

SOME PROPERTIES OF LINEAR OPERATORS DEFINED BY HYPERGEOMETRIC FUNCTIONS

Z. Orouji and R. Aghalary

Department of Mathematics, Faculty of Science, Urmia University, Urmia, Iran

e-mails: z.ouuji@urmia.ac.ir; z.ouuji@yahoo.com, raghalary@yahoo.com

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This paper is concerned with the connection of certain operators $\Phi_{b,c}$, defined by hypergeometric functions, with the α -logarithmic Bloch spaces $\mathcal{B}_{\log,\alpha}$ of analytic functions.

Key words : Fractional derivative; α -logarithmic Bloch spaces; nontangential limits; Pochhammer symbol; uniformly locally univalent.

1. INTRODUCTION

Let $\mathcal{H}(\mathbb{D})$ denote the class of analytic functions f on the open unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Also denote by \mathcal{A} the subclass of $\mathcal{H}(\mathbb{D})$ consisting of functions normalized by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n.$$

For $\alpha > 0$, the α -logarithmic Bloch space $\mathcal{B}_{\log,\alpha}$ is the Banach space of those functions $f \in \mathcal{H}(\mathbb{D})$ which satisfy

$$\|f\|_{\log,\alpha} := |f(0)| + \sup_{z \in \mathbb{D}} (1 - |z|^2) \left(\log \frac{2}{1 - |z|^2} \right)^\alpha |f'(z)| < \infty.$$

For simplicity, the space $\mathcal{B}_{\log,1}$ will be denoted by \mathcal{B}_{\log} .

The logarithmic Bloch spaces were studied by many authors (see for example [3] and [4]).

A typical example of an unbounded univalent function in the space \mathcal{B}_{\log} is the function

$$f(z) = \log \log \frac{2}{1 - z}, \quad (z \in \mathbb{D}).$$

Let us mention also that the logarithmic Bloch spaces play a basic role studying the boundedness of Hankel and Toeplitz operators in Bergman and Bloch-type spaces (see. e.g., [1] and [12]).

Various operators of fractional calculus (that is, fractional integral and fractional derivative) have been studied in the literature (cf. [5, 9-11]). We use the following definition used recently by Srivastava and Owa [10]. The fractional integral of order λ is defined, for a function f , by

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\lambda}} d\zeta, \quad (\lambda > 0),$$

where f is an analytic function in a simply-connected region of the z -plane containing the origin, and the multiplicity of $(z-\zeta)^{\lambda-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $z-\zeta > 0$.

It is easy to check that for $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$ we have

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(2+\lambda)} z^{\lambda+1} + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)}{\Gamma(\lambda+n+1)} a_n z^{n+\lambda}, \quad (z \in \mathbb{D}). \quad (1.1)$$

Also the fractional derivative of order λ is defined, for a function f , by

$$D_z^{\lambda} f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\lambda}} d\zeta, \quad (0 \leq \lambda < 1),$$

where f is a above, and the multiplicity of $(z-\zeta)^{-\lambda}$ is removed, as in previous definition.

Also it is interesting to note that for $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$ we have

$$D_z^{\lambda} f(z) = \frac{1}{\Gamma(2-\lambda)} z^{1-\lambda} + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n+1-\lambda)} a_n z^{n-\lambda}, \quad (z \in \mathbb{D}). \quad (1.2)$$

Jung *et al.* [5] investigated the linear operator $\mathfrak{S}_{\beta}^{\alpha}$ which is defined by

$$\mathfrak{S}_{\beta}^{\alpha} f(z) = \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)} \frac{\alpha}{z^{\beta}} \int_0^z \left(1 - \frac{t}{z}\right)^{\alpha-1} t^{\beta-1} f(t) dt, \quad (f \in \mathcal{H}(\mathbb{D}), z \in \mathbb{D})$$

$$\alpha > 0; \beta > -1;$$

and they proved for $f \in \mathcal{A}$ with $\operatorname{Re} f'(z) > 0$, $\mathfrak{S}_{\beta}^{\alpha} f$ is bounded at least for $\alpha > 1$ and $\beta > -1$. In view of ([5], Lemma 4), for function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$ we have

$$\mathfrak{S}_{\beta}^{\alpha} f(z) = z + \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)} \sum_{n=2}^{\infty} \frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)} a_n z^n, \quad (\alpha > 0, \beta > -1). \quad (1.3)$$

Definition 1.1 — For $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with $f \in \mathcal{H}(\mathbb{D})$, we define

$$\Phi_{b,c} f(z) = \sum_{n=0}^{\infty} \frac{(b)_n}{(c)_n} a_n z^n, \quad (z \in \mathbb{D}).$$

Here b, c are complex numbers such that $c \neq 0, -1, -2, \dots, (b)_0 = 1$ for $b \neq 0$, and for each positive integer $n, (b)_n = b(b + 1)(b + 2)\dots(b + n - 1)$ is the Pochhammer symbol.

