frac{1}{\prod_{r=1}^{k} (nk+j+1-r)} = k! \sum_{n=1}^{\infty} \frac{1}{(nk+j+1-k)_k}$$

$$= \sum_{n=0}^{\infty} \frac{1}{\prod_{r=1}^{k} \left(n+1+\frac{j+1-r}{k} \right)}$$

where Pochhammer's symbol $(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)}$. By partial fraction decomposition we have

$$T(j,k) = \sum_{r=1}^{k} (-1)^r \binom{k-1}{r-1} \sum_{n=0}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+\frac{j-r}{k}} \right)$$

and applying Lemma 1 we conclude

$$T(j,k) = \sum_{r=1}^{k} (-1)^r \binom{k-1}{r-1} \psi\left(\frac{r+j}{k}\right)$$

and (2.1) follows. For $j=0$ (2.2) follows and we notice that $T(0,k) = k!S(k)$ which is the sum (1.1). □

It is possible to express $T(j,k)$ in terms of basic trigonometric functions and we show the result in the next remark

Remark 1 : Gauss's Digamma theorem states that for $0 < a < b$

$$\psi\left(\frac{a}{b}\right) = -\gamma - \ln(2b) - \frac{\pi}{2}\cot\left(\frac{\pi a}{b}\right) + 2\sum_{\mu=1}^{\left[\frac{b}{2}\right]-1} \cos\left(\frac{2\pi a \mu}{b}\right) \ln\left(\sin\left(\frac{\pi\mu}{b}\right)\right).$$

Applying Gauss's Digamma theorem to (2.1) we have

$$\begin{aligned} T(j,k) &= (-1)^k \ln(2k) + \sum_{r=1}^{k} (-1)^r \binom{k-1}{r-1} \left(-\frac{\pi}{2}\cot\left(\frac{(r+j)\pi}{k}\right)\right) \\ &\quad + 2\sum_{r=1}^{k} (-1)^r \binom{k-1}{r-1} \left(\sum_{\mu=1}^{\left[\frac{k}{2}\right]-1} \cos\left(\frac{2\pi(r+j)\mu}{k}\right) \ln\left(\sin\left(\frac{\pi\mu}{k}\right)\right)\right), \end{aligned}$$

where $[x]$ is the integer part of x and $r+j<k$. The case $j=0$ follows simply.

Some examples follow. The case $j=k$ is interesting and we see that

$$T(k,k) = \sum_{n=1}^{\infty} \frac{1}{\binom{nk+k}{k}} = \sum_{r=1}^{k} (-1)^r \binom{k-1}{r-1} \psi\left(\frac{r+k}{k}\right).$$

Now since $\psi(1+z)=\psi(z)+\frac{1}{z}$, we have that

$$\begin{aligned} T(k,k) &= \sum_{n=1}^{\infty}\frac{1}{\binom{nk+k}{k}}=\sum_{r=1}^{k}(-1)^{r}\binom{k-1}{r-1}\left(\frac{k}{r}+\psi(\frac{r}{k})\right) \\ &= -1+T(0,k). \end{aligned}$$

Also

$$\begin{aligned} T(0,6) &= 32\ln 2-\frac{27}{2}\ln 3-\frac{7\sqrt{3}\pi}{6}, \\ T(3,6) &= 47-32\ln 2-\frac{27}{2}\ln 3-\frac{11\sqrt{3}\pi}{6}, \\ T(8,6) &= -\frac{757}{28}+32\ln 2+\frac{27}{2}\ln 3-\frac{11\sqrt{3}\pi}{6}, \\ T\left(\frac{3}{2},4\right) &= \pi\left(4+2\sqrt{2}\right)-\frac{64}{3}. \end{aligned}$$

In the next section we give an extension to Theorem 1 by incorporating harmonic numbers to the sum $T(j,k)$ and associating the sum with hypergeometric and integral representation.

3. Extension

We begin with the proof of the following Theorem.

