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BOREL'S FIXED POINT THEOREM FOR FINITE DIMENSIONAL COMPACT ABELIAN GROUPS

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In this paper, we extend some results of Borel's localization theorems to actions of finite dimensional compact abelian groups which are extensions of a p-group by finite dimensional compact connected abelian groups.

Key words : Equivariant cohomology; compact group; fixed point.

1. Introduction

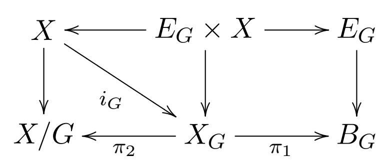
This article deals with Borel's equivariant cohomology theory and its application to transformation group theory. The purpose of this article is to give an extension of a result of Borel.

Unless expressly stated otherwise, k will be assume to be a field of characteristic zero, and all spaces will be assumed to be Hausdorff.

First of all, let us restate Milnor's construction: For any topological group G, by introducing a suitable topology in the $(m+1)$-join $E_G^m = G * ... * G$ and letting G act on it naturally, and the orbit space $B_G^m = E_G^m/G$ is called as m-classifying space of G, so we obtain an m-universal G-bundle (E_G^m, p, B_G^m, G). Also, we obtain a weakly contractible space $E_G = \lim_m E_G^m$, by taking direct limit, on which G acts freely and properly. We denote the quotient space by B_G which is called a classifying space of G. So we have principal G-bundle $E_G \to B_G$ called the universal G-bundle [6,7]. If G is a compact group, then $H^i\left(E_G^m; k\right) = H^i\left(pt; k\right)$ for all $i < m$ and $H^i\left(E_G; k\right) = H^i\left(pt; k\right)$ for every i [8].

Note that for any compact group G, E_G and B_G are not necessarily compact but E_G^m and B_G^m are compact. Since some results in Sheaf cohomology, such as the continuity property, is in need of compactness assumptions, it is often convenient to consider B_G^m instead of B_G. Since the morphisms $H^i\left(B_G^{m+1};k\right) \to H^i\left(B_G^m;k\right)$ are isomorphisms for $i < m$, then $H^i\left(B_G;k\right)$ is canonically isomorphic to $H^i\left(B_G^m;k\right)$ for all $i < m$. In order to avoid some technical difficulties, $H^*\left(B_G^m;k\right)$ can be taken into $H^*\left(B_G;k\right)$ for sufficiently large m without loss of generality (see [3, 5, 8]).

In the cohomology theory of transformation groups, the Borel construction is a powerful tool. It is defined as follows: Let X be a G-space. A technique due to Borel for studying G-actions is the construction of the so-called Borel space $E_G \times_G X = (E_G \times X)/G$ associated to the G-space X (on $E_G \times X$, there is the diagonal action given by $g(e,x) = (ge, gx)$). This leads to the commutative diagram:



where π_1 is a fiber bundle mapping with fiber X and structure group G/K where K is the ineffective kernel of the G action on X, π_2 is a mapping such that $\pi_2^{-1}(x^*) = B_{G_x}$, where $x^* \in X/G$, and $x \in x^*$. Then the equivariant graded cohomology algebra of X with coefficient k is defined by $H_G^*(X;k) = H^*(X_G;k)$ [1].

The natural projection $X_G \to B_G$ makes the equivariant cohomology into a module over $H^*(B_G;k)$. Therefore the algebra structure of $H^*(B_G;k)$ is important to determine equivariant cohomology. In the case, $G = T^r$, a finite dimensional connected torus, it is well known that $H^*(B_G;k)$ is the polynomial algebra in r variables of homogeneous degree 2.

It is well known fact that the localization theorem of Borel-Segal-Quillen-Hsiang, introduced in [1, 2, 12, 16] is a powerful tool for cohomology theory of compact, especially compact abelian, transformation groups. One of the most profound result of localization theorem is to determine the cohomology structure of fixed point set by the equivariant cohomology of the space for the torus or p-torus actions.

The following well known Borel's fixed point theorem is one of the important consequences of the localization theorem [16].

Here, the cohomological dimension of a space X is defined as the natural number (or ∞)

$\max\{n : H^n(X) \neq 0\}$.

