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ON THE MULTIPLIER SEMIGROUP OF A WEIGHTED ABELIAN SEMIGROUP[1]

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Let (S,ω) be a weighted abelian semigroup. We show that a ω-bounded semigroup multiplier on S is a multiplication by a bounded function on the space of ω-bounded generalized semicharacters on S; and discuss a converse. Given a ω-bounded multiplier α on S, we investigate the induced weighted semigroup (S_α,ω_α). We show that the ω_α-bounded generalized semicharacters on S_α are scalar multiples of ω-bounded generalized semicharacters on S. Moreover, if (S_0,ω_0) is another weighted semigroup formed with some other operation on set S such that ω_0-bounded generalized semicharacters on S_0 are scalar multiples of ω-bounded generalized semicharacters on S, then it is shown that $S_0 = S_\alpha$ under some natural conditions. A number of examples and counter examples are discussed. The paper strengthens the idea that a weighted semigroup provides a semigroup analogue of a normed algebra for which a Gelfand duality may be searched.

Key words : Multipliers on commutative Banach algebra; weighted semigroup; multipliers on a semigroup.

1. INTRODUCTION

A *weighted semigroup* (S,ω) is a semigroup S with a *weight* $\omega : S \to (0,\infty)$ satisfying $\omega(st) \leq \omega(s)\omega(t)$ $(s,t \in S)$. The present paper contributes to the multiplier problem on weighted semigroup (S,ω) (which is S when $\omega = 1$) [1, 2] that is aimed at understanding the multiplier semigroup $M_\omega(S)$ of all ω-bounded (semigroup) multipliers on S and its relation with the (algebra) multipliers of the associated Beurling algebra $\ell^1(S,\omega)$. By the classical result of Wang [10, Theorem 3.1], a multiplier on a semisimple commutative Banach algebra $\mathcal{A}$ is achieved, at the level of Gelfand transform, by

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multiplication with a bounded continuous function φ on the Gelfand space; and conversely, such a function φ leaving Gelfand transforms $\widehat{\mathcal{A}} = \{\widehat{a} : a \in \mathcal{A}\}$ invariant results in a multiplier on $\mathcal{A}$. Thus a multiplier is a multiplication by a bounded continuous function on the Gelfand space. In Section 2, we show that analogously a ω-bounded semigroup multiplier on S is a multiplication by a bounded function on the space of ω-bounded generalized semicharacters on S. The converse holds when ω is a uniform weight on S.

A multiplier T on a commutative Banach algebra $(\mathcal{A}, ||\cdot||)$ produces a Banach algebra $\mathcal{A}_T$ which is $\mathcal{A}$ with multiplication $a \circ b = aTb \ (a, b \in \mathcal{A})$ resulting into an important tool in the study of multipliers [9]. Analogously, given a ω-bounded semigroup multiplier α on S, we consider the weighted semigroup $(S_\alpha, \omega_\alpha)$ with $S = S_\alpha$ having multiplication $s \circ t = s\alpha(t) \ (s, t \in S)$ and weight $\omega_\alpha(s) = \widetilde{\omega}(\alpha)\omega(s) \ (s \in S)$, where $\widetilde{\omega}(\alpha) = \sup\{\frac{\omega(\alpha(s))}{\omega(s)} : s \in S\}$. We compute the space $\widehat{(S_\alpha, \omega_\alpha)}$ of ω_α-bounded generalized semicharacters on S_α as $\widehat{(S_\alpha, \omega_\alpha)} = \{\psi(\chi)\chi : \chi \in \widehat{(S, \omega)}\}$, where ψ is given by $\widehat{\alpha s} = \psi \widehat{s} \ (s \in S)$, where $\widehat{s}$ is the "semigroup Gelfand transform" of s, $\widehat{s}(\chi) = \chi(s) \ (\chi \in \widehat{(S, \omega)})$. A number of examples and counter examples are discussed. The paper strengthen the point of view initiated in [1] that a weighted semigroup (S, ω) is a semigroup analogue of a normed algebra to which a Gelfand duality can be searched.

A semigroup S is *faithful* if for $s, t \in S$, $su = tu$ for all $u \in S$ or $us = ut$ for all $u \in S$ implies $s = t$. A *multiplier* [8] on semigroup S is a map $\alpha : S \to S$ such that $\alpha(st) = \alpha(s)t = s\alpha(t)$ $(s, t \in S)$. Let $M(S)$ be the set of all multipliers on S. A multiplier α on S is *ω-bounded* if there exists $K_\alpha > 0$ such that $\omega(\alpha(s)) \le K_\alpha \omega(s) (s \in S)$. Let $M_\omega(S)$ be the set of all ω-bounded multipliers on S. Then $M_\omega(S)$ is a semigroup with operation composition. Also with help of weight ω on S we can define a weight $\widetilde{\omega}$ on $M_\omega(S)$ as $\widetilde{\omega}(\alpha) = \sup\limits_{s \in S} \dfrac{\omega(\alpha s)}{\omega(s)}$.

A *generalized semicharacter* on semigroup S is a nonzero map $\chi : S \to \mathbb{C}$ satisfying $\chi(st) = \chi(s)\chi(t) \ (s, t \in S)$. A generalized semicharacter χ is *ω-bounded* if $|\chi(s)| \le \omega(s) \quad (s \in S)$. Let $\widehat{(S, \omega)}$ represent the set of all ω-bounded generalized semicharacters on S. Every $s \in S$ gives a mapping $\widehat{s} : \widehat{(S, \omega)} \to \mathbb{C}$ defined as $\widehat{s}(\chi) = \chi(s) (\chi \in \widehat{(S, \omega)})$.