The above series converges absolutely for all $z \in \mathbb{D}$, and hence represents an analytic function in the unit disc \mathbb{D} . It should be remarked that by specializing the parameters b, c and using (1.1), (1.2) and (1.3) one can obtain that

$$D_z^{-\lambda} f(z) = \frac{z^\lambda}{\Gamma(1 + \lambda)} \Phi_{1,1+\lambda} f(z), \quad (z \in \mathbb{D}, \lambda > 0), \tag{1.4}$$

$$D_z^\lambda f(z) = \frac{z^{-\lambda}}{\Gamma(1 - \lambda)} \Phi_{1,1-\lambda} f(z), \quad (z \in \mathbb{D}, 0 \leq \lambda < 1), \tag{1.5}$$

and

$$\mathfrak{S}_\beta^\alpha f(z) = \frac{\alpha + \beta}{\beta} \Phi_{\beta,\alpha+\beta} f(z), \quad (z \in \mathbb{D}, \alpha > 0, \beta > 0), \tag{1.6}$$

where $f \in \mathcal{A}$.

It is natural to ask that for which functions f and parameters $b, c, \Phi_{b,c} f \in \mathcal{B}_{\log,\alpha} (\alpha > 0)$. The main purpose of this paper is finding conditions on the parameters b, c such that when f is chosen from suitable classes, $\Phi_{b,c} f \in \mathcal{B}_{\log,\alpha} (\alpha > 0)$.

Remark 1.1 : It is easy to see that the function $g(x) = (1 - x) \left(\log \frac{2}{1-x} \right)^\alpha$ ($0 < x < 1$) is bounded and decreasing for $0 \leq \alpha < \ln 2$.

Proving our results we shall use the following well known result

$$\int_0^1 t^{p-1} (1 - t)^{q-1} dt = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p + q)}$$

where $\text{Re } p > 0$ and $\text{Re } q > 0$.

2. MAIN RESULTS

In the following Theorem we prove:

Theorem 2.1 — *Let b, c be complex numbers such that $-1 < \text{Re } c < \text{Re } b$. Also let $f \in \mathcal{H}(\mathbb{D})$ and $\Phi_{b,c} f \in \mathcal{B}_{\log,\alpha}$ where $0 \leq \alpha < \ln 2$. Then $f \in \mathcal{B}_{\log,\alpha}$. Moreover*

$$\|f\|_{\log,\alpha} \leq |f(0)| + \frac{|\Gamma(b)|\Gamma(\text{Re}(b - c))\Gamma(\text{Re } c + 1)}{|\Gamma(c)||\Gamma(b - c)|\Gamma(\text{Re } b + 1)} \|\Phi_{b,c} f\|_{\log,\alpha}.$$

PROOF : Let $f(z) = \sum_{n=0}^\infty a_n z^n \in \mathcal{H}(\mathbb{D})$ and $\Phi_{b,c} f \in \mathcal{B}_{\log,\alpha}$. By the hypotheses of the theorem,

for $z \in \mathbb{D}$ we have

$$\begin{aligned}
 \int_0^1 (1-t)^{b-c-1} t^{c-1} \Phi_{b,c} f(tz) dt &= \frac{\Gamma(c)}{\Gamma(b)} \int_0^1 (1-t)^{b-c-1} t^{c-1} \sum_{n=0}^{\infty} \frac{\Gamma(b+n)}{\Gamma(c+n)} a_n t^n z^n dt \\
 &= \frac{\Gamma(c)}{\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(b+n)}{\Gamma(c+n)} a_n z^n \int_0^1 (1-t)^{b-c-1} t^{n+c-1} dt \\
 &= \frac{\Gamma(c)\Gamma(b-c)}{\Gamma(b)} \sum_{n=0}^{\infty} a_n z^n \\
 &= \frac{\Gamma(c)\Gamma(b-c)}{\Gamma(b)} f(z). \tag{2.1}
 \end{aligned}$$

