EULER RELATED BINOMIAL SUMS

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We develop new closed form representations of sums of reciprocal binomial coefficients. We also identify new integral and hypergeometric representation for the binomial-harmonic number sums.

Key words : Binomial coefficients; harmonic numbers; combinatorial series identities; summation formulas; partial fraction approach.

1. INTRODUCTION AND PRELIMINARIES

In the interesting paper [6], Nimbran considers the representation of

\[ S(k) = \sum_{n=1}^{\infty} \frac{(nk-k)!}{(nk)!}, \]

(1.1)

for \( k \in \mathbb{N} \setminus \{1\} \), in closed form and evaluates \( S(k) \) for \( k = \{2, 3, 4, 5, 6, 8, 10, 12\} \). In particular \( S(2) = \ln 2 \) is listed in [4], \( S(3) = \frac{\sqrt{3\pi}}{12} - \frac{1}{4} \ln 3 \) and \( S(4) = \frac{1}{4} \ln 2 - \frac{\pi}{24} \) are listed in [5]. Nimbran’s search of the literature yields no other evaluation of \( S(k) \) for \( k \geq 5 \) and then sets out to evaluate \( S(k) \) for \( k = \{5, 6, 8, 10, 12\} \). Nimbran claims \( S(10) \) is difficult to evaluate and finds it impossible to evaluate \( S(k) \) for any other values of \( k \). Nimbran’s method of evaluating \( S(k) \) is indeed ingenious and relies on the representation

\[ \ln p = \sum_{m=1}^{\infty} \left( \sum_{r=1}^{p-1} \left( \frac{1}{mp+r-m} - \frac{1}{mp} \right) \right) \]

which is a generalization of an identity given by Euler in 1734, [3]. As a by-product of Nimbran’s investigations, he also obtains some rather interesting representations of \( \pi \) including

\[ \pi = \frac{22}{7} - \sum_{n=1}^{\infty} \frac{60}{(4n^2 - 1)(16n^2 - 1)(16n^2 - 9)}. \]
In this paper we shall investigate (1.1) and give a general identity for $S(k)$ for every $k \in \mathbb{N} \setminus \{1\}$. Furthermore we shall extend our investigation of (1.1) and evaluate representations for harmonic number sums of the form $H_{kn} S(k)$. First we recall some definitions of some special functions that will be useful throughout this paper. The Gamma function, for $z \in \mathbb{C}$, as given by Euler in integral form is

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, \Re(z) > 0,$$

the special case for $z \in \mathbb{N}$ reduces to, from the recurrence relation, $\Gamma(n+1) = n \Gamma(n) = n!$. The Pochhammer, or shifted factorial is defined by

$$(\lambda)_\nu = \frac{\Gamma(\lambda + \nu)}{\Gamma(\lambda)}.$$

The Beta function, or Euler integral of the first kind is

$$B(z, w) = \int_0^1 t^{z-1} (1 - t)^{w-1} dt = \frac{\Gamma(z) \Gamma(w)}{\Gamma(z+w)}, \Re(z) > 0, \Re(w) > 0.$$

Let $H_n = \sum_{r=1}^n \frac{1}{r} = \int_0^1 \frac{1-t^n}{1-t} dt = \gamma + \psi(n+1) = \sum_{j=1}^{\infty} \frac{n}{j(j+n)}$, $H_0 := 0$

be the $n$th harmonic number, where $\gamma$ denotes the Euler-Mascheroni constant, $H_n^{(m)} = \sum_{r=1}^n \frac{1}{r^m}$ is the $m$th order harmonic number and $\psi(z)$ is the Digamma (or Psi) function defined by

$$\psi(z) := \frac{d}{dz} \{\log \Gamma(z)\} = \frac{\Gamma'(z)}{\Gamma(z)} \text{ and } \psi(1 + z) = \psi(z) + \frac{1}{z},$$

moreover

$$\psi(z) = -\gamma + \sum_{n=0}^{\infty} \left( \frac{1}{n+1} - \frac{1}{n+z} \right).$$