A weight ω on semigroup S is *uniform* if $\omega(s^2) = \omega(s)^2$ for all $s \in S$ and ω is *semisimple* [4] if $\lim_{n\to\infty} \omega(s^n)^{1/n} > 0$ for all $s \in S$. Note that a uniform weight is always semisimple. A semigroup S is *separating* [6] if $s^2 = t^2 = st$ implies $s = t$ for all $s, t \in S$.

Let (S, ω) be a weighted abelian semigroup. The Beurling algebra on (S, ω) is the set

$$\ell^1(S, \omega) = \{f : S \to \mathbb{C} : \|f\|_\omega := \sum_{s \in S} |f(s)|\omega(s) < \infty\}.$$

An element of $\ell^1(S,\omega)$ is of the form $f = \sum_{s\in S} f(s)\delta_s$, where $f(s) \in \mathbb{C}$ and $\delta_s : S \to \mathbb{C}$ is defined as $\delta_s(t) = 0$ if $t \neq s$ and $\delta_s(s) = 1$. The space $\ell^1(S,\omega)$ is a commutative Banach algebra with the norm $\|\cdot\|_\omega$ and the convolution multiplication defined as follows. If $f, g \in \ell^1(S,\omega)$ and $s \in S$, then $(f \star g)(s) = \sum\limits_{uv=s} f(u)g(v)$ if $uv = s$ has a solution in S and $(f \star g)(s) = 0$ if $uv = s$ has no solution in S.

For a commutative Banach algebra $\mathcal{A}$, let $\Delta(\mathcal{A})$ represent the set of all multiplicative linear functionals on $\mathcal{A}$. The spectral radius of an element $a \in \mathcal{A}$ is given by $r(a) = \sup\limits_{\varphi\in\Delta(\mathcal{A})} |\varphi(a)|$. Note that $r(a) \leq \|a\|$ for all $a \in \mathcal{A}$. A commutative Banach algebra $\mathcal{A}$ is *semisimple* [7] if $\bigcap_{\varphi\in\Delta(\mathcal{A})} \ker\varphi = \{0\}$. Note that $\ell^1(S,\omega)$ is semisimple if and only if S is separating and ω is semisimple [4, Proposition 4.8]. Also note that $\widehat{(S,\omega)}$ is homeomorphically isomorphic to $\Delta(\ell^1(S,\omega))$ [1, Corollary 3.4] through the correspondence $\chi \mapsto \varphi_\chi$, where $\varphi_\chi(\sum\limits_{s\in S} f(s)\delta_s) = \sum\limits_{s\in S} f(s)\chi(s)$ $(f \in \ell^1(S,\omega))$. For a Banach algebra $\mathcal{A}$, a mapping $T : \mathcal{A} \to \mathcal{A}$ is a *multiplier* [9] if $T(ab) = aT(b) = T(a)b$ $(a, b \in \mathcal{A})$. Let $M(\mathcal{A})$ be the set of all multipliers on $\mathcal{A}$.

2. Multipliers of a Semigroup

A Banach algebra $\mathcal{A}$ is *without order* [9] if $a \in \mathcal{A}$ and $a\mathcal{A} = \{0\}$ or $\mathcal{A}a = \{0\}$ implies $a = 0$. By [9, Theorem 1.1.5], if $\mathcal{A}$ is a Banach algebra without order, then $\mathcal{A}$ is commutative if and only if $M(\mathcal{A})$ is a maximal commutative subalgebra of $E(\mathcal{A})$, where $E(\mathcal{A})$ is the collection of all bounded linear operators on $\mathcal{A}$. The following is analogous result for semigroup.

***Theorem* 2.1** — *Let S be a faithful semigroup, and let ω be any weight on S. For $u \in S$ define the maps L_u and R_u from S into itself as $L_u(s) = us$ and $R_u(s) = su$ for all $s \in S$. Then the following are equivalent.*

(i) S is abelian.

(ii) $\{L_u : u \in S\} \subset M_\omega(S)$.

(iii) $\{R_u : u \in S\} \subset M_\omega(S)$.

(iv) $M_\omega(S)$ is a maximal abelian subsemigroup of $F_\omega(S)$, where $F_\omega(S) = \{f : S \to S : \omega(f(s)) \leq K_f\omega(s)\ (s \in S)\}$.

PROOF : Let S be abelian. Then, for each $u \in S$, L_u is in $M(S)$. Moreover, since $\omega(us) \leq \omega(u)\omega(s)$ for all $s \in S$, $L_u \in M_\omega(S)$. This proves implication (i) $\Rightarrow$ (ii).

Suppose that (ii) holds. Given any $s,t \in S$, $(ts)w = t(sw) = t(L_s(w)) = L_s(t)w = (st)w$ for all $w \in S$. Since S is faithful, we have $st = ts$ and hence S is abelian. Thus (ii) ⇒ (i). Similar arguments show that (i) ⇔ (iii).

To show that (iv) ⇒ (i), suppose that $M_\omega(S)$ is a maximal abelian subsemigroup of $F_\omega(S)$. If $s,t \in S$ and $\alpha \in M_\omega(S)$, then $(L_s\alpha)(t) = s(\alpha t) = \alpha(st) = \alpha(L_s(t))$ for all $t \in S$. Thus $L_s\alpha = \alpha L_s$ for all $s \in S, \alpha \in M_\omega(S)$. Since $M_\omega(S)$ is a maximal abelian subsemigroup of $F_\omega(S)$, $\{L_u : u \in S\} \subset M_\omega(S)$, and hence S is abelian.