By Remark 1.1 and relation (2.1), we obtain

$$\begin{aligned}
 (1-|z|^2) \left(\log \frac{2}{1-|z|^2} \right)^\alpha |f'(z)| &\leq \frac{|\Gamma(b)|}{|\Gamma(c)||\Gamma(b-c)|} \int_0^1 (1-t)^{Re(b-c)-1} t^{Re c} (1-|z|^2) \left(\log \frac{2}{1-|z|^2} \right)^\alpha |(\Phi_{b,c} f)'(tz)| dt \\
 &\leq \frac{|\Gamma(b)|}{|\Gamma(c)||\Gamma(b-c)|} \int_0^1 (1-t)^{Re(b-c)-1} t^{Re c} (1-|tz|^2) \left(\log \frac{2}{1-|tz|^2} \right)^\alpha |(\Phi_{b,c} f)'(tz)| dt \\
 &\leq \frac{|\Gamma(b)||\Phi_{b,c} f|_{\log, \alpha}}{|\Gamma(c)||\Gamma(b-c)|} \int_0^1 (1-t)^{Re(b-c)-1} t^{Re c} dt \\
 &< \infty, \quad (z \in \mathbb{D}),
 \end{aligned}$$

since $-1 < Re c < Re b$. Therefore $f \in \mathcal{B}_{\log, \alpha}$ and

$$\|f\|_{\log, \alpha} \leq |f(0)| + \frac{|\Gamma(b)|\Gamma(Re(b-c))\Gamma(Re c + 1)}{|\Gamma(c)||\Gamma(b-c)|\Gamma(Re b + 1)} \|\Phi_{b,c} f\|_{\log, \alpha}. \square$$

By putting $b = 1$ and $c = 1 - \lambda$ in the Theorem 2.1 and using (1.5) we obtain the following corollary:

Corollary 2.1 — Let $f \in \mathcal{A}$ and $z^\lambda D_z^\lambda f \in \mathcal{B}_{\log, \alpha}$ where $0 \leq \lambda < 1$ and $0 \leq \alpha < \ln 2$. Then $f \in \mathcal{B}_{\log, \alpha}$ and $\|f\|_{\log, \alpha} = (1 - \lambda) \|z^\lambda D_z^\lambda f\|_{\log, \alpha}$.

A function $f \in \mathcal{H}(\mathbb{D})$ is called uniformly locally univalent if there exists a constant $\rho > 0$ such that f is univalent on the hyperbolic disc $|(z - a)/(1 - \bar{a}z)| < \tanh \rho$ of radius ρ for every $a \in \mathbb{D}$. It is known that a non-constant analytic function f is uniformly locally univalent if and only if the norm

$$\|f''/f'\| = \sup_{z \in \mathbb{D}} (1 - |z|^2) \left| \frac{f''(z)}{f'(z)} \right|$$

of the pre-Schwarzian derivative f''/f' of f is finite. Let

$$B(\lambda) = \{f \in \mathcal{H}(\mathbb{D}); \| f''/f' \| \leq 2\lambda\}.$$

Kim and Sugawa [6, 7] investigated various properties of the functions which belong to the class $B(\lambda)$. In the next theorem we prove:

Theorem 2.2 — *Let $\lambda > 0$ and let b, c be complex numbers such that $-1 < \operatorname{Re} b < \operatorname{Re} c - \frac{\lambda}{2}$. Also suppose that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is uniformly locally univalent with $\| f''/f' \| \leq \lambda$. Then $\Phi_{b,c}f \in \mathcal{B}_{\log,\alpha}$ for $0 \leq \alpha < \ln 2$.*

PROOF : Let $|z| = r$ ($0 < r < 1$) and $\| f''/f' \| \leq \lambda$. Then we have

$$\begin{aligned} \log \left| \frac{f'(z)}{f'(0)} \right| &\leq \left| \log \frac{f'(z)}{f'(0)} \right| \\ &= \left| \int_0^z \frac{f''(w)}{f'(w)} dw \right| \\ &\leq r \int_0^1 \left| \frac{f''(tz)}{f'(tz)} \right| dt \\ &\leq r \int_0^1 \frac{\lambda}{1-r^2t^2} dt \\ &= \lambda \log \sqrt{\frac{1+r}{1-r}}. \end{aligned}$$