A generalized hypergeometric function is defined by

$$pF_q [z] = \left[ a_1, a_2, \ldots, a_p \right| b_1, b_2, \ldots, b_q \left| z \right. = \sum_{n\geq0} \frac{(a_1)_n \ldots (a_p)_n}{(b_1)_n \ldots (b_q)_n} \frac{z^n}{n!}$$

$$= \sum_{n\geq0} \frac{(a_1)_n \ldots (a_p)_n}{(b_1)_n \ldots (b_q)_n} \frac{z^n}{n!} = \frac{\prod_{j=1}^p (a_j)_n}{\prod_{j=1}^q (b_j)_n} \frac{z^n}{n!} \quad \text{(1.2)}.$$
for \(b_j\) non-negative integers or zero. When \(p \leq q\); \(pF_q[z]\) converges for all complex values of \(z\), \(pF_q[z]\) is an entire function. When \(p > q + 1\); \(pF_q[z]\) converge for \(z = 0\), unless it terminates, which it does when one of the parameters \(a_j\) is a negative integer, hence \(pF_q[z]\) is a polynomial in \(z\). When \(p = q + 1\) the series converges in the unit disc \(|z| < 1\), and also for \(|z| = 1\) provided that \(\Re \left( \sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j \right) > 0\). When \(p = 2\), \(q = 1\) we have the familiar Gauss hypergeometric function

\[
\mathbf{2F1} \left[ \begin{array}{c|c} a, b \\ \hline c \\ \hline z \end{array} \right] = \frac{\Gamma (c)}{\Gamma (b) \Gamma (c-b)} \int_0^1 \frac{t^{b-1} (1-t)^{c-b-1}}{(1-zt)^a} dt,
\]

where \(|z| < 1\), \(\Re (c - b) > 0\) and \(\Re (b) > 0\). The following Lemma will be useful in the development of the main Theorem.

\[\text{Lemma 1} \quad \text{Let} \quad p(n) \quad \text{and} \quad q(n) \quad \text{be polynomials in} \quad n \quad \text{where all the roots of} \quad q(n) \quad \text{are simple. No root of} \quad q(n) \quad \text{is in} \quad \mathbb{N} \quad \text{and let the} \quad \deg (p(n)) \leq \deg (q(n) - 2). \quad \text{Let} \quad v_n = \frac{p(n)}{q(n)}. \quad \text{Then}
\]

\[
\sum_{n=0}^{\infty} v_n = - \sum_{r=1}^{k} \alpha_r \psi (\beta_r)
\]

(1.3)

where

\[
v_n = \frac{p(n)}{q(n)} = \sum_{r=1}^{k} \frac{\alpha_r}{n + \beta_r}.
\]

(1.4)

\[\text{PROOF : From} \quad v_n = \frac{p(n)}{q(n)} \quad \text{we have} \quad \sum_{n=0}^{\infty} v_n = \sum_{n=0}^{\infty} \frac{p(n)}{q(n)}. \quad \text{By partial fraction expansion} \quad v_n = \sum_{r=1}^{k} \frac{\alpha_r}{n + \beta_r} \quad \text{since all the roots of} \quad q(n) \quad \text{are simple. For the series} \quad \sum_{n=0}^{\infty} v_n \quad \text{to converge it suffices to have} \quad \lim_{n \to \infty} n v_n = 0, \quad \text{in which case} \sum_{r=1}^{k} \alpha_r = 0. \quad \text{Now}
\]

\[
\sum_{n=0}^{\infty} v_n = \sum_{n=0}^{\infty} \sum_{r=1}^{k} \frac{\alpha_r}{n + \beta_r} = \sum_{n=0}^{\infty} \sum_{r=1}^{k} \alpha_r \left( \frac{1}{n + \beta_r} - \frac{1}{n + 1} \right)
\]

\[= \sum_{n=0}^{\infty} \sum_{r=1}^{k} \alpha_r \left( \frac{1}{n + \beta_r} - \frac{1}{n + 1} \right)
\]
\[ \sum_{r=1}^{k} \alpha_r \sum_{n=0}^{\infty} \left( \frac{1}{n + \beta_r} - \frac{1}{n + 1} \right) = - \sum_{r=1}^{k} \alpha_r (\gamma + \psi(\beta_r)) = - \sum_{r=1}^{k} \alpha_r \psi(\beta_r) \]

and the Lemma is proved. \(\square\)

2. Closed form Summation

We now prove the following theorem.