To prove the implication (i) ⇒ (iv), assume that S is abelian. Then $\{L_u : u \in S\} \subset M_\omega(S)$. If $M_\omega(S)$ were not a maximal abelian subsemigroup of $F_\omega(S)$, then there exists an abelian subsemigroup, say $E_\omega(S)$, of $F_\omega(S)$ which contains $M_\omega(S)$ properly. But if $\gamma \in E_\omega(S)$, then for each $s,t \in S$, $s(\gamma t) = L_s(\gamma t) = (L_s\gamma)(t) = (\gamma L_s)(t) = \gamma(st)$ and hence $\gamma \in M_\omega(S)$. This contradiction proves our claim. □

By [3, Theorem 4], if $\mathcal{A}$ is a semisimple commutative Banach algebra and $T : \mathcal{A} \to \mathcal{A}$ is a linear map, then T is a multiplier if and only if $T(a^2) = aTa \ (a \in \mathcal{A})$. The analogous result for semigroups is not true. If S is a separating abelian semigroup and α is a multiplier on S, then $\alpha(s^2) = s\alpha(s) \ (s \in S)$. However, the converse need not be true. For instance consider semigroup $\mathbb{N} \cup \{0\}$ with usual multiplication and a map $f : \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$ defined as $f(2^k) = 0$ for all $k \geq 0$ and $f(n) = n$ otherwise. Then $f(n^2) = nf(n)$ for all $n \in \mathbb{N}$ but $f \notin M(\mathbb{N})$ as $f(6) = 6$ while $3f(2) = 0$.

As shown in [10, Theorem 3.1] if $\mathcal{A}$ is a commutative Banach algebra without order and if $T \in M(\mathcal{A})$, then there is a unique bounded continuous function $\psi : \Delta(\mathcal{A}) \to \mathbb{C}$ such that $\widehat{Ta} = \psi\widehat{a} \ (a \in \mathcal{A})$ and $\| \psi \|_\infty \leq \| [T] \|$. Analogously, we have the following result.

Theorem 2.2 — *Let (S,ω) be a weighted abelian semigroup. If $\alpha \in M_\omega(S)$, then there is a unique bounded function $\psi : \widehat{(S,\omega)} \to \mathbb{C}$ such that $\widehat{\alpha(s)} = \psi\widehat{s}$ for all $s \in S$ and $\| \psi \|_\infty \leq \widetilde{\omega}(\alpha)$.*

PROOF : Define $\psi : \widehat{(S,\omega)} \to \mathbb{C}$ as $\psi(\chi) = \frac{\widehat{\alpha s}(\chi)}{\widehat{s}(\chi)} \ (\chi \in \widehat{(S,\omega)})$, where s is such that $\chi(s) \neq 0$. The function ψ so defined is independent of the choice of such s. Indeed, let $t \in S$ be such that $\chi(t) \neq 0$. Then $\chi(\alpha(s)t) = \chi(s\alpha(t))$ gives $\chi(\alpha s)\chi(t) = \chi(s)\chi(\alpha(t))$, which gives $\frac{\widehat{\alpha s}(\chi)}{\widehat{s}(\chi)} = \frac{\widehat{\alpha t}(\chi)}{\widehat{t}(\chi)}$. Let $\chi \in \widehat{(S,\omega)}$. If $s \in S$, then $|\chi(s)| \leq \omega(s)$, i.e., $\frac{|\chi(s)|}{\omega(s)} \leq 1$. Thus $0 < K_\chi = \sup_{s\in S} |\frac{\chi(s)}{\omega(s)}| \leq 1$. This gives $|\psi(\chi)\widehat{s}(\chi)| = |\widehat{\alpha s}(\chi)| \leq K_\chi\omega(\alpha s) \leq K_\chi\widetilde{\omega}(\alpha)\omega(s)$, which implies $|\frac{\psi(\chi)\widehat{s}(\chi)}{\omega(s)}| \leq K_\chi\widetilde{\omega}(\alpha)$. Since this is true for all $s \in S$, we have $\sup_{s\in S} |\frac{\psi(\chi)\widehat{s}(\chi)}{\omega(s)}| \leq K_\chi\widetilde{\omega}(\alpha)$. Thus $|\psi(\chi)K_\chi| \leq K_\chi\widetilde{\omega}(\alpha)$. Since this holds for all $\chi \in \widehat{(S,\omega)}$, $\|\psi\|_\infty \leq \widetilde{\omega}(\alpha)$. The function ψ so obtained is unique. If $\eta : \widehat{(S,\omega)} \to \mathbb{C}$ is a bounded function satisfying $\eta\widehat{s}(\chi) = \widehat{\alpha s}(\chi)$ for all $\chi \in \widehat{(S,\omega)}$, then $\psi(\chi)\widehat{s}(\chi) =$

$\eta(\chi)\widehat{s}(\chi)$ for all $\chi \in \widehat{(S,\omega)}$. Since for each χ there exists $s \in S$ such that $\widehat{s}(\chi) \neq 0$, $\eta = \psi$. □

As discussed by Helgason in [5] (formally put in [9, Corollary 1.2.1]), if $\mathcal{A}$ is a semisimple commutative Banach algebra, and if $\psi : \Delta(\mathcal{A}) \to \mathbb{C}$ is a bounded continuous function such that $\psi\widehat{a} \in \widehat{\mathcal{A}}$ for all $a \in \mathcal{A}$, then the relation $\widehat{Ta} = \psi\widehat{a}$ $(a \in \mathcal{A})$ defines a multiplier on $\mathcal{A}$. If ω is a uniform weight, then we have the following analogous result.

