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SOME CONGRUENCES ON q-FRANEL NUMBERS AND q-CATALAN NUMBERS

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We present certain q-analogues of Franel numbers and Catalan numbers, and establish several congruences on these q-numbers.

Key words : q-Franel numbers; q-Catalan numbers; q-congruence.

1. INTRODUCTION

The q-analogue of a positive integer n can be defined by

$$[n]_q = \frac{1-q^n}{1-q}.$$

It is clear that $\lim_{q\to 1}[n]_q = n$. Supposing that $a \equiv b \pmod{n}$, we have

$$[a]_q = \frac{1-q^a}{1-q} = \frac{1-q^b+q^b(1-q^{a-b})}{1-q} \equiv \frac{1-q^b}{1-q} = [b]_q \pmod{[n]_q}.$$

The Gaussian binomial coefficients, also called the q-binomial coefficients, are q-analogues of the binomial coefficients, which are given by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{cases} \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}, & \text{if } 0 \leq k \leq n, \\ 0, & \text{otherwise}, \end{cases}$$

where $(a;q)_k$ is defined by

$$(a;q)_0 = 1,\ (a;q)_k = \prod_{j=0}^{k-1} (1 - aq^j) \text{ for } k \geq 1.$$

Franel in [3] noted that the numbers

$$f_n = \sum_{k=0}^{n} \binom{n}{k}^3$$

satisfy the recurrence relation:

$$(n+1)^2 f_{n+1} = (7n^2 + 7n + 2) f_n + 8n^2 f_{n-1}$$

for $n = 1, 2, \ldots$. Now such numbers are usually called Franel numbers. The generalized Franel numbers are defined by

$$f_n^{(r)} = \sum_{k=0}^{n} \binom{n}{k}^r.$$

Then $f_n^{(3)} = f_n$ are the Franel numbers. In [2, Proposition 3]C Calkin proved the following congruence:

$$f_n^{(2r)} \equiv 0 \pmod{p},$$

where p is a prime such that $\frac{n}{m} < p < \frac{n+1}{m} + \frac{n+1-m}{m(2mr-1)}$ for some positive integer m. Guo and Zeng [5, Theorem 4.4] established that, for any positive integer n,

$$f_n^{(2r)} \equiv 0 \pmod{n+1}.$$

It was Sun [10] who initiated the systematic investigation of fundamental congruences for the Franel numbers. In particular, he obtained the following congruence: for any prime $p > 3$,

$$f_{p-1} \equiv 1 + 3pq_p(2) + 3p^2 q_p(2)^2 \pmod{p^3}, \tag{1.1}$$

where $q_p(2) = \frac{2^{p-1}-1}{p}$. Other congruences on the Franel numbers can be found in [4, 7].

In this paper, we will establish a q-analogue of (1.1) in the modulus p^2. We define the q-analogues of the generalized Franel numbers as

$$f_n^{(2r+1)}(q) := \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q^{2r+1} q^{(2r+1)\binom{k+1}{2}}, \ f_n^{(2r)}(q) := \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q^{2r} q^{2r\binom{k+1}{2}+k}.$$

***Theorem* 1.1** — *Let p be an odd prime and r a nonnegative integer. Then*

$$f_{p-1}^{(2r)}(q) \equiv (1-2r+2rp)[p]_q \pmod{[p]_q^2},$$
$$f_{p-1}^{(2r+1)}(q) \equiv 1+(2r+1)Q_p(2,q)[p]_q \pmod{[p]_q^2}.$$

In [11] Sun proposed several conjectured congruences on the Catalan numbers. For example, let $p > 3$ be a prime, then

$$\sum_{k=0}^{\frac{p-1}{2}} \frac{C_k C_{k+1}}{16^k} \equiv 8 \pmod{p^2} \tag{1.2}$$

and if $p \equiv 1 \pmod 4$, then

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{2k}{k+1}}{8^k} \equiv 0 \pmod{p}. \tag{1.3}$$

Here and below, the notation $A \equiv B \pmod{p^l}$ denotes that $(A-B)/p^l$ is a p-adic integer for $A, B \in \mathbb{Q}$. The above congruences were confirmed by Zhang [12].

In this paper, we will give a q-analogue of (1.2) in the modulus p case. Before stating our main results, we need to mention the following q-analogue of Catalan numbers:

$$C_n(q) = \frac{\begin{bmatrix} 2n \\ n \end{bmatrix}_q}{[n+1]_{q^2}}.$$

***Theorem* 1.2** — *Let $p \geq 5$ be a prime. Then*

$$\sum_{k=0}^{\frac{p-1}{2}} \frac{C_k(q) C_{k+1}(q)}{(-q;q)_k^2 (-q^2;q)_k^2} q^{4k+3} \equiv (1+q)^3 \pmod{[p]_q}.$$

In addition, we set up a more general q-analogue of (1.3).