***Theorem* 2** — *Under the assumptions of Theorem 1 and let* $m\in\mathbb{N}$ *then,*

$$\begin{aligned} T^{(m)}(j,k) &= \sum_{n=1}^{\infty}\frac{d^{(m)}}{dj^{(m)}}\left(\binom{nk+j}{k}^{-1}\right)=\sum_{n=1}^{\infty}Q^{(m)}(j,k) \\ &= \frac{1}{k^{m}}\sum_{r=1}^{k}(-1)^{r}\binom{k-1}{r-1}\psi^{(m)}(\frac{r+j}{k}) \qquad (3.1) \\ &= \frac{m!}{k^{m}}\sum_{r=1}^{k}(-1)^{r+m}\binom{k-1}{r-1}H^{(m+1)}_{\frac{r+j-k}{k}}, \qquad (3.2) \end{aligned}$$

where

$$Q^{(m)}(j,k)=\frac{d^{(m)}}{dj^{(m)}}\left(\binom{nk+j}{k}^{-1}\right).$$

PROOF : From the identity (2.1) we differentiate both sides "m" times with respect to j so that

$$\begin{aligned} T^{(m)}(j,k) &= \sum_{n=1}^{\infty} \frac{d^{(m)}}{dj^{(m)}} \left(\binom{nk+j}{k}^{-1} \right) = \sum_{n=1}^{\infty} Q^{(m)}(j,k) \\ &= \frac{1}{k^m} \sum_{r=1}^{k} (-1)^r \binom{k-1}{r-1} \psi^{(m)}\left(\frac{r+j}{k}\right) \end{aligned}$$

and (3.1) follows. From the known identity, relating polygamma functions with harmonic numbers

$$\psi^{(m)}(1+z) = (-1)^m\, m! \left(H_z^{(m+1)} - \zeta(1+z) \right),$$

then

$$T^{(m)}(j,k) = \frac{m!}{k^m} \sum_{r=1}^{k} (-1)^{r+m} \binom{k-1}{r-1} H^{(m+1)}_{\frac{r+j-k}{k}}$$

since

$$\sum_{r=1}^{k} (-1)^r \binom{k-1}{r-1} = 0, \text{ for } k \geq 2,$$

hence (3.2) follows. For completeness we detail some values of $Q^{(m)}(j,k)$:

$$Q^{(1)}(j,k) = \frac{1}{\binom{nk+j}{k}} \left(H_{kn+j-k} - H_{kn+j} \right)$$

and

$$Q^{(2)}(j,k) = \frac{1}{\binom{nk+j}{k}} \left((H_{kn+j-k} - H_{kn+j})^2 - \left(H^{(2)}_{kn+j-k} - H^{(2)}_{kn+j} \right) \right),$$

some more details on the function $Q^{(m)}(j,k)$ are given in the paper [9]. □

The cases $j = 0$ and $j = k$ are interesting and the results are given in the next corollary.

Corollary 1 — For $j = 0$

$$\begin{aligned} T^{(m)}(0,k) &= \sum_{n=1}^{\infty} Q^{(m)}(0,k) \\ &= \frac{1}{k^m} \sum_{r=1}^{k} (-1)^r \binom{k-1}{r-1} \psi^{(m)}\left(\frac{r}{k}\right) \\ &= \frac{m!}{k^m} \sum_{r=1}^{k-1} (-1)^{r+m} \binom{k-1}{r-1} H^{(m+1)}_{\frac{r-k}{k}}, \end{aligned} \tag{3.3}$$

where

$$\sum_{n=1}^{\infty} Q^{(m)}(0,k) = \sum_{n=1}^{\infty} \lim_{j\to 0}\left(\frac{d^{(m)}}{dj^{(m)}}\left(\binom{nk+j}{k}^{-1}\right)\right).$$

For $j=k$

$$T^{(m)}(k,k) = T^{(m)}(0,k) + (-1)^{m+1}\Lambda^{(m)}(k) \tag{3.4}$$

where

$$\Lambda^{(m)}(k) = \lim_{\alpha\to 0}\left(\frac{d^{(m)}}{dj^{(m)}}\left(\binom{k+\alpha}{\alpha}^{-1}\right)\right)$$