***Theorem* 2.3** — *Let S be a separating abelian semigroup and ω be a uniform weight on S. If $\psi : \widehat{(S,\omega)} \to \mathbb{C}$ is a bounded function with the property that given $s \in S$ there exists $t \in S$ such that $\psi(\chi)\widehat{s}(\chi) = \widehat{t}(\chi)$ for all $\chi \in \widehat{(S,\omega)}$, then there exists unique $\alpha \in M_\omega(S)$ such that $\psi\widehat{s} = \widehat{\alpha s}$ for all $s \in S$.*

PROOF : Here $\psi : \widehat{(S,\omega)} \to \mathbb{C}$ is a bounded function such that given $s \in S$ there is $t \in S$ such that $\psi(\chi)\widehat{s}(\chi) = \widehat{t}(\chi)$ for all $\chi \in \widehat{(S,\omega)}$. We define $\alpha : S \to S$ as $\alpha(s) = t$ if $\psi(\chi)\widehat{s}(\chi) = \widehat{t}(\chi)$ for all $\chi \in \widehat{(S,\omega)}$. The map α is well defined since if $\alpha(s) = t_1$ and $\alpha(s) = t_2$, then $\widehat{t_1}(\chi) = \widehat{t_2}(\chi)$ for all $\chi \in \widehat{(S,\omega)}$. Therefore $\varphi(\delta_{t_1}) = \varphi(\delta_{t_2})$ for all $\varphi \in \Delta(\ell^1(S,\omega))$. Since S is separating and ω is a uniform weight, $\ell^1(S,\omega)$ is semisimple, and hence $t_1 = t_2$. Let $\alpha(s) = t$ and $\alpha(u) = v$. Then $\widehat{\alpha(su)}(\chi) = \psi(\chi)\widehat{(su)}(\chi) = \psi(\chi)\widehat{s}(\chi)\widehat{u}(\chi) = \widehat{t}(\chi)\widehat{u}(\chi) = \widehat{\alpha(s)u}(\chi)$ for all $\chi \in \widehat{(S,\omega)}$. Thus $\alpha(su) = \alpha(s)u$, and hence α is a multiplier on S. Also since $\widehat{\alpha s}(\chi) = (\psi\widehat{s})(\chi)$ for all $\chi \in \widehat{(S,\omega)}$, $|\varphi(\delta_{\alpha s})| \leq \|\psi\|_\infty |\varphi(\delta_s)|$ for all $\varphi \in \Delta(\ell^1(S,\omega))$. Thus $r(\delta_{\alpha s}) \leq \|\psi\|_\infty r(\delta_s)$. Since ω is a uniform weight, for every $s \in S$, $r(\delta_s) = \lim_{n\to\infty} \|(\delta_s)^n\|_\omega^{1/n} = \lim_{n\to\infty} \omega(s^n)^{1/n} = \lim_{n\to\infty} \omega(s^{2^n})^{1/2^n} = \omega(s)$. Thus $\omega(\alpha s) \leq \|\psi\|_\infty \omega(s)$ for all $s \in S$. Therefore $\alpha \in M_\omega(S)$. Uniqueness of the map α follows from the semisimplicity of $\ell^1(S,\omega)$. □

3. THE α-MULTIPLIER SEMIGROUP

Let $\mathcal{A}$ be a semisimple commutative Banach algebra, and let $T \in M(\mathcal{A})$. Then the multiplication $a \circ b = aTb$ $(a, b \in \mathcal{A})$ makes $\mathcal{A}$ an algebra. This algebra is denoted by $\mathcal{A}_T$. Some of the properties of the algebra $\mathcal{A}_T$ are investigated in [3]. Analogously, we form a semigroup S_α by using a multiplier on a semigroup S.

Let (S,ω) be a weighted abelian semigroup. Let $\alpha \in M_\omega(S)$. For $s, t \in S$ define $s \circ t = s\alpha t$. The set S with this operation forms an abelian semigroup S_α, which we shall call the *α-multiplier semigroup* of S. The weight ω on S gives a natural weight ω_α on S_α defined as $\omega_\alpha(s) = \widetilde{\omega}(\alpha)\omega(s)$, where $\widetilde{\omega}(\alpha) = \sup\{\frac{\omega(\alpha(s))}{\omega(s)} : s \in S\}$. Indeed, if $s, t \in S$, then $\omega_\alpha(s \circ t) = \widetilde{\omega}(\alpha)\omega(s \circ t) = \widetilde{\omega}(\alpha)\omega(s\alpha t) \leq \widetilde{\omega}(\alpha)\omega(s)\widetilde{\omega}(\alpha)\omega(t) = \omega_\alpha(s)\omega_\alpha(t)$. For a semigroup S, a subset X of S is a *set of relative units* [6] if for each $s \in S$ there exists $x \in X$ such that $xs = sx = s$.

***Theorem* 3.1** — *Let (S,ω) be a weighted semigroup, and let $\alpha \in M_\omega(S)$. Then the following statements hold.*

(i) If S_α has an identity (respectively a set of relative units), then S has an identity (respectively a set of relative units).

(ii) If S_α is separating, then S is separating.

PROOF : (i) Given any $s, e \in S$, $s \circ e = e \circ s = s$ implies $s\alpha(e) = \alpha(e)s = s$. Take $e' = \alpha(e) \in S$. Then $se' = e's = s$, hence the claim.