***Theorem* 1** — *Let G be a finite dimensional connected torus or finite abelian p-group (p prime number). Suppose that a G-space X is either a compact G-space, or X be a paracompact G-space with finite cohomological dimension, and with finitely many orbit types. Then the following statements are equivalent;*

1. *The fixed point set $F(G, X)$ is nonempty.*
2. *$\pi_1 : X_G \to B_G$ has a section.*
3. *$\pi_1^* : H^*(B_G; k) \to H^*_G(X; k)$ is injective.*

Dieck extended this result to the following case; let $G = T \times H$ be a compact abelian Lie group where T is a torus and H is a finite abelian p-group, and let X be a compact G-space, then statements (1), (2), and (3) of Theorem 1 are equivalent, considering in (3) unitary cobordism theory instead of singular cohomology [15]. Moreover, Jackowski [14] generalized the results of Hsiang and Dieck for a finite p-group (not necessarily abelian) with zero-dimensional stable cohomotopy replacing singular cohomology in (3).

Definition 2 — We say that G is n-dimensional compact connected group if G is the projective limit of n-dimensional compact connected Lie groups. We shall say that G is finite dimensional if G is an n-dimensional for some $n \in \mathbb{N}$.

We say that G is an n-dimensional compact connected abelian group if G is the projective limit of n-dimensional tori. G is called as solenoid if G is the projective limit of the circle groups. If G is n-dimensional compact connected abelian group, then it is considered $G = \varprojlim_{N \in \mathcal{N}} G/N$ where $\mathcal{N}$ is a filter basis of compact normal zero dimensional subgroups of G such that $\bigcap \mathcal{N} = \{1\}$ and G/N is n-dimensional torus for each $N \in \mathcal{N}$.

Let G be an n-dimensional compact connected abelian group and X be a compact G-space, then the transformation group is represented as a limit

$$(G, X) = \varprojlim_{N \in \mathcal{N}} (G/N, X/N)$$

where $\mathcal{N}$ is a filter basis of compact normal zero dimensional subgroups of G such that $\bigcap \mathcal{N} = \{1\}$ and G/N is n-dimensional torus for each $N \in \mathcal{N}$.

If the set $\{[G_x] : x \in X\}$ is finite, where $[G_x]$ denotes the conjugacy class of the isotropy subgroup G_x in G, then the group G is said to act on a space X with finitely many orbit types (FMOT). Let X be a G-space and N be a closed normal subgroup of G, then there exists a canonically induced action of the quotient group G/N on the orbit space X/N and a natural homeomorphism from X/G to $(X/N)\,/\,(G/N)$. Moreover, one can easily see that if a compact connected group G acts on a space X and N is any totally disconnected closed normal subgroup of G, then the orbit map $X \to X/N$ induces a homeomorphism $F(G,X) \approx F(G/N, X/N)$ for each $N \in \mathcal{N}$ (for details see [4, 11]).

On the other hand, we can construct effective compact transformation groups by taking inverse limits of some inverse systems of effective compact Lie transformation groups with bonding maps satisfying certain equivariant properties (see [13, Lemma 1]).

The main interest of this paper is cohomology theory of effective (locally) compact transformation groups, because effective (locally) compact transformation group theory is motivated by Hilbert-Smith conjecture which asserts that if a locally compact group acts effectively on a connected finite dimensional manifold then it is a Lie group.

Throughout this paper, cohomology will be taken to be sheaf-theoretic cohomology with rational coefficients, since we need the continuity property that is not satisfied for such theories as singular cohomology. Moreover, one can use Alexander-Spanier cohomology or Cech cohomology which have continuity property. And Sheaf cohomoloy, Alexander-Spanier cohomology and Cech cohomology are equivalent for paracompact spaces (see [5]).

2. Main Result

For any locally compact abelian group G, the group $Hom\,(G,T)$ of all continuous homomorphisms from G to the circle group T given with compact-open topology is called the character group of G and written $\widehat{G}$. The character group of locally compact abelian group is locally compact abelian group (see [9, 10] for more details).

***Theorem* 3** — *[8, p. 206, 212]. For any compact abelian group G, the homogeneous component $H^2(B_G;\mathbb{Z})$ of degree 2 of $H^*(B_G;\mathbb{Z})$ is naturally isomorphic to the character group $\widehat{G}$. If G is connected, the entire cohomology algebra $H^*(B_G;\mathbb{Z})$ is the symmetric $\mathbb{Z}$-algebra $P(\widehat{G})$ generated by this component of homogeneous degree 2. Also, in this case, the homogeneous component $H^1(G;\mathbb{Z})$ of degree 1 of $H^*(G;\mathbb{Z})$ is also naturally isomorphic to $\widehat{G}$, and the entire cohomology algebra $H^*(G;\mathbb{Z})$ is the exterior algebra $\bigwedge(H^1(G;\mathbb{Z}))$.*

In particular, for any commutative unital ring R, and compact connected abelian G

$$H^*(B_G;R) \cong R \otimes P(\widehat{G}) \cong P(R \otimes \widehat{G})$$

We have the next lemma from this theorem

Lemma 4 — Let G be a finite dimensional compact connected abelian group and N be a totally disconnected closed subgroup of G. Then the map

$$H^*(B_{G/N};k) \to H^*(B_G;k)$$

which induced by the quotient morphism $G \to G/N$ is an isomorphism.