PROOF : From (3.1) we have

$$\begin{aligned} T^{(m)}(0,k) &= \frac{1}{k^m}\sum_{r=1}^{k}(-1)^r\binom{k-1}{r-1}\psi^{(m)}(\frac{r}{k}) \\ &= \frac{m!}{k^m}\sum_{r=1}^{k}(-1)^{r+m}\binom{k-1}{r-1}H_{\frac{r-k}{k}}^{(m+1)}, \end{aligned}$$

since for $r=k,\ H_0^{(m+1)}=0$, then (3.3) follows. For the case $j=k$,

$$T^{(m)}(k,k) = \sum_{n=1}^{\infty} Q^{(m)}(k,k)$$

$$= \frac{1}{k^m}\sum_{r=1}^{k}(-1)^r\binom{k-1}{r-1}\psi^{(m)}(1+\frac{r}{k})$$

where

$$Q^{(m)}(k,k) = \lim_{j\to k}\left(\frac{d^{(m)}}{dj^{(m)}}\left(\binom{nk+j}{k}^{-1}\right)\right).$$

By the property of the polygamma function

$$\psi^{(m)}(1+z) = \psi^{(m)}(1+z) + \frac{(-1)^m\, m!}{z^{m+1}}$$

$$T^{(m)}(k,k) = \frac{1}{k^m}\sum_{r=1}^{k}(-1)^r\binom{k-1}{r-1}\left(\psi^{(m)}(\frac{r}{k}) + \frac{(-1)^m\, m!k^{m+1}}{r^{m+1}}\right)$$

$$= T^{(m)}(0,k) + (-1)^m\, m!k\sum_{r=1}^{k}(-1)^r\binom{k-1}{r-1}\frac{1}{r^{m+1}}.$$

From the paper [9], we have

$$\sum_{r=1}^{k}(-1)^{r}\binom{k-1}{r-1}\frac{1}{r^{m+1}}=-\frac{\Lambda^{(m)}(k)}{m!k}$$

hence

$$T^{(m)}(k,k)=T^{(m)}(0,k)+(-1)^{m+1}\Lambda^{(m)}(k),$$

hence (3.4) follows. Some values of $\Lambda^{(m)}(k)$ are

$$\Lambda^{(1)}(k) = H_k\,,\ \Lambda^{(2)}(k)=H_k^2+H_k^{(2)}$$

$$\Lambda^{(3)}(k) = H_k^3+3H_kH_k^2+2H_k^{(3)}.\square$$

Example 1 : Some illustrative examples follow.

$$\begin{aligned} T^{(1)}(j,k) &= \sum_{n=1}^{\infty}\frac{H_{nk-k+j}-H_{nk+j}}{\binom{nk+j}{k}} \\ &= \frac{m!}{k^m}\sum_{r=1}^{k}(-1)^{r+1}\binom{k-1}{r-1}H^{(2)}_{\frac{r+1-k}{k}}, \\ T^{(1)}(1,4) &= \frac{3}{16}\zeta(2)-\frac{1}{6}G-\frac{1}{6},\ T^{(1)}(0,4)=\frac{1}{6}G-\frac{7}{48}\zeta(2) \end{aligned}$$

where G is Catalan's constant.

$$\begin{aligned} T^{(3)}(0,2) &= -\frac{21}{2}\zeta(4), \\ T^{(4)}(4,4) &= -\frac{2835}{16}\zeta(5)-\frac{5\pi^5}{16}-\frac{76111}{864}. \end{aligned}$$

The expression $T(j,k)$ and $T^{(m)}(j,k)$ can also be represented in integral and hypergeometric form and for completeness the following is recorded.

***Theorem* 3** — *Let the assumptions of Theorem 1 apply, then*

$$T(j,k) = k\int_0^1\frac{x^j(1-x)^{k-1}}{(1-x^k)}dx, \tag{3.5}$$

$$T^{(m)}(j,k) \;=\; k\int_0^1 \frac{x^j (1-x)^{k-1} \ln^m x}{(1-x^k)} dx \tag{3.6}$$

and

$$T(j,k) = \frac{1}{\binom{k+j}{k}} \;{}_{1+k}F_k\left[\begin{array}{c} \frac{1+j}{k}, \frac{2+j}{k}, ..., \frac{k+j}{k}, 1 \\ \frac{1+j+k}{k}, \frac{2+j+k}{k}, ..., \frac{2k+j}{k} \end{array}\middle| 1\right]. \tag{3.7}$$