(ii) Let S_α be separating, and let $s, t \in S$ be such that $ss = tt = st$. Then $\alpha(ss) = \alpha(st) = \alpha(tt)$, which gives $s\alpha(s) = t\alpha(t) = s\alpha(t)$ and hence $s = t$. □

The converse of Theorem 3.1(i) is not true. Consider the semigroup $S = \mathbb{N} \cup \{0\}$ with usual addition operation and take any weight ω on $\mathbb{N} \cup \{0\}$. Take $\alpha = L_3$. Then S_α has no identity even though S has 0 as identity. By [6, Theorem 3.3, Theorem 3.5], a commutative Beurling algebra $\ell^1(S,\omega)$ has an identity if and only if semigroup S contains a finite set of relative units. This along with Theorem 3.1(i) shows that Beurling algebra $\ell^1(S,\omega)$ has identity if $\ell^1(S_\alpha,\omega_\alpha)$ has identity. However the converse is not true.

The converse of Theorem 3.1(ii) is not true. For instance, consider semigroup $\mathbb{N}$ with operation defined as $mn = \max\{m,n\}$ and weight $\omega(n) = 1$ for all $n \in \mathbb{N}$. Then $(\mathbb{N},\omega)$ is separating. However taking $\alpha = L_5$, we get $2 \circ 2 = 2 \circ 3 = 3 \circ 3 = 5$, showing $\mathbb{N}_\alpha$ is not separating. Thus Theorem 3.1(ii) along with [4, Proposition 4.8] shows that if $\ell^1(S_\alpha,\omega_\alpha)$ is semisimple, then $\ell^1(S,\omega)$ is semisimple, but the converse is not true.

Theorem 3.2 and Theorem 3.5, which follow, contain results analogous to those in [3, Theorem 5] for commutative Banach algebras.

***Theorem* 3.2** — *Let (S,ω) be a weighted abelian semigroup. Let $\alpha \in M_\omega(S)$ be such that if $s \in S$, $\chi \in \widehat{(S,\omega)}$ and $\chi(s) \neq 0$, then $\chi(\alpha s) \neq 0$. Let (S_α,ω_α) be the weighted α-multiplier semigroup of S corresponding to the multiplier α. Let $\psi : \widehat{(S,\omega)} \to \mathbb{C}$ be a map satisfying $\widehat{\alpha s} = \psi\widehat{s}$ for all $s \in S$. Then $\widehat{(S_\alpha,\omega_\alpha)} = \{\psi(\chi)\chi : \chi \in \widehat{(S,\omega)}\}$.*

PROOF : Let $\chi \in \widehat{(S,\omega)}$. As shown in Theorem 2.2, corresponding to α there exists a bounded function $\psi : \widehat{(S,\omega)} \to \mathbb{C}$ such that $\psi(\chi) = \frac{\chi(\alpha s)}{\chi(s)}$. Also, the condition that $\chi(\alpha(s))$ is nonzero whenever $\chi(s)$ is nonzero implies that $\psi(\chi) \neq 0$ for all $\chi \in \widehat{(S,\omega)}$. We define $\overline{\chi} : S_\alpha \to \mathbb{C}$ as $\overline{\chi}(s) = \psi(\chi)\chi(s)$. Then $\overline{\chi}(s \circ t) = \psi(\chi)\chi(s\alpha t) = \psi(\chi)\psi(\chi)\chi(s)\chi(t) = \overline{\chi}(s)\overline{\chi}(t)$. Thus $\overline{\chi}$ is a generalized semicharacter on S_α. Moreover, if $|\chi(s)| \neq 0$, then $|\overline{\chi}(s)| = |\psi(\chi)||\chi(s)| = |\frac{\chi(\alpha s)}{\chi(s)}||\chi(s)| =$

$|\chi(\alpha s)| \leq \omega(\alpha s) \leq \widetilde{\omega}(\alpha)\omega(s) = \omega_\alpha(s)$, while if $\chi(s) = 0$, then $|\overline{\chi}(s)| = |\psi(\chi)||\chi(s)| = 0 \leq \omega_\alpha(s)$. Thus $\overline{\chi} \in \widehat{(S_\alpha, \omega_\alpha)}$.