PROOF : The quotient morphism $q : G \to G/N$ induces an injection

$$\lambda_{G,N} : \widehat{G/N} \cong N^{\perp} \hookrightarrow \widehat{G}, \qquad \lambda_{G,N}(\chi)(g) = \chi(gN)$$

where $N^{\perp} = \{\chi \in \widehat{G} : \chi(n) = 0 \text{ for all } n \in N\}$ is called the annihilator of N in $\widehat{G}$.

Now, let us consider the following exact sequence.

$$0 \to N \to G \to G/N \to 0$$

Thus, we have the following exact sequence of the character groups.

$$0 \to \widehat{G/N} \to \widehat{G} \to \widehat{N} \to 0$$

The symmetric algebra functor P preserves exactness, we get the next exact sequence of the symmetric algebras

$$0 \to P\left(\widehat{G/N}\right) \to P\left(\widehat{G}\right) \to P\left(\widehat{N}\right) \to 0$$

Hence, we have

$$q^* : H^*(B_{G/N};\mathbb{Z}) \to H^*(B_G;\mathbb{Z})$$

is injective from the graded cohomology structure of $H^*(B_G;\mathbb{Z})$ for compact connected abelian groups.

Furthermore, tensoring with k preserves exactness, because of the above exact sequence of the symmmetric algebras, we obtain the exact sequence

$$0 \to P\left(\widehat{G/N} \otimes k\right) \to P\left(\widehat{G} \otimes k\right) \to P\left(\widehat{N} \otimes k\right) \to 0$$

On the other hand, the character group of the compact, totally disconnected group is a torsion group. It follows that $\widehat{N} \otimes k = 0$. So, we have the isomorphism $P\left(\widehat{G/N} \otimes k\right) \to$

$P\left(\widehat{G}\otimes k\right)$. Because of the natural isomorphisms $H^*(B_G;k)\cong P(k\otimes\widehat{G})$ and $H^*(B_{G/N};k)\cong P(k\otimes\widehat{G/N})$, we get that the map

$$H^*(B_{G/N};k)\to H^*(B_G;k)$$

is an isomorphism. □

To prove our theorems, we will need the next theorems.

***Theorem* 5** — *[4, Chapter III, Theorem 7]. Let G be a finite group, X be a paracompact G-space, and let $\pi : X\to X/G$ be the orbit map. If k is a field of characteristic zero or prime to $|G|$, then*

$$\pi^* : H^*\left(X/G;k\right)\to\left(H^*\left(X;k\right)\right)^G$$

is an isomorphism, where $\left(H^*\left(X;k\right)\right)^G$ is the subspace of elements which are invariant under the action of G on $H^*\left(X;k\right)$.

Let G_0 be the identity component of G.

***Theorem* 6** — *[8]. For any compact abelian group G, the natural morphism*

$$H^*\left(B_G;k\right)\to H^*\left(B_{G_0};k\right)$$

which induced by the inclusion $G_0\subseteq G$ is an isomorphism and the natural morphism

$$H\left(\pi,k\right) : H^*\left(B_{G/G_0};k\right)\to H^*\left(B_G;k\right)$$

which induced by the quotient map $\pi : G\to G/G_0$ is an injection.

Now, we can prove our main theorems:

***Theorem* 7** — *Let G be a finite dimensional compact connected abelian group and X be a compact G-space. Then statements (1), (2), and (3) of Theorem 1 are equivalent.*

Proof : Implications (1)⇒ (2) ⇒ (3) are obvious. We shall prove (3)⇒ (1).