PROOF : Consider

$$\begin{aligned} T(j,k) &= \sum_{n=1}^{\infty} \frac{1}{\binom{nk+j}{k}} = \sum_{n=1}^{\infty} \frac{\Gamma(nk+j-k+1)\,\Gamma(k+1)}{\Gamma(nk+j+1)} \\ &= k\sum_{n=1}^{\infty} B(k, nk-k+j+1), \end{aligned}$$

where $\Gamma(\cdot)$ is the gamma function and $B(\cdot,\cdot)$ is the beta function. Now

$$T(j,k) = k\int_0^1 \frac{x^j (1-x)^{k-1}}{x^k} \sum_{n=1}^{\infty} \left(x^k\right)^n \, dx,$$

and (3.5) follows. Now differentiating m times with respect to j results in

$$T^{(m)}(j,k) = k\int_0^1 \frac{x^j (1-x)^{k-1} \ln^m x}{(1-x^k)} dx$$

hence (3.6). For the hypergeometric function we consider the definition (1.2) above and write

$$T(j,k) = \sum_{n=1}^{\infty} \frac{1}{\binom{nk+j}{k}} = \sum_{n=0}^{\infty} \frac{1}{\binom{nk+k+j}{k}}$$

therefore (3.7) follows. □

Remark 2 : It is straightforward to see, from (3.3) and (3.6), that

$$\begin{aligned} T^{(m)}(0,2) &= 2\int_0^1 \frac{\ln^m x}{1+x}dx = \sum_{n=1}^{\infty}\lim_{j\to 0}\left(\frac{d^{(m)}}{dj^{(m)}}\left(\binom{2n+j}{2}^{-1}\right)\right) \\ &= 2(-1)^m m!\left(1-2^{-m}\right)\zeta(m+1) \\ &= \frac{(-1)^{m+1} m!}{2^m}H_{-\frac{1}{2}}^{(m+1)} \\ &= 2m!\sum_{n=1}^{\infty}\frac{(-1)^{m+n+1}}{n^{m+1}}. \end{aligned}$$

Many other examples of binomial sums, harmonic number sums, integral representations and hypergeometric summation are available in [1, 2, 8, 10-15]. Some interesting binomial series are also investigated in [7].

References

1. W. Chu, Finite differences and Dixon-like Binomial sums, *Integral Transforms Spec. Funct.*, **23**(4) (2012), 251-261.
2. M. W. Coffey and S. J. Johnston, Some results involving series representations of hypergeometric functions, *J. Comput. Appl. Math.*, **233**(3) (2009), 674-679.
3. L. Euler, De progressionibus harmonicis observationes, *Comm. aca. sci. Petrop.*, **7** (1740), 150-161. *Opera omnia*, **14**(1), 87-100. http://www.math.dartmouth.edu/euler/pages/E043.
4. W. L. Ferrar, *A textbook of convergence*, Oxford University Press, 1947, p.74.
5. K. Knopp, Theory and applications of infinite series, Dover Publications, New York, 1990 reprint of Blackie and son, London 2^{nd} ed. 1951, p268, ex. 107(g) and (h).
6. A. Nimbran, Euler's constant, harmonic series and related infinite sums, *Math. Student*, **82**(1-4) (2013), 177-182.
7. A. Nimbran and P. Levrie, Some continued fractions for π and G. *Indian J. Pure Appl. Math.*, **48**(2) (2017), 187-204.
8. R. Roy, Binomial identities and hypergeometric series, *Amer. Math. Monthly*, **94**(1) (1987), 36-46.
9. A. Sofo, New classes of harmonic number identities, *J. Integer Seq.*, **15**(7) (2012), Article 12.7.4, 12 pp.
10. A. Sofo, Quadratic alternating harmonic number sums, *J. Number Theory*, **154** (2015), 144-159.

11. A. Sofo, Integrals of logarithmic and hypergeometric functions, *Commun. Math.*, **24**(1) (2016), 7-22.

12. A. Sofo, A master integral in four parameters, *J. Math. Anal. Appl.*, **448**(1) (2017), 81-92.

13. A. Sofo, Integral identities for sums, *Math. Commun.*, **13**(2) (2008), 303-309.

14. V. Strehl, Binomial identities-combinatorial and algorithmic aspects. Trends in discrete mathematics, *Discrete Math.*, **136**(1-3) (1994), 309-346.

15. E. W. Weisstein, “Binomial Sums”, From MathWorld-A Wolfram Web Resource, http://mathworld.wolfram.com/BinomialSums.html.