**Theorem 1** — Let \( k \in \mathbb{N} \setminus \{1\} \) and \( j \in \mathbb{R}^+ \) then we have the novel representation

\[ T(j, k) = \sum_{n=1}^{\infty} \frac{1}{nk + j} = \sum_{r=1}^{k} (-1)^r \binom{k-1}{r-1} \psi\left(\frac{r+j}{k}\right). \]  

(2.1)

The case \( j = 0 \) reduces to

\[ T(0, k) = \sum_{n=1}^{\infty} \frac{1}{nk} = \sum_{r=1}^{k} (-1)^r \binom{k-1}{r-1} \psi\left(\frac{r}{k}\right). \]  

(2.2)

**Proof**: Consider the expansion

\[ T(j, k) = \sum_{n=1}^{\infty} \frac{1}{nk + j} = \sum_{n=1}^{\infty} \frac{k! \ (nk - k + j)!}{(nk + j)!}. \]

\[ = k! \sum_{n=1}^{\infty} \prod_{r=1}^{k} \frac{1}{nk + j + 1 - r} = k! \sum_{n=1}^{\infty} \frac{1}{(nk + j + 1 - k)k}. \]

\[ = \sum_{n=0}^{\infty} \frac{k \prod_{r=1}^{k} \left( n + 1 + \frac{j+1-r}{k} \right)}{n} \]
where Pochhammer’s symbol \((x)_n = \frac{\Gamma(x+n)}{\Gamma(x)}\). By partial fraction decomposition we have

\[
T(j, k) = \sum_{r=1}^{k} (-1)^r \left( \begin{array}{c} k-1 \\ r-1 \end{array} \right) \sum_{n=0}^{\infty} \left( \frac{1}{n+1} - \frac{1}{n + \frac{j-r}{k}} \right)
\]

and applying Lemma 1 we conclude

\[
T(j, k) = \sum_{r=1}^{k} (-1)^r \left( \begin{array}{c} k-1 \\ r-1 \end{array} \right) \psi\left( \frac{r+j}{k} \right)
\]

and (2.1) follows. For \(j = 0\) (2.2) follows and we notice that \(T(0, k) = k! S(k)\) which is the sum (1.1). \(\square\)

It is possible to express \(T(j, k)\) in terms of basic trigonometric functions and we show the result in the next remark

**Remark 1**: Gauss’s Digamma theorem states that for \(0 < a < b\)

\[
\psi\left( \frac{a}{b} \right) = -\gamma - \ln (2b) - \frac{\pi}{2} \cot \left( \frac{\pi a}{b} \right) + 2 \sum_{\mu=1}^{[\frac{b}{a}]-1} \cos \left( \frac{2\pi \mu}{b} \right) \ln \left( \sin \left( \frac{\pi \mu}{b} \right) \right).
\]

Applying Gauss’s Digamma theorem to (2.1) we have

\[
T(j, k) = (-1)^k \ln (2k) + \sum_{r=1}^{k} (-1)^r \left( \begin{array}{c} k-1 \\ r-1 \end{array} \right) \left( -\frac{\pi}{2} \cot \left( \frac{(r+j)\pi}{k} \right) \right)
\]

\[
+ 2 \sum_{r=1}^{k} (-1)^r \left( \begin{array}{c} k-1 \\ r-1 \end{array} \right) \left( \sum_{\mu=1}^{[\frac{r+j}{k}]-1} \cos \left( \frac{2\pi (r+j) \mu}{k} \right) \ln \left( \sin \left( \frac{\pi \mu}{k} \right) \right) \right),
\]

where \([x]\) is the integer part of \(x\) and \(r + j < k\). The case \(j = 0\) follows simply.