Now, assume that $H^*\left(B_G;k\right)\to H^*_G\left(X;k\right)$ is injective. Since G is a finite dimensional compact abelian group, there exists a totally disconnected closed normal subgroup N of G such that G/N is a torus [9]. We get that the map $H^*\left(B_{G/N},k\right)\to H^*_{G/N}\left(X/N,k\right)$ is injective from the following commutative diagram:

$$\begin{array}{ccc} H^*\left(B_{G/N};k\right) & \longrightarrow & H^*_{G/N}\left(X/N;k\right) \\ \simeq\downarrow & & \downarrow \\ H^*(B_G;k) & \longrightarrow & H^*_G\left(X;k\right) \end{array}$$

Therefore, by Theorem 1, we have that $F(G/N, X/N) \neq \emptyset$ which implies $F(G, X) = F(G/N, X/N) \neq \emptyset$ □

Remark 8 : Since the map $q^* : H^*(B_{G/N}; \mathbb{Z}) \to H^*(B_G; \mathbb{Z})$ is injective, it is clear from the above commutative diagram that Theorem 7 remains true for cohomology with integer coefficients.

Now, we shall express an extension of Theorem 7.

***Theorem* 9** — *Let G be a finite dimensional compact abelian group such that G/G_0 is a finite p-group and X be a compact G-space. Then statements (1), (2), and (3) of Theorem 1 are equivalent.*

Proof : Implications (1)⇒ (2) ⇒ (3) are obvious. We shall prove (3)⇒ (1). Assume that $H^*(B_G; k) \to H^*_G(X; k)$ is injective.

It is well known that if X is a paracompact G-space, X_G is also paracompact space. Let us consider the orbit map $X_{G_0} \to X_G \simeq (X_{G_0}) / (G/G_0)$. Since G/G_0 is finite, we have that the map

$$H^*_G(X; k) \to H^*_{G_0}(X; k)$$

which induced by the inclusion $G_0 \subseteq G$ is injective by Theorem 5. On the other hand if we consider G/G_0 action on the orbit space X/G_0 then we have the following commutative diagram,

$$\begin{array}{ccc} H^*\left(B_{G/G_0}; k\right) & \longrightarrow & H^*_{G/G_0}(X/G_0; k) \\ \downarrow & & \downarrow \\ H^*(B_G; k) & \longrightarrow & H^*_G(X; k) \end{array}$$

From this commutative diagram we have

$$H^*\left(B_{G/G_0}; k\right) \to H^*_{G/G_0}(X/G_0; k)$$

is injective, and from the following commutative diagram,

$$\begin{array}{ccc} H^*(B_G; k) & \longrightarrow & H^*_G(X; k) \\ \simeq\downarrow & & \downarrow \\ H^*(B_{G_0}; k) & \longrightarrow & H^*_{G_0}(X; k) \end{array}$$

we have that the map

$$H^*(B_{G_0}; k) \to H^*_{G_0}(X; k)$$

is injective. Thus, we get $F(G_0, X) \neq \emptyset$ from Theorem 7 and since G/G_0 is a finite abelian p-group, $F(G/G_0, X/G_0) \neq \emptyset$ by Theorem 1. Now consider the action of G/G_0 on space $F(G_0, X)$. It is clear that the fixed point set of this action is $F(G/G_0, F(G_0, X)) = F(G, X)$.

Then, we must show that $F(G/G_0, F(G_0, X)) \neq \emptyset$. It is equivalent to show that the map $H^*\left(B_{G/G_0}; k\right) \to H^*_{G/G_0}(F(G_0, X); k)$ is injective, since G/G_0 is a finite abelian p-group. Let's consider the trivial actions of G_0 on space $F(G_0, X)$ and of G on space $F(G_0, X)$. So the map

$$H^*(B_{G_0}; k) \to H^*_{G_0}(F(G_0, X); k)$$

is injective. Therefore, it is obtained that the map

$$H^*(B_G; k) \to H^*_G(F(G_0, X); k)$$

is injective by the next commutaive diagram;

$$\begin{array}{ccc} H^*(B_G; k) & \longrightarrow & H^*_G(F(G_0, X); k) \\ \simeq\downarrow & & \downarrow \\ H^*(B_{G_0}; k) & \longrightarrow & H^*_{G_0}(F(G_0, X); k) \end{array}$$

Thus, from the following commutative diagram;

$$\begin{array}{ccc} H^*\left(B_{G/G_0}; k\right) & \longrightarrow & H^*_{G/G_0}(F(G_0, X); k) \\ \downarrow & & \downarrow \\ H^*(B_G; k) & \longrightarrow & H^*_G(F(G_0, X); k) \end{array}$$

we get that the map $H^*\left(B_{G/G_0}; k\right) \to H^*_{G/G_0}(F(G_0, X); k)$ is injective. □

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