This implies

$$|f'(z)| \leq |f'(0)| \left(\frac{1+r}{1-r} \right)^{\frac{\lambda}{2}}, \quad (|z| = r < 1). \tag{2.2}$$

On the other hand, we have

$$\begin{aligned} \Phi_{b,c}f(z) &= \frac{\Gamma(c)}{\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(b+n)}{\Gamma(c+n)} \cdot \frac{c+n}{c+n} a_n z^n \\ &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b+1)} \sum_{n=0}^{\infty} \frac{\Gamma(b+n)\Gamma(c-b+1)}{\Gamma(c+n+1)} (c+n) a_n z^n \\ &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b+1)} \int_0^1 t^{b-1}(1-t)^{c-b} \sum_{n=0}^{\infty} (c+n) a_n z^n t^n dt \\ &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b+1)} \int_0^1 t^{b-1}(1-t)^{c-b} (cf(tz) + tzf'(tz)) dt. \end{aligned} \tag{2.3}$$

Set $h(z) = cf(z) + zf'(z)$. Since $\| f''/f' \| \leq \lambda$, we obtain $|f''(z)| \leq \frac{\lambda|f'(z)|}{1-|z|^2}$ ($z \in \mathbb{D}$), and so

$$|h'(z)| \leq |f'(z)| \left(\frac{\lambda|z|}{1-|z|^2} + |c| + 1 \right), \quad (z \in \mathbb{D}). \tag{2.4}$$

Then in view of (2.3) and (2.4) we have

$$|(\Phi_{b,c}f)'(z)| \leq \frac{|\Gamma(c)|}{|\Gamma(b)||\Gamma(c-b+1)|} \int_0^1 t^{\operatorname{Re} b} (1-t)^{\operatorname{Re}(c-b)} |f'(tz)| \left(\frac{\lambda t|z|}{1-t^2|z|^2} + |c| + 1 \right) dt$$

and by relation (2.2), for $|z| = r$ ($0 < r < 1$), we obtain

$$\begin{aligned} (1-|z|^2) \left(\log \frac{2}{1-|z|^2} \right)^\alpha |(\Phi_{b,c}f)'(z)| &\leq \frac{|\Gamma(c)|}{|\Gamma(b)||\Gamma(c-b+1)|} \int_0^1 t^{\operatorname{Re} b} (1-t)^{\operatorname{Re}(c-b)} (1-|z|^2) \left(\log \frac{2}{1-|z|^2} \right)^\alpha |f'(0)| \\ &\quad \times \left(\frac{1+tr}{1-tr} \right)^{\frac{\lambda}{2}} \left(\frac{\lambda t|z|}{1-t^2|z|^2} + |c| + 1 \right) dt \\ &\leq \frac{|\Gamma(c)||f'(0)|}{|\Gamma(b)||\Gamma(c-b+1)|} (\log 2)^\alpha \int_0^1 t^{\operatorname{Re} b} (1-t)^{\operatorname{Re}(c-b)} \left(\frac{2}{1-t} \right)^{\frac{\lambda}{2}} \left(\frac{\lambda t}{1-t} + |c| + 1 \right) dt \\ &\leq \frac{|\Gamma(c)||f'(0)| 2^{\frac{\lambda}{2}}}{|\Gamma(b)||\Gamma(c-b+1)|} \lambda (\log 2)^\alpha \int_0^1 t^{\operatorname{Re} b+1} (1-t)^{\operatorname{Re}(c-b)-\frac{\lambda}{2}-1} \\ &\quad + \frac{|\Gamma(c)||f'(0)| 2^{\frac{\lambda}{2}}}{|\Gamma(b)||\Gamma(c-b+1)|} (|c|+1) (\log 2)^\alpha \int_0^1 t^{\operatorname{Re} b} (1-t)^{\operatorname{Re}(c-b)-\frac{\lambda}{2}} \\ &< \infty \end{aligned}$$

since $-1 < \operatorname{Re} b < \operatorname{Re} c - \frac{\lambda}{2}$ and so $\Phi_{b,c}f \in \mathcal{B}_{\log, \alpha}$.