Some examples follow. The case \(j = k\) is interesting and we see that

\[
T(k, k) = \sum_{n=1}^{\infty} \frac{1}{nk+k} = \sum_{r=1}^{k} (-1)^r \left( \begin{array}{c} k-1 \\ r-1 \end{array} \right) \psi\left( \frac{r+k}{k} \right).
\]
Now since $\psi(1 + z) = \psi(z) + \frac{1}{z}$, we have that

$$T(k, k) = \sum_{n=1}^{\infty} \frac{1}{nk + k} = \sum_{r=1}^{k} (-1)^r \binom{k-1}{r-1} \left( \frac{k}{r} + \psi\left(\frac{r}{k}\right) \right)$$

$$= -1 + T(0, k).$$

Also

$$T(0, 6) = 32 \ln 2 - \frac{27}{2} \ln 3 - \frac{7\sqrt{3}\pi}{6},$$

$$T(3, 6) = 47 - 32 \ln 2 - \frac{27}{2} \ln 3 - \frac{11\sqrt{3}\pi}{6},$$

$$T(8, 6) = -\frac{757}{28} + 32 \ln 2 + \frac{27}{2} \ln 3 - \frac{11\sqrt{3}\pi}{6},$$

$$T\left(\frac{3}{2}, 4\right) = \pi \left( 4 + 2\sqrt{2} \right) - \frac{64}{3}.$$

In the next section we give an extension to Theorem 1 by incorporating harmonic numbers to the sum $T(j, k)$ and associating the sum with hypergeometric and integral representation.

3. Extension

We begin with the proof of the following Theorem.

**Theorem 2** — Under the assumptions of Theorem 1 and let $m \in \mathbb{N}$ then,

$$T^{(m)}(j, k) = \sum_{n=1}^{\infty} \frac{d^{(m)}}{dj^{(m)}} \left( \binom{nk + j}{k} \right)^{-1} = \sum_{n=1}^{\infty} Q^{(m)}(j, k)$$

$$= \frac{1}{k^m} \sum_{r=1}^{k} (-1)^r \binom{k-1}{r-1} \psi^{(m)}\left(\frac{r + j}{k}\right)$$

$$= \frac{m!}{k^m} \sum_{r=1}^{k} (-1)^{r+m} \binom{k-1}{r-1} H^{(m+1)}_{r+j-k}, \quad (3.1)$$

where

$$Q^{(m)}(j, k) = \frac{d^{(m)}}{dj^{(m)}} \left( \binom{nk + j}{k} \right)^{-1}. \quad (3.2)$$
PROOF: From the identity (2.1) we differentiate both sides "m" times with respect to \( j \) so that

\[
T^{(m)}(j, k) = \sum_{n=1}^{\infty} d^{(m)}(nk + j) \binom{nk + j}{k}^{-1} = \sum_{n=1}^{\infty} Q^{(m)}(j, k) = \frac{1}{k^{m}} \sum_{r=1}^{k} (-1)^{r} \binom{k - 1}{r - 1} \psi^{(m)}(\frac{r + j}{k})
\]

and (3.1) follows. From the known identity, relating polygamma functions with harmonic numbers

\[
\psi^{(m)}(1 + z) = (-1)^{m} m! \left( H^{(m+1)}_{2} z - \zeta(1 + z) \right),
\]

then

\[
T^{(m)}(j, k) = \frac{m!}{k^{m}} \sum_{r=1}^{k} (-1)^{r+m} \binom{k - 1}{r - 1} \frac{H^{(m+1)}_{r+j-k}}{k^{r}}
\]

since

\[
\sum_{r=1}^{k} (-1)^{r} \binom{k - 1}{r - 1} = 0, \text{ for } k \geq 2,
\]

hence (3.2) follows. For completeness we detail some values of \( Q^{(m)}(j, k) \):

\[
Q^{(1)}(j, k) = \frac{1}{\binom{nk + j}{k}} (H_{kn+j-k} - H_{kn+j})
\]

and

\[
Q^{(2)}(j, k) = \frac{1}{\binom{nk + j}{k}} \left( (H_{kn+j-k} - H_{kn+j})^{2} - \left( H^{(2)}_{kn+j-k} - H^{(2)}_{kn+j} \right) \right),
\]

some more details on the function \( Q^{(m)}(j, k) \) are given in the paper [9].