***Theorem* 1.3** — *Let p be an odd prime and $d \in \{0, 1, \ldots, \frac{p-1}{2}\}$. Then*

$$\sum_{k=0}^{p-1} \frac{\begin{bmatrix} 2k \\ k \end{bmatrix}_q \begin{bmatrix} 2k \\ k+d \end{bmatrix}_{q^2} (-q^{2(k+1)};q^2)_{\frac{p-1}{2}-k}}{(-q;q)_k^2} q^{2k+\frac{(p-1)(p-3)}{4}}$$
$$\equiv \begin{cases} 0 & \text{if } \frac{p-1}{2} \not\equiv d \pmod 2 \\ (-1)^{\frac{p-1}{2}} q^{\frac{(p-1)^2}{4}-d^2} \begin{bmatrix} \frac{p-1}{2} \\ \frac{p-1-2d}{4} \end{bmatrix}_{q^4} & \text{if } \frac{p-1}{2} \equiv d \pmod 2 \end{cases} \pmod{[p]_q}.$$

Throughout this article, we use the notation $P(q) \equiv Q(q) \pmod{[p]_q}$, where $P(q)$ and $Q(q)$ are rational functions in q, to denote that $\frac{P(q)-Q(q)}{[p]_q} = \frac{A(q)}{B(q)}$ for some polynomials $A(q)$ and $B(q)$ with rational coefficients and $\gcd(B(q), [p]_q) = 1$.

In the next section, we will provide some lemmas which are crucial in the proofs of Theorems 1.1-1.3. Section 3 is devoted to our proofs of Theorems 1.1-1.3.

2. Some Auxiliary Results

In order to prove Theorems 1.1-1.3, we need the following results.

Lemma 2.1 — Let p be an odd prime. Then for any integer $k \in \{1, 2, \ldots, p-1\}$, we have

$$(-1)^k \begin{bmatrix} p-1 \\ k \end{bmatrix}_q q^{\binom{k+1}{2}} \equiv 1 - [p]_q H_k(q) + [p]_q^2 \sum_{1 \le i < j \le k} \frac{1}{[i]_q [j]_q} \pmod{[p]_q^3},$$

where $H_k(q)$ is defined by

$$H_k(q) = \sum_{j=1}^{k} \frac{1}{[j]_q}.$$

Proof : For $k \in \{1, 2, \ldots, p-1\}$,

$$\begin{aligned} (-1)^k \begin{bmatrix} p-1 \\ k \end{bmatrix}_q q^{\binom{k+1}{2}} &= \frac{(q^p - q)(q^p - q^2) \cdots (q^p - q^k)}{(q;q)_k} \\ &= (1 - [p]_q) \left(1 - \frac{[p]_q}{[2]_q}\right) \cdots \left(1 - \frac{[p]_q}{[k]_q}\right). \end{aligned}$$

Then

$$(-1)^k \begin{bmatrix} p-1 \\ k \end{bmatrix}_q q^{\binom{k+1}{2}} \equiv 1 - [p]_q H_k(q) + [p]_q^2 \sum_{1 \le i < j \le k} \frac{1}{[i]_q [j]_q} \pmod{[p]_q^3}.$$

This proves Lemma 2.1. □

The following result (see [9, Theorem 1.1] for a generalization) is also very important.

Lemma 2.2 — Let p be an odd prime. Then

$$\sum_{j=1}^{\frac{p-1}{2}} \frac{1}{[2j]_q} \equiv -Q_p(2, q) \pmod{[p]_q},$$

where $Q_p(2, q) = \frac{(-q;q)_{p-1} - 1}{[p]}$.

Lemma 2.3 — (q-binomial Theorem, see [1]). Let n be a positive integer and z, q two complex numbers with $|z| < 1$ and $|q| < 1$. Then

$$(z;q)_n = \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}} z^k.$$

Lemma 2.4 — (q-Lucas Theorem, see [8]). Let m, k, d be positive integers with $m = ad + b$ and $k = rd + s$, where $0 \le b, s \le d-1$. Let $\Phi_d(q)$ be the d-th cyclotomic polynomial in q. Then

$$\begin{bmatrix} m \\ k \end{bmatrix}_q \equiv \binom{a}{r} \begin{bmatrix} b \\ s \end{bmatrix}_q \pmod{\Phi_d(q)}.$$

We also need the following result which follows from direct computations.