By putting $b = 1$ and $c = 1 + \lambda$ in the Theorem 2.2 and using (1.4) we obtain the following corollary:

Corollary 2.2 — Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in B(p)$ and $0 < p < \lambda$. Then $z^{-\lambda} D_z^{-\lambda} f \in \mathcal{B}_{\log, \alpha}$ for $0 \leq \alpha < \ln 2$.

Also by putting $b = \beta$ and $c = \gamma + \beta$ in the Theorem 2.2 and using (1.6) we obtain:

Corollary 2.3 — Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in B(p)$ and $0 < p < \gamma$. Then $\mathfrak{S}_\beta^\gamma f \in \mathcal{B}_{\log, \alpha}$ for $0 \leq \alpha < \ln 2$ and $\beta > 0$.

Theorem 2.3 — Let $0 \leq \alpha < \ln 2$ and let $0 < b < c - 1$. Also suppose $f(z) = \sum_{n=1}^{\infty} a_n z^n$ and $g(z) = \sum_{n=1}^{\infty} \frac{a_n}{n^{c-b}} z^n$ be analytic in \mathbb{D} where $\{a_n\}$ is a bounded sequence in \mathbb{C} . Then $g \in \mathcal{B}_{\log, \alpha}$ if and only if $\Phi_{b,c}f \in \mathcal{B}_{\log, \alpha}$.

PROOF : From Stirlings formula, we have the following asymptotic expansion for the gamma function ($|\arg z| \leq \pi - \epsilon, \epsilon > 0$) ([8], p. 88):

$$\Gamma(z) \approx e^{-z} z^z \sqrt{\frac{2\pi}{z}} \left(1 + \frac{1}{12z} + \frac{1}{288z^2} + \dots \right) \quad (z \rightarrow \infty).$$

Hence we obtain (as $n \rightarrow \infty$)

$$\frac{\Gamma(b+n)}{\Gamma(c+n)} \approx e^{c-b} \left(\frac{b+n}{c+n}\right)^{b+n-\frac{1}{2}} (c+n)^{b-c} \left(1 + \frac{c-b}{12(b+n)(c+n)} + \dots\right). \quad (2.5)$$

Note that (as $n \rightarrow \infty$)

$$\left(\frac{b+n}{c+n}\right)^{b+n-\frac{1}{2}} \approx e^{-(c-b)} \left(1 + \frac{c-b}{2(b+n)} + \dots\right) \quad (2.6)$$

and

$$(c+n)^{b-c} = n^{b-c} \left(1 - \frac{-c}{n}\right)^{-(c-b)} = n^{b-c} \left(1 + \frac{-c(c-b)}{n} + \dots\right). \quad (2.7)$$

Therefore by relations (2.5) to (2.7), we have

$$\frac{\Gamma(b+n)}{\Gamma(c+n)} \approx n^{b-c} \sum_{i=0}^{\infty} A_i n^{-i} \quad (n \rightarrow \infty) \quad (2.8)$$

where the A_i are constants depending on b, c with $A_0 = 1$. From (2.8) we obtain

$$\begin{aligned} \Phi_{b,c}f(z) &= \frac{\Gamma(c)}{\Gamma(b)} \sum_{n=1}^{\infty} \frac{\Gamma(b+n)}{\Gamma(c+n)} a_n z^n \\ &\approx \frac{\Gamma(c)}{\Gamma(b)} \sum_{n=1}^{\infty} n^{b-c} \left\{ \sum_{i=0}^{\infty} A_i n^{-i} \right\} a_n z^n, \end{aligned}$$

and so

$$\frac{\Gamma(b)}{\Gamma(c)} \Phi_{b,c}f(z) = g(z) + \sum_{n=1}^{\infty} n^{b-c} O\left(\frac{1}{n}\right) a_n z^n \quad (2.9)$$

where $g(z) = \sum_{n=1}^{\infty} \frac{a_n}{n^{c-b}} z^n$. Since $a_n = O(1)$, we have

$$\begin{aligned} \left| \sum_{n=1}^{\infty} n^{b-c+1} O\left(\frac{1}{n}\right) a_n z^{n-1} \right| &\leq \left(\max_{n \in \mathbb{N}} |a_n| \right) \sum_{n=1}^{\infty} \left| O\left(\frac{1}{n^{c-b}}\right) \right| \\ &< \infty \end{aligned}$$

since $0 < b < c - 1$ and so there exists $N > 0$ such that

$$\left| \sum_{n=1}^{\infty} n^{b-c+1} O\left(\frac{1}{n}\right) a_n z^{n-1} \right| < N, \quad (z \in \mathbb{D}). \quad (2.10)$$