Conversely let $\overline{\chi} \in \widehat{(S_\alpha, \omega_\alpha)}$. Since $\alpha(s \circ t) = \alpha(s\alpha t) = \alpha(s)\alpha(t) = \alpha(s) \circ t$ and $\omega_\alpha(\alpha s) = \widetilde{\omega}(\alpha)\omega(s) \leq \widetilde{\omega}(\alpha)\widetilde{\omega}(\alpha)\omega(s) = \widetilde{\omega}(\alpha)\omega_\alpha(s)$, $\alpha \in M_{\omega_\alpha}(S_\alpha)$. Hence, there exists a bounded function $\overline{\psi} : \widehat{(S_\alpha, \omega_\alpha)} \to \mathbb{C}$ such that $\widehat{\alpha s} = \overline{\psi}\widehat{s} \quad (s \in S)$. If $s \in S$, then $\overline{\chi}(s)\overline{\chi}(s) = \overline{\chi}(s \circ s) = \overline{\chi}(s\alpha s) = \overline{\chi}(\alpha(s^2)) = \overline{\psi}(\overline{\chi})\overline{\chi}(s^2)$. Since for each $\overline{\chi} \in \widehat{(S_\alpha, \omega_\alpha)}$, there exists $s \in S$ such that $\overline{\chi}(s) \neq 0$, $\overline{\psi}(\overline{\chi}) \neq 0$ for all $\overline{\chi} \in \widehat{(S_\alpha, \omega_\alpha)}$. We define $\chi : S \to \mathbb{C}$ as $\chi(s) = \frac{1}{\overline{\psi}(\overline{\chi})}\overline{\chi}(s)$. If $s, t \in S$, then $\chi(st) = \frac{1}{\overline{\psi}(\overline{\chi})}\overline{\chi}(st) = \frac{1}{\overline{\psi}(\overline{\chi})}\overline{\chi}(s)\frac{1}{\overline{\psi}(\overline{\chi})}\overline{\chi}(t) = \chi(s)\chi(t)$. Also, $|\chi(s)| = \frac{1}{|\overline{\psi}(\overline{\chi})|}|\overline{\chi}|(s) \leq \frac{\omega_\alpha(s)}{|\overline{\psi}(\overline{\chi})|} = \frac{\widetilde{\omega}(\alpha)\omega(s)}{|\overline{\psi}(\overline{\chi})|}$. Since this is true for all $s \in S$, we have $|\chi(s^n)| = |\chi(s)|^n = \frac{\widetilde{\omega}(\alpha)}{|\overline{\psi}(\overline{\chi})|}\omega(s^n) \leq \frac{\widetilde{\omega}(\alpha)}{|\overline{\psi}(\overline{\chi})|}\omega(s)^n$ for all $n \in \mathbb{N}$. This gives $\lim_{n\to\infty} |\chi(s^n)|^{1/n} = \lim_{n\to\infty} |\chi(s)|^{n/n} \leq \lim_{n\to\infty} \left(\frac{\widetilde{\omega}(\alpha)}{|\psi_\alpha(\overline{\chi})|}\right)^{1/n} \omega(s)^{n/n}$. Hence, $|\chi(s)| \leq \omega(s)$ and thus $\chi \in \widehat{(S, \omega)}$. Let $s \in S$. Then $\overline{\psi}(\overline{\chi})^2\chi(s)^2 = \overline{\chi}(s)^2 = \overline{\chi}(s \circ s) = \overline{\psi}(\overline{\chi})\chi(s\alpha s) = \overline{\psi}(\overline{\chi})\chi(s)\psi(\chi)\chi(s) = \overline{\psi}(\overline{\chi})\psi(\chi)\chi(s)^2$. Since $\overline{\psi}(\overline{\chi}) \neq 0$ for all $\overline{\chi} \in \widehat{(S_\alpha, \omega_\alpha)}$ and for each χ there exists $s \in S$ such that $\chi(s) \neq 0$, we get $\overline{\psi}(\overline{\chi}) = \psi(\chi)$ for all $\overline{\chi} \in \widehat{(S_\alpha, \omega_\alpha)}$. □

The following example illustrates above theorem.

Example 3.3 : Consider $\mathbb{N}$ with addition. Then $M(\mathbb{N}) = \{\alpha_m : m \in \mathbb{N} \cup \{0\}\}$, where $\alpha_m(n) = m + n \quad (n \in \mathbb{N})$. Note that for any $\varphi \in \widehat{\mathbb{N}}, \varphi(n) = \varphi(1)^n$. Let $\alpha \in M(\mathbb{N})$. We will find all the generalized semicharacters on $\mathbb{N}_\alpha$. Since α is a multiplier on $\mathbb{N}$, $\alpha(n) = n + m$ for some fixed $m \in \mathbb{N} \cup \{0\}$. If $\phi \in \widehat{\mathbb{N}_\alpha}$, for any $n > 1$, $\phi(n)^2 = \phi(n \circ n) = \phi((n-1) \circ (n+1)) = \phi(n-1)\phi(n+1)$. Thus $\phi(n) \neq 0$ for all $n \in \mathbb{N}$. Hence, $\frac{\phi(n+1)}{\phi(n)} = \frac{\phi(n)}{\phi(n-1)} = k$ for some $k \in \mathbb{C} \setminus \{0\}$, which gives $\phi(n) = k\phi(n-1)$ for all $n \in \mathbb{N}$. Proceeding in this manner we get $\phi(n) = k^{n-1}\phi(1)$. To find the value of k we note that $\phi(2 \circ 1) = k\phi(1 \circ 1) = k\phi(1)^2$. Also, $\phi(2 \circ 1) = \phi(3 + m) = k^{3+m-1}\phi(1)$. Using above equations we get $\phi(1) = k^{m+1}$,i.e., $k = \phi(1)^{\frac{1}{m+1}}$. Thus we have $\phi(n) = k^{n-1}\phi(1) = \phi(1)^{\frac{n-1}{m+1}}\phi(1)$. Taking $\phi(1)^{\frac{1}{m+1}} = z$, we have $\phi(n) = z^{n+m}$.

Note that the condition $\chi(\alpha s) \neq 0$ whenever $\chi(s) \neq 0$ is necessary in Theorem 3.2. Indeed, if each $\overline{\chi} \in \widehat{(S_\alpha, \omega_\alpha)}$ is of the form $M_\chi\chi$ for some $\chi \in \widehat{(S, \omega)}$ and some constant $M_\chi \in \mathbb{C} \setminus \{0\}$, then $\overline{\chi}(s \circ t) = M_\chi\chi(s \circ t)$ for all $s, t \in S$. This gives $\overline{\chi}(s)\overline{\chi}(t) = M_\chi\chi(s\alpha t)$ for all $s, t \in S$. Thus $M_\chi\chi(s)M_\chi\chi(t) = M_\chi\chi(\alpha s)\chi(t)$ for all $s, t \in S$. This in turn shows that $M_\chi\chi(s)\chi(t) = \chi(\alpha s)\chi(t)$ for all $s, t \in S$. Taking t such that $\chi(t) \neq 0$ we get $M_\chi\chi(s) = \chi(\alpha s)$ for all $s \in S$. Since $M_\chi \neq 0$, whenever $\chi(s) \neq 0$, $\chi(\alpha s) \neq 0$. The following example illustrates the necessity of this condition on α.