Lemma 2.5 — Let p be an odd prime. Then for $0 \le k \le (p-1)/2$,

$$\begin{bmatrix} (p-1)/2 \\ k \end{bmatrix}_{q^2} \equiv \frac{(-1)^k q^{-k^2}}{(-q;q)_k^2} \begin{bmatrix} 2k \\ k \end{bmatrix}_q \pmod{[p]_q}.$$

3. Proof of Theorems 1.1-1.3

PROOF OF THEOREM 1.1 : By Lemma 2.1, we have

$$\begin{aligned} f_{p-1}^{(2r+1)}(q) - 1 &= \sum_{k=1}^{p-1} \begin{bmatrix} p-1 \\ k \end{bmatrix}_q^{2r+1} q^{(2r+1)\binom{k+1}{2}} \\ &\equiv \sum_{k=1}^{p-1} (-1)^k \left(1 - [p]_q H_k(q)\right)^{2r+1} \\ &\equiv \sum_{k=1}^{p-1} (-1)^k \left(1 - (2r+1)[p]_q H_k(q)\right) \\ &= -(2r+1)[p]_q \sum_{k=1}^{p-1} (-1)^k H_k(q) \pmod{[p]_q^2}. \end{aligned}$$

Since

$$\sum_{k=1}^{p-1} (-1)^k H_k(q) = \sum_{k=1}^{p-1} (-1)^k \sum_{j=1}^{k} \frac{1}{[j]_q} = \sum_{j=1}^{p-1} \frac{1}{[j]_q} \sum_{k=j}^{p-1} (-1)^k = \sum_{j=1}^{\frac{p-1}{2}} \frac{1}{[2j]_q},$$

we use Lemma 2.2 to get

$$f_{p-1}^{(2r+1)}(q) \equiv 1 + (2r+1) Q_p(2,q) [p]_q \pmod{[p]_q^2}.$$

Similarly,

$$\begin{aligned} f_{p-1}^{(2r)}(q)-1 &= \sum_{k=1}^{p-1} {\begin{bmatrix} p-1 \\ k \end{bmatrix}}_q^{2r} q^{2r\binom{k+1}{2}+k} \\ &\equiv \sum_{k=1}^{p-1} (1-[p]_q H_k(q))^{2r} q^k \equiv \sum_{k=1}^{p-1}(1-2r[p]_q H_k(q))q^k \\ &= \frac{q(1-q^{p-1})}{1-q} - 2r[p]_q \sum_{k=1}^{p-1} H_k(q)q^k \pmod{[p]_q^2}. \end{aligned}$$

Since

$$\begin{aligned} \sum_{k=1}^{p-1} H_k(q)q^k &= \sum_{k=1}^{p-1} q^k \sum_{j=1}^{k} \frac{1}{[j]_q} = \sum_{j=1}^{p-1} \frac{1}{[j]_q} \sum_{k=j}^{p-1} q^k \\ &= \sum_{j=1}^{p-1} \frac{1}{[j]_q} \frac{q^j - q^p}{1-q} \equiv 1-p \pmod{[p]_q}, \end{aligned}$$

we see that

$$f_{p-1}^{(2r)}(q) \equiv (1-2r+2rp)[p]_q \pmod{[p]_q^2}.$$

This concludes the proof of Theorem 1.1. □

PROOF OF THEOREM 1.2 : We denote $[z^l]f(z)$ as the coefficient of z^l in the polynomial $f(z)$. By the q-binomial theorem, we have

$$\begin{aligned} (-z;q^2)_{n+1} &= \sum_{k=0}^{n+1} {\begin{bmatrix} n+1 \\ k \end{bmatrix}}_{q^2} q^{k(k-1)} z^k, \\ (-zq^{2(n+1)};q^2)_{n+1} &= \sum_{k=0}^{n+1} {\begin{bmatrix} n+1 \\ k \end{bmatrix}}_{q^2} q^{k(k-1)+2(n+1)k} z^k, \\ (-z;q^2)_{2n+2} &= \sum_{k=0}^{2n+2} {\begin{bmatrix} 2n+2 \\ k \end{bmatrix}}_{q^2} q^{k(k-1)} z^k. \end{aligned}$$