The cases \( j = 0 \) and \( j = k \) are interesting and the results are given in the next corollary.

Corollary 1 — For \( j = 0 \)

\[
T^{(m)}(0, k) = \sum_{n=1}^{\infty} Q^{(m)}(0, k) = \frac{1}{k^{m}} \sum_{r=1}^{k} (-1)^{r} \binom{k - 1}{r - 1} \psi^{(m)}(\frac{r}{k}) = \frac{m!}{k^{m}} \sum_{r=1}^{k-1} (-1)^{r+m} \binom{k - 1}{r - 1} \frac{H^{(m+1)}_{r+j-k}}{k^{r}}, \quad (3.3)
\]
where
\[
\sum_{n=1}^{\infty} Q^{(m)}(0, k) = \sum_{n=1}^{\infty} \lim_{j \to 0} \left( \frac{d^{(m)}}{dj^{(m)}} \left( \binom{n k + j}{k}^{-1} \right) \right).
\]

For \( j = k \)
\[
T^{(m)}(k, k) = T^{(m)}(0, k) + (-1)^{m+1} \Lambda^{(m)}(k)
\]  
(3.4)

where
\[
\Lambda^{(m)}(k) = \lim_{\alpha \to 0} \left( \frac{d^{(m)}}{dj^{(m)}} \left( \binom{k + \alpha}{\alpha}^{-1} \right) \right)
\]

**Proof:** From (3.1) we have
\[
T^{(m)}(0, k) = \frac{1}{k^m} \sum_{r=1}^{k} (-1)^r \binom{k-1}{r-1} \psi^{(m)}(\frac{r}{k})
\]
\[
= \frac{m!}{k^m} \sum_{r=1}^{k} (-1)^{r+m} \binom{k-1}{r-1} H_{\frac{r}{k}}^{(m+1)},
\]
since for \( r = k, \ H_{0}^{(m+1)} = 0, \) then (3.3) follows. For the case \( j = k, \)
\[
T^{(m)}(k, k) = \sum_{n=1}^{\infty} Q^{(m)}(k, k)
\]
\[
= \frac{1}{k^m} \sum_{r=1}^{k} (-1)^r \binom{k-1}{r-1} \psi^{(m)}(1 + \frac{r}{k})
\]
where
\[
Q^{(m)}(k, k) = \lim_{j \to k} \left( \frac{d^{(m)}}{dj^{(m)}} \left( \binom{n k + j}{k}^{-1} \right) \right).
\]

By the property of the polygamma function
\[
\psi^{(m)}(1 + z) = \psi^{(m)}(1 + z) + \frac{(-1)^{m}}{z^{m+1}} m!
\]
\[
T^{(m)}(k, k) = \frac{1}{k^m} \sum_{r=1}^{k} (-1)^r \binom{k-1}{r-1} \left( \psi^{(m)}(\frac{r}{k}) + \frac{(-1)^{m} m! k^{m+1}}{r^{m+1}} \right)
\]
\[
= T^{(m)}(0, k) + (-1)^{m} m! k \sum_{r=1}^{k} (-1)^r \binom{k-1}{r-1} \frac{1}{r^{m+1}}.
\]
From the paper [9], we have
\[ \sum_{r=1}^{k} (-1)^r \binom{k-1}{r-1} \frac{1}{r^{m+1}} = -\frac{\Lambda^{(m)}(k)}{m!k} \]

hence
\[ T^{(m)}(k, k) = T^{(m)}(0, k) + (-1)^{m+1} \Lambda^{(m)}(k), \]

hence (3.4) follows. Some values of \( \Lambda^{(m)}(k) \) are
\[ \Lambda^{(1)}(k) = H_k, \quad \Lambda^{(2)}(k) = H_k^2 + H_k^{(2)} \]
\[ \Lambda^{(3)}(k) = H_k^3 + 3H_kH_k^2 + 2H_k^{(3)}. \]

Example 1: Some illustrative examples follow.