Now, suppose $g \in \mathcal{B}_{\log, \alpha}$ where $0 \leq \alpha < \ln 2$. From (2.9) we have

$$(\Phi_{b,c}f)'(z) = \frac{\Gamma(c)}{\Gamma(b)} \left(g'(z) + \sum_{n=1}^{\infty} n^{b-c+1} O\left(\frac{1}{n}\right) a_n z^{n-1} \right), \quad (z \in \mathbb{D}),$$

and so by (2.10)

$$\begin{aligned} (1 - |z|^2) \left(\log \frac{2}{1 - |z|^2} \right)^\alpha |(\Phi_{b,c}f)'(z)| &\leq \frac{|\Gamma(c)|}{|\Gamma(b)|} \left(\|g\|_{\log,\alpha} + N(1 - |z|^2) \left(\log \frac{2}{1 - |z|^2} \right)^\alpha \right) \\ &\leq \frac{|\Gamma(c)|}{|\Gamma(b)|} (\|g\|_{\log,\alpha} + N(\log 2)^\alpha) \\ &< \infty, \quad (z \in \mathbb{D}). \end{aligned}$$

Therefore $\Phi_{b,c}f \in \mathcal{B}_{\log,\alpha}$.

Conversely, suppose $\Phi_{b,c}f \in \mathcal{B}_{\log,\alpha}$. From (2.9),

$$g'(z) = \frac{\Gamma(b)}{\Gamma(c)} (\Phi_{b,c}f)'(z) - \sum_{n=1}^{\infty} n^{b-c+1} O\left(\frac{1}{n}\right) a_n z^{n-1}, \quad (z \in \mathbb{D}),$$

and therefore we obtain

$$\begin{aligned} \|g\|_{\log,\alpha} &\leq \frac{|\Gamma(b)|}{|\Gamma(c)|} \|\Phi_{b,c}f\|_{\log,\alpha} + \sup_{z \in \mathbb{D}} (1 - |z|^2) \left(\log \frac{2}{1 - |z|^2} \right)^\alpha N \\ &\leq \frac{|\Gamma(b)|}{|\Gamma(c)|} \|\Phi_{b,c}f\|_{\log,\alpha} + N(\log 2)^\alpha. \end{aligned}$$

So $g \in \mathcal{B}_{\log,\alpha}$ and the proof is completed. \square

By putting $b = 1$ and $c = 1 + \lambda$ in the Theorem 2.3 and using (1.4) we obtain the following corollary:

Corollary 2.4 — Let $0 \leq \alpha < \ln 2$ and $\lambda > 1$. Also let $f(z) = \sum_{n=1}^{\infty} a_n z^n \in \mathcal{A}$ where $\{a_n\}$ is a bounded sequence in \mathbb{C} . Then $\sum_{n=1}^{\infty} n^{-\lambda} a_n z^n \in \mathcal{B}_{\log,\alpha}$ if and only if $z^{-\lambda} D_z^{-\lambda} f \in \mathcal{B}_{\log,\alpha}$.

Also by putting $b = \beta$ and $c = \gamma + \beta$ in the Theorem 2.3 and using (1.6) we obtain:

Corollary 2.5 — Let $0 \leq \alpha < \ln 2$ and $\gamma > 1$. Also let $f(z) = \sum_{n=1}^{\infty} a_n z^n \in \mathcal{A}$ where $\{a_n\}$ is a bounded sequence in \mathbb{C} . Then $\sum_{n=1}^{\infty} n^{-\gamma} a_n z^n \in \mathcal{B}_{\log,\alpha}$ if and only if $\mathfrak{S}_\beta^\gamma f \in \mathcal{B}_{\log,\alpha}$.

Corollary 2.6 — Suppose $0 < b < c - 1$ and $f(z) = \sum_{n=1}^{\infty} a_n z^n \in \mathcal{H}(\mathbb{D})$ where $\{a_n\}$ is a bounded sequence in \mathbb{C} . Then $\Phi_{b,c}f$ has nontangential limits in almost every direction.