Example 3.4 : Consider $\mathbb{N}$ with operation $nm = \min\{n, m\}$ for all $n, m \in \mathbb{N}$. First we find $\widehat{\mathbb{N}}$. Let $\varphi \in \widehat{\mathbb{N}}$. If $n \in \mathbb{N}$, then $\varphi(n) = \varphi(nn) = \varphi(n)^2$. Thus $\varphi(1) = 0$ or 1. If $\varphi(1) = 1$, then

$1 = \varphi(1) = \varphi(1n) = \varphi(n)\varphi(1) = \varphi(n)$ for all $n \in \mathbb{N}$. We denote this φ by φ_1. If $\varphi(1) = 0$, then there is an integer n_0 such that $\varphi(n) = 0$ if $n < n_0$ and $\varphi(n) = 1$ if $n \geq n_0$. We denote such φ by φ_{n_0}. Thus if $\varphi \in \widehat{\mathbb{N}}$, then $\varphi = \varphi_{n_0}$ for some $n_0 \in \mathbb{N}$. Also given any $n_0 \in \mathbb{N}$ if we define $\varphi_{n_0} : \mathbb{N} \to \mathbb{C}$ as $\varphi_{n_0}(n) = 0$ if $n < n_0$, and $\varphi_{n_0}(n) = 1$ if $n \geq n_0$, then $\varphi_{n_0} \in \widehat{\mathbb{N}}$. Thus $\widehat{\mathbb{N}} = \{\varphi_{n_0} : n \in \mathbb{N}\}$.

Consider the multiplier $\alpha = L_5$ on $\mathbb{N}$, where $L_5(n) = 5n = \min\{5, n\}$ $(n \in \mathbb{N})$. Note that $\varphi_6(10) = 1$ but $\varphi_6(\alpha(10)) = \varphi_6(L_5(10)) = \varphi_6(5) = 0$. Let $m, n \in \mathbb{N}$. Then

$$n \circ m = \min\{n, \min\{m, 5\}\}.$$

Therefore $n \circ m = \min\{n, 5\}$ if $m \geq 5$ and $nom = \min\{n, m\}$ if $m < 5$.

If $\chi \in \widehat{\mathbb{N}_\alpha}$, then $\chi(n)^2 = \chi(n \circ n) = \chi(n)$ if $n < 5$ and $\chi(n)^2 = \chi(5)$ if $n \geq 5$. Thus if $n \geq 5$, then $\chi(n) = \chi(5)^{\frac{1}{2}}$. Also from the above equation we have $\chi(1) = 1$ or 0. If $\chi(1) = 1$, then $1 = \chi(1) = \chi(1 \circ n) = \chi(n)\chi(1) = \chi(n)$ for all $n \in \mathbb{N}$. If $\chi(1) = 0$, then choose smallest n_0 such that $\chi(n_0) = 1$. Then $\chi(n) = 0$ if $n < n_0$ and $\chi(n) = 1$ if $n \geq n_0$. Note that $n_0 \leq 5$ because if $\chi(5) = 0$ then $\chi(n) = \chi(5)^{\frac{1}{2}} = 0$ for all $n > 5$. Thus there does not exist any $K \in \mathbb{C} \setminus \{0\}$ such that $K\varphi_{10} \in \widehat{\mathbb{N}_\alpha}$.

A semigroup S is said to *admit factorization* if for each $s \in S$ there exists $u, v \in S$ such that $s = uv$. Note that a semigroup with identity always admits factorization.

***Theorem* 3.5** — *Let S be a separating abelian semigroup which admits factorization, and let ω be a uniform weight on S. Let S_0 be a semigroup obtained from the set S with some operation, and ω_0 be a weight on S_0. Let $\psi : \widehat{(S,\omega)} \to \mathbb{C} \setminus \{0\}$ be a bounded function such that $\widehat{(S_0,\omega_0)} = \{\psi(\chi)\chi : \chi \in \widehat{(S,\omega)}\}$. Then $S_0 = S_\alpha$ for some $\alpha \in M_\omega(S)$ for which $\chi(\alpha s) \neq 0$ whenever $\chi(s) \neq 0$ for all $s \in S$ and $\chi \in \widehat{(S,\omega)}$.*

PROOF : Let $s \in S$. Since S admits factorization, there are $u, v \in S$ such that $s = uv$. Let $\chi \in \widehat{(S,\omega)}$. Then by assumption $\overline{\chi} = \psi(\chi)\chi \in \widehat{(S_0,\omega_0)}$. This gives $\psi(\chi)\widehat{s}(\chi) = \psi(\chi)\widehat{u}(\chi)\widehat{v}(\chi) = \frac{1}{\psi(\chi)}\widehat{u}(\overline{\chi})\widehat{v}(\overline{\chi}) = \frac{1}{\psi(\chi)}\widehat{u \circ v}(\overline{\chi}) = \widehat{u \circ v}(\chi)$. Define $\alpha : S \to S$ as $\alpha(s) = \alpha(uv) = u \circ v$. Then by Theorem 2.3, α is ω-bounded multiplier on S. Clearly $s \circ t = \alpha(st) = s\alpha(t)$ for all $s, t \in S$. Thus $S_0 = S_\alpha$. □

The following theorem is analogous to Corollary 2 of [3, Theorem 5].