\[ T^{(1)}(j, k) = \sum_{n=1}^{\infty} \frac{H_{nk-k+j} - H_{nk+j}}{(nk+j)} \left( \frac{1}{nk+j} \right) \]
\[ = \frac{m!}{k^m} \sum_{r=1}^{k} (-1)^{r+1} \binom{k-1}{r-1} H_{r+1, k-1}^{(2)} \]
\[ T^{(1)}(1, 4) = \frac{3}{16} \zeta(2) - \frac{1}{6} G - \frac{1}{6}, \quad T^{(1)}(0, 4) = \frac{1}{6} G - \frac{7}{48} \zeta(2) \]

where \( G \) is Catalan’s constant.
\[ T^{(3)}(0, 2) = -\frac{21}{2} \zeta(4), \]
\[ T^{(4)}(4, 4) = -\frac{2835}{16} \zeta(5) - \frac{5\pi^5}{16} - \frac{76111}{864}. \]

The expression \( T(j, k) \) and \( T^{(m)}(j, k) \) can also be represented in integral and hypergeometric form and for completeness the following is recorded.

**Theorem 3** — Let the assumptions of Theorem 1 apply, then

\[ T(j, k) = k \int_{0}^{1} x^j (1-x)^{k-1} \frac{dx}{(1-x^k)}, \quad (3.5) \]
\[ T^{(m)}(j, k) = k \int_0^1 x^j (1 - x)^{k-1} \ln^m x \frac{1}{(1 - x^k)} \, dx \]  \hspace{1cm} (3.6) \\
and \\
\[ T(j, k) = \frac{1}{\binom{k+j}{k}} \left( \begin{array}{c} 1+j \frac{k}{k}, \frac{2+j}{k}, \ldots, \frac{k+j}{k}, 1 \\ \frac{1+j+k}{k}, \frac{2+j+k}{k}, \ldots, \frac{2k+j}{k} \end{array} \right) \binom{1}{1} \right) . \]  \hspace{1cm} (3.7) \\

**Proof:** Consider \\
\[ T(j, k) = \sum_{n=1}^{\infty} \binom{1}{\frac{nk+j}{k}} = \sum_{n=1}^{\infty} \frac{\Gamma(nk+j-k+1) \Gamma(k+1)}{\Gamma(nk+j+1)} \]
\[ = k \sum_{n=1}^{\infty} B(k, nk - k + j + 1), \]

where \( \Gamma(\cdot) \) is the gamma function and \( B(\cdot, \cdot) \) is the beta function. Now \\
\[ T(j, k) = k \int_0^1 x^j (1 - x)^{k-1} \ln^m x \frac{1}{(1 - x^k)} \, dx, \]

and (3.5) follows. Now differentiating \( m \) times with respect to \( j \) results in \\
\[ T^{(m)}(j, k) = k \int_0^1 x^j (1 - x)^{k-1} \ln^m x \frac{1}{(1 - x^k)} \, dx \]

hence (3.6). For the hypergeometric function we consider the definition (1.2) above and write \\
\[ T(j, k) = \sum_{n=1}^{\infty} \binom{1}{\frac{nk+j}{k}} = \sum_{n=0}^{\infty} \frac{1}{\binom{nk+k+j}{k}} \]

therefore (3.7) follows.  \( \square \)
Remark 2 : It is straightforward to see, from (3.3) and (3.6), that

\[ T^{(m)}(0, 2) = 2 \int_0^1 \frac{\ln^m x}{1 + x} dx = \sum_{n=1}^{\infty} \lim_{j \to 0} \frac{d^{(m)}(j)}{dj^{(m)}} \left( \frac{2n + j}{2} \right)^{-1} \]

\[ = 2 (-1)^m \frac{m!}{1 - 2^{-m}} \zeta(m + 1) \]

\[ = \frac{(-1)^{m+1} m! H^{(m+1)}}{2m} \]

\[ = 2m! \sum_{n=1}^{\infty} \frac{(-1)^{m+n+1}}{n^{m+1}}. \]

Many other examples of binomial sums, harmonic number sums, integral representations and hypergeometric summation are available in [1, 2, 8, 10-15]. Some interesting binomial series are also investigated in [7].

REFERENCES

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