PROOF : Set $g(z) = \sum_{n=1}^{\infty} \frac{a_n}{n^{c-b}} z^n$, ($z \in \mathbb{D}$). By relation (2.9) from Theorem 2.3, we have

$$\frac{\Gamma(b)}{\Gamma(c)} \Phi_{b,c}f(z) = g(z) + \sum_{n=1}^{\infty} n^{b-c} O\left(\frac{1}{n}\right) a_n z^n, \quad (z \in \mathbb{D}).$$

Since $a_n = O(1)$ and $c - b > 1$, we obtain

$$|g(z)| \leq \left(\max_{n \in \mathbb{N}} |a_n| \right) \sum_{n=1}^{\infty} \frac{1}{n^{c-b}} < \infty, \quad (z \in \mathbb{D})$$

and

$$\left| \sum_{n=1}^{\infty} n^{b-c} O\left(\frac{1}{n}\right) a_n z^n \right| \leq \left(\max_{n \in \mathbb{N}} |a_n| \right) \sum_{n=1}^{\infty} \left| O\left(\frac{1}{n^{c-b+1}}\right) \right| < \infty, \quad (z \in \mathbb{D}).$$

Therefore $\Phi_{b,c}f$ is bounded and so has nontangential limits in almost every direction. □

The next theorem deals with the case $0 < b < c + 1$. For its proof we shall need the following result due to Clunie and Macgregor [2].

Theorem 2.4 — *Let f be analytic and univalent in \mathbb{D} and let $\gamma > 1/2$. Then there is a set E of measure 2π such that for all $\theta \in E$, if \angle_α is a Stolz angle with vertex $\alpha = e^{i\theta}$,*

$$\lim_{z \rightarrow \alpha} \frac{\log |f'(z)|}{\left(\log \frac{1}{1-|z|}\right)^\gamma} = 0, \quad (z \in \angle_\alpha).$$

Theorem 2.5 — *Suppose f is analytic and univalent in \mathbb{D} and $0 < b < c + 1$. Then $\Phi_{b,c}f$ has nontangential limits in almost every direction.*

PROOF : Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be analytic and univalent in \mathbb{D} and fix γ in the interval $(1/2, 1)$. By Theorem 2.4, there exists some constant A and a set E of measure 2π such that for all $\theta \in E$

$$\log |f'(z)| \leq A \left(\log \frac{1}{1-|z|} \right)^\gamma$$

whenever $\alpha = e^{i\theta}$ and $z \in \angle_\alpha$. For any $\delta > 0$ we have

$$\begin{aligned} (1-|z|)^\delta |f'(z)| &\leq (1-|z|)^\delta \exp \left\{ A \left(\log \frac{1}{1-|z|} \right)^\gamma \right\} \\ &= \exp \left\{ A \left(\log \frac{1}{1-|z|} \right)^\gamma - \delta \log \frac{1}{1-|z|} \right\}. \end{aligned}$$

Since $\gamma < 1$, it is clear that this last expression approaches zero as $|z| \rightarrow 1$. Hence

$$|f'(z)| \leq \frac{B}{(1-|z|)^\delta} \tag{2.11}$$

for some constant B whenever z lies in the Stolz angle \angle_α . If $g(z) = \sum_{n=0}^{\infty} (c+n)a_n z^n = cf(z) + zf'(z)$, then from (2.11) we obtain

$$\begin{aligned} |g(z)| &\leq |cf(0)| + |c|rB \int_0^1 \frac{1}{(1-tr)^\delta} dt + r \frac{B}{(1-r)^\delta} \\ &\leq |cf(0)| + \frac{B|c|}{1-\delta} + \frac{rB}{(1-r)^\delta} \end{aligned} \tag{2.12}$$

for all $z \in \angle_\alpha(|z| = r)$. Using the same argument as in the proof of Theorem 2.2 we know that

$$\Phi_{d,c}f(z) = \frac{\Gamma(c)}{\Gamma(d)\Gamma(c-d+1)} \int_0^1 t^{d-1}(1-t)^{c-d}g(tz)dt. \quad (2.13)$$