Then

$$\begin{aligned} {\begin{bmatrix} 2n+2 \\ n \end{bmatrix}}_{q^2} q^{n(n-1)} &= [z^n](-z;q^2)_{2n+2} \\ &= [z^n]\{(-z;q^2)_{n+1} \cdot (-zq^{2(n+1)};q^2)_{n+1}\} \\ &= \sum_{k=0}^{n} \left({\begin{bmatrix} n+1 \\ k \end{bmatrix}}_{q^2} q^{k(k-1)+2(n+1)k} \cdot {\begin{bmatrix} n+1 \\ n-k \end{bmatrix}}_{q^2} q^{(n-k)(n-k-1)} \right) \\ &= \sum_{k=0}^{n} {\begin{bmatrix} n+1 \\ k \end{bmatrix}}_{q^2} {\begin{bmatrix} n+1 \\ n-k \end{bmatrix}}_{q^2} q^{2k^2+2k+n^2-n}, \end{aligned}$$

namely,

$$\begin{bmatrix} 2n+2 \\ n \end{bmatrix}_{q^2} = \sum_{k=0}^{n} \begin{bmatrix} n+1 \\ k \end{bmatrix}_{q^2} \begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_{q^2} q^{2k^2+2k}. \tag{3.1}$$

Below we set $n = \frac{p-1}{2}$. By the q-Lucas theorem,

$$\begin{bmatrix} 2n+2 \\ n+1 \end{bmatrix}_{q} \equiv 0 \pmod{[p]_q} \tag{3.2}$$

and

$$\begin{bmatrix} 2n+2 \\ n \end{bmatrix}_{q^2} \equiv 0 \pmod{[p]_q} \text{ for } p \geq 5. \tag{3.3}$$

It follows from (3.2) and Lemma 2.5 that

$$\begin{aligned}
\sum_{k=0}^{n} \frac{C_k(q)C_{k+1}(q)}{(-q;q)_k^2(-q^2;q)_k^2} q^{4k+3} &\equiv -(1+q)^2 \sum_{k=0}^{n-1} \frac{\begin{bmatrix} n \\ k \end{bmatrix}_{q^2} \begin{bmatrix} n \\ k+1 \end{bmatrix}_{q^2}}{[k+1]_{q^2}[k+2]_{q^2}} q^{2k^2+6k+4} \\
&= -\frac{(1+q)^2}{[n+1]_{q^2}^2} \sum_{k=0}^{n-1} \begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_{q^2} \begin{bmatrix} n+1 \\ k+2 \end{bmatrix}_{q^2} q^{2k^2+6k+4} \\
&\equiv -(1+q)^4 \sum_{k=1}^{n} \begin{bmatrix} n+1 \\ k \end{bmatrix}_{q^2} \begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_{q^2} q^{2k^2+2k} \pmod{[p]_q}.
\end{aligned}$$

From (3.1) and (3.3), we know that

$$\begin{aligned}
\sum_{k=1}^{n} \begin{bmatrix} n+1 \\ k \end{bmatrix}_{q^2} \begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_{q^2} q^{2k^2+2k} &= \sum_{k=0}^{n} \begin{bmatrix} n+1 \\ k \end{bmatrix}_{q^2} \begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_{q^2} q^{2k^2+2k} - [n+1]_{q^2} \\
&= \begin{bmatrix} 2n+2 \\ n \end{bmatrix}_{q^2} - [n+1]_{q^2} \\
&\equiv -\frac{1}{1+q} \pmod{[p]_q}.
\end{aligned}$$

Hence

$$\sum_{k=0}^{n} \frac{C_k(q)C_{k+1}(q)}{(-q;q)_k^2(-q^2;q)_k^2} q^{4k+3} \equiv (1+q)^3 \pmod{[p]_q}.$$

This finishes the proof of Theorem 1.2. □

PROOF OF THEOREM 1.3 : By the q-Lucas theorem,

$$\begin{bmatrix} 2k \\ k \end{bmatrix}_{q} \equiv 0 \pmod{[p]_q} \text{ for } n < k < p.$$

Then by Lemma 2.5 and [6, (4.1)],

$$\sum_{k=0}^{p-1} \frac{{2k \brack k}_q {2k \brack k+d}_{q^2} (-q^{2(k+1)};q^2)_{n-k}}{(-q;q)_k^2} q^{2k+\frac{(p-1)(p-3)}{4}}$$
$$\equiv \sum_{k=0}^{n} (-1)^k q^{2\binom{n-k}{2}} {n \brack k}_{q^2} {2k \brack k+d}_{q^2} (-q^{2(k+1)};q^2)_{n-k}$$
$$= \begin{cases} 0 & \text{if } n \not\equiv d \pmod 2 \\ (-1)^n q^{n^2-d^2} {n \brack \frac{n-d}{2}}_{q^4} & \text{if } n \equiv d \pmod 2 \end{cases} \pmod{[p]_q}.$$

This ends the proof of Theorem 1.3. □

Remark 3.1 : Similarly, we can use the method which is used in the proof of Theorem 1.3 and the identities (4.2), (4.7) and (4.8) in [6] to deduce another three q-analogues of (1.3).

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