***Theorem* 3.6** — *Let (S,ω) be a separating weighted abelian semigroup with semisimple weight. Let $\alpha \in M_\omega(S)$ and $(S_\alpha, \omega_\alpha)$ be the weighted α-multiplier semigroup corresponding to the multiplier α. Then $M_\omega(S) = M_{\omega_\alpha}(S_\alpha)$.*

PROOF : Let $\beta \in M_\omega(S)$. If $s, t \in S$, then $\beta(s\alpha t) = \beta(s)\alpha(t)$ and $\omega_\alpha(\beta s) = \widetilde{\omega}(\alpha)\omega(\beta s)$ $\leq \widetilde{\omega}(\alpha)\widetilde{\omega}(\beta)\omega(s) = \widetilde{\omega}(\beta)\omega_\alpha(s)$. Therefore $\beta \in M_{\omega_\alpha}(S_\alpha)$.

Conversely, let $\gamma \in M_{\omega_\alpha}(S_\alpha)$. Then by Theorem 2.2, there exists a bounded function $\overline{\psi} : \widehat{(S_\alpha, \omega_\alpha)} \to \mathbb{C}$ such that $\widehat{\gamma s}(\overline{\chi}) = \overline{\psi}(\overline{\chi})\widehat{s}(\overline{\chi})$ for all $s \in S, \overline{\chi} \in \widehat{(S_\alpha, \omega_\alpha)}$. Given $\overline{\chi} \in \widehat{(S_\alpha, \omega_\alpha)}$, as shown in converse part in the proof of Theorem 3.2, there exist $\chi \in \widehat{(S, \omega)}$ and constant $M_\chi \in \mathbb{C} \setminus \{0\}$ such that $\overline{\chi} = M_\chi \chi$. This gives $\dfrac{\overline{\chi}(\gamma s)}{\overline{\chi}(s)} = \dfrac{M_\chi \chi(\gamma s)}{M_\chi \chi(s)} = \dfrac{\chi(\gamma s)}{\chi(s)}$ for all s for which $\chi(s) \neq 0$. Define map $\psi : \widehat{(S, \omega)} \to \mathbb{C}$ as $\psi(\chi) = \overline{\psi}(\overline{\chi})$. Then ψ is a bounded function and $\widehat{\gamma s}(\chi) = \psi(\chi)\widehat{s}(\chi)$ for all $\chi \in \widehat{(S, \omega)}$ and $s \in S$, which defines a multiplier on S. Moreover, since $\gamma \in M_{\omega_\alpha}(S_\alpha)$, $\sup_{s \in S} \dfrac{\omega_\alpha(\gamma s)}{\omega_\alpha(s)} < \infty$. Thus $\sup_{s \in S} \dfrac{\omega(\gamma s)}{\omega(s)} = \sup\limits_{s \in S} \dfrac{\widetilde{\omega}(\alpha)\omega(\gamma s)}{\widetilde{\omega}(\alpha)\omega(s)} = \sup\limits_{s \in S} \dfrac{\omega_\alpha(\gamma s)}{\omega_\alpha(s)} < \infty$, and hence $\gamma \in M_\omega(S)$. □

The following example illustrates the theorem.

Example 3.7 : Let S be the semigroup $\mathbb{N}$ with usual addition. Let weight ω be the constant function 1 on S. The ω-bounded multiplier on S are of the form α_m for some $m \in \mathbb{N} \cup \{0\}$, where $\alpha_m(n) = n + m$. Now let $\gamma \in M(S_{\alpha_m})$, then $\gamma(j \circ k) = \gamma(j) \circ k = j \circ \gamma(k)$ for all $j, k \in \mathbb{N}$. This gives $\gamma(j) + m + k = j + m + \gamma(k)$ for all $j, k \in \mathbb{N}$. Thus $\gamma(j) - j = \gamma(k) - k$ for all $j, k \in \mathbb{N}$. This gives $\gamma(k) = k + l$ for some $l \in \mathbb{N} \cup \{0\}$. Since weight $\widetilde{\omega}$ on S_{α_m} is constant function 1, $M(S_{\alpha_m}) = M_{\widetilde{\omega}}(S_{\alpha_m})$ and hence $M_\omega(S) = M_{\widetilde{\omega}}(S_{\alpha_m})$.

As shown in Corollary 1 of [3, Theorem 5], if $\mathcal{A}$ is a semisimple commutative Banach algebra admitting factorization, $T \in M(\mathcal{A})$, then $\mathcal{A}_T$ admits factorization if and only if T is invertible. The following gives semigroup analogue of this result.

***Theorem* 3.8** — *Let S be a separating abelian semigroup admitting factorization, and let ω be a uniform weight on S. Let $\alpha \in M_\omega(S)$. Then α is invertible if and only if S_α admits factorization.*

PROOF : Suppose that S_α admits factorization. By Theorem 3.5, there exists $\gamma \in M_{\omega_\alpha}(S_\alpha) = M_\omega(S)$ such that $st = s \circ \gamma t$ for all $s, t \in S$. Let $u \in S$. Since S and S_α both admit factorization, there exists $s, t, v, w \in S$ such that $u = st$ and $u = v \circ w$. Thus $u = st = s \circ \gamma t = \gamma(s \circ t) = \gamma(s\alpha t) = \gamma\alpha(st) = \gamma\alpha(u)$. Also, $u = v \circ w = v\alpha w = \alpha(vw) = \alpha(v \circ \gamma w) = \alpha\gamma(u)$. Hence, α is invertible.

Conversely, let α be invertible. Take $u = st \in S$, $u = \alpha^{-1}\alpha(st) = \alpha^{-1}(s(\alpha t)) = (\alpha^{-1}s) \circ t$ as $\alpha^{-1} \in M(S)$. Thus S_α admits factorization. □

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