Let d be such that $b < d < c + 1$, and choose $\delta > 0$ so that $\delta < c - d + 1$. Then in view of (2.12) and (2.13), for all $z \in \angle_\alpha(|z| = r)$, we have

$$\begin{aligned} |\Phi_{d,c}f(z)| &\leq \frac{|\Gamma(c)|}{\Gamma(d)\Gamma(c-d+1)} \int_0^1 t^{d-1}(1-t)^{c-d}|g(tz)|dt \\ &\leq \frac{|\Gamma(c)|}{\Gamma(d)\Gamma(c-d+1)} \int_0^1 t^{d-1}(1-t)^{c-d} \left(|cf(0)| + \frac{B|c|}{1-\delta} + \frac{Btr}{(1-tr)^\delta} \right) dt \\ &\leq \frac{|\Gamma(c)|}{\Gamma(d)\Gamma(c-d+1)} \left(\left(|cf(0)| + \frac{B|c|}{1-\delta} \right) \int_0^1 t^{d-1}(1-t)^{c-d}dt + B \int_0^1 t^d(1-t)^{c-d-\delta}dt \right) \\ &= K < \infty. \end{aligned}$$

Hence $\Phi_{d,c}f$ is bounded inside \angle_α . Also

$$\begin{aligned} \Phi_{b,c}f(z) &= \frac{\Gamma(c)}{\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(b+n)}{\Gamma(d+n)} \cdot \frac{\Gamma(d+n)}{\Gamma(c+n)} a_n z^n \\ &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(d-b)} \int_0^1 t^{b-1}(1-t)^{d-b-1} \Psi(tz)dt \end{aligned} \quad (2.14)$$

where

$$\Psi(z) = \sum_{n=0}^{\infty} \frac{\Gamma(d+n)}{\Gamma(c+n)} a_n z^n, \quad (z \in \mathbb{D}).$$

Therefore

$$|\Psi(z)| = \frac{\Gamma(d)}{|\Gamma(c)|} |\Phi_{d,c}f(z)| < \frac{\Gamma(d)}{|\Gamma(c)|} K. \quad (2.15)$$

Let

$$h(t) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(d-b)} t^{b-1}(1-t)^{d-b-1}.$$

Then $h > 0$, $\int_0^1 h(t)dt = \frac{\Gamma(c)}{\Gamma(d)}$, and by (2.14),

$$\Phi_{b,c}f(z) = \int_0^1 h(t)\Psi(tz)dt. \quad (2.16)$$

These properties imply that $\Phi_{b,c}f$ is uniformly continuous on the set \angle_α . To see this, let $z_1, z_2 \in \angle_\alpha$ and let $\epsilon > 0$ be given. Since $\int_0^1 h(t)dt$ exists, there is an $x \in (0, 1)$ such that

$$\int_{1-x}^1 h(t)dt < \frac{\epsilon}{4K} \frac{|\Gamma(c)|}{\Gamma(d)}. \quad (2.17)$$

By the uniform continuity of Ψ on compact subset of \mathbb{D} , there exists $\lambda > 0$ such that

$$|\Psi(tz_2) - \Psi(tz_1)| < \frac{\epsilon\Gamma(d)}{2|\Gamma(c)|} \tag{2.18}$$

whenever $|z_2 - z_1| < \lambda$ with $z_1, z_2 \in \angle_\alpha$ and for all $t \in [0, 1 - x]$. Therefore, by relations (2.15) to (2.18), for $|z_2 - z_1| < \lambda$ we have

$$\begin{aligned} |\Phi_{b,c}f(z_2) - \Phi_{b,c}f(z_1)| &\leq \int_0^{1-x} h(t)|\Psi(tz_2) - \Psi(tz_1)|dt + \int_{1-x}^1 h(t)|\Psi(tz_2) - \Psi(tz_1)|dt \\ &< \frac{\epsilon\Gamma(d)}{2|\Gamma(c)|} \int_0^{1-x} h(t)dt + 2K \frac{\Gamma(d)}{|\Gamma(c)|} \int_{1-x}^1 h(t)dt \\ &< \frac{\epsilon\Gamma(d)}{2|\Gamma(c)|} \int_0^1 h(t)dt + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Hence $\Phi_{b,c}f$ is uniformly continuous inside \angle_α and so can be extended continuously to the boundary. This implies that

$$\lim_{z \rightarrow \alpha} \Phi_{b,c}f(z) \quad (z \in \angle_\alpha),$$

exists for every $\theta \in E$, where $\alpha = e^{i\theta}$. Since E has measure 2π , this proves the theorem. □

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