Indian J. Pure Appl. Math., **50**(1): 213-224, March 2019

DOI: 10.1007/s13226-019-0319-6

A FAMILY OF WAITING TIME DISTRIBUTIONS ARISING FROM A BIVARIATE BERNOULLI SCHEME

H. M. Barakat

Department of Mathematics, Faculty of Science, Zagazig University,
Zagazig, Egypt
e-mail: hbarakat2@hotmail.com

(*Received* 11 *October* 2016; *after final revision* 27 *February* 2018;
accepted 24 *April* 2018)

A new univariate five-parameter generalized negative binomial distribution based on the bivariate Bernoulli scheme is introduced. This distribution produces a new three-parameter generalized geometric distribution in a natural manner using probabilistic properties of the four-outcome model. Some basic statistical properties of the new distribution are studied. In addition, estimation of the unknown parameters is illustrated. Moreover, a new univariate three-parameter generalized exponential distribution is derived as a limit of the proposed three-parameter generalized geometric distribution. Finally, a generalization of the proposed distributions based on trivariate Bernoulli scheme is introduced.

Key words : Bernoulli scheme; bivariate binomial distribution; exponential distribution; geometric distribution; lifetime distributions.

1. Introduction

In the literature the sequence of trials in sampling with replacement has been referred to as Bernoulli trials. Bivariate extension of the Bernoulli scheme is defined by considering two characteristics A_1 and A_2 (e.g., hair color and eye color) that may be studied simultaneously on each individual under the following three conditions:

1. Each trial has four possible outcomes $A_{11}, A_{10}; A_{21}, A_{20}$, e.g., A_{11} is the event that the individual has dark hair, labeled success, or 1, and A_{10} is the event that the individual has not dark hair, labeled failure, or 0. On the other hand, A_{21} is the event that the individual has dark eyes, labeled success, or 1, and A_{20} is the event that the individual has not dark eyes, labeled failure, or 0.

2. The probabilities $p_{00} = P(A_{10}A_{20})$, $p_{11} = P(A_{11}A_{21})$, $p_{01} = P(A_{10}A_{21})$ and $p_{10} = P(A_{11}A_{20})$, where $p_{00} + p_{01} + p_{10} + p_{11} = 1$, remain constant over the trials.

3. The trials are independent.

Under the aforesaid conditions we call (A_1, A_2) the bivariate Bernoulli scheme and $(A_{11}, A_{10}; A_{21}, A_{20})$ the fourfold Bernoulli model. The bivariate Bernoulli model enables us to obtain several interesting trivariate and bivariate distributions in natural manner such as the trivariate binomial distribution, bivariate binomial, hypergeometric, negative binomial and geometric distributions (see, Marshal and Olkin [14], Kocherlakota and Kocherlakota [13], and Bairamov and Gultekin [1]).

A simple trivariate distribution in the described fourfold model is defined by sampling n times. Furthermore, let X_1, X_2 and X_{12} be the number of occurrences of A_{11}, A_{21} and $A_{11}A_{21}$, respectively. The joint probability density function (jpdf) of the random variables (rv's) X_1, X_2 and X_{12} is given by (cf. Bairamov and Gultekin [1])

$$B(k,s,r|n;p_{10},p_{01},p_{11}) = P(X_1 = k, X_2 = s, X_{12} = r) = \frac{n!}{(k-r)!r!(s-r)!(n-k-s+r)!}$$

$$.p_{10}^{k-r}p_{11}^{r}p_{01}^{k-r}p_{00}^{n-k-s+r},\ k,s = 0,2,...,n; r = 0 \vee (k+s-n),...,k \wedge s, \tag{1.1}$$

where $a \vee b = \max(a,b)$ and $a \wedge b = \min(a,b)$. The corresponding moment generating function (mgf) is $M_{X_1,X_2,X_3}(t_1,t_2,t_3|p_{10},p_{01},p_{11}) = (p_{11}e^{t_1+t_2+t_3} + p_{10}e^{t_1} + p_{01}e^{t_2} + p_{00})^n$. Chandrasekar and Balakrishnan [7] considered the trivariate binomial distribution (1.1) and obtained regression equations of this distribution. Moreover, they provided a set of necessary and sufficient conditions for the regression to be linear. It is clear that the univariate marginals of the random vector (X_1, X_2, X_{11}) are binomial, with success probabilities $p_1 = p_{10} + p_{11}$ (where $q_1 = 1 - p_1 = p_{01} + p_{00}$), $p_2 = p_{01} + p_{11}$ (where $q_2 = 1 - p_2 = 1 - p_{10} + p_{00}$) and p_{11}, respectively. Moreover, the joint distribution of (X_1, X_2) is obviously the bivariate binomial distribution with pdf

$$B(k,s|n;p_{10},p_{01},p_{11}) = P(X_1 = k, X_2 = s)$$

$$= \sum_{r=0\vee(k+s-n)}^{k\wedge s} \frac{n!}{(k-r)!r!(s-r)!(n-k-s+r)!} p_{10}^{k-r}p_{11}^{r}p_{01}^{k-r}p_{00}^{n-k-s+r}. \tag{1.2}$$

The corresponding mgf is

$$M_{X_1,X_2}(t_1,t_2|p_{10},p_{01},p_{11}) = M_{X_1,X_2,X_3}(t_1,t_2,0|p_{10},p_{01},p_{11}) = (p_{11}e^{t_1+t_2} + p_{10}e^{t_1} + p_{01}e^{t_2} + p_{00})^n.$$

Therefore, the two rv's X_1 and X_2 are independent if and only if $p_{10} = p_1q_2, p_{11} = p_1p_2, p_{01} = q_1p_2$ and $p_{00} = q_1q_2$, since in this case, we have

$$M_{X_1,X_2}(t_1,t_2|p_{10},p_{01},p_{11}) = (p_1p_2e^{t_1+t_2}+p_1q_2e^{t_1}+q_1p_2e^{t_2}+q_1q_2)^n = (p_1e^{t_1}+q_1)^n(p_2e^{t_2}+q_2)^n.$$

This means that $B(k,s|n;p_{10},p_{01},p_{11}) = B(k|n,p_1)B(s|n,p_2), \forall k,s = 1,2,...,n$, where $B(.|n,p)$ is the binomial distribution function (df) with success probability p. An important application of the df (1.2) is the possibility of the expression of the joint df (jdf) of the bivariate $(n-k+1, n-s+1)$th order statistics $(X_{1,n-k+1:n}, X_{2,n-s+1})$ based on the bivariate df $F(x_1,x_2)$ by

$$F_{n-k+1,n-s+1:n}(x_1,x_2) = \sum_{i=1}^{k-1}\sum_{j=1}^{s-1} B(i,j|n;G_1(x_1)-G(x_1,x_2),G_2(x_2)-G(x_1,x_2),F(x_1,x_2)),$$

where $G(x_1,x_2)$ is the survivor function of $F(x_1,x_2)$, while $G_1(x_1)$ and $G_2(x_2)$ are the two marginals of $G(x_1,x_2)$. The corresponding mgf is

$$M_{X_{1,n-k+1:n},X_{2,n-s+1}}(t_1,t_2|G_1(x_1)-G(x_1,x_2),G_2(x_2)-G(x_1,x_2),F(x_1,x_2)).$$

Moreover, $X_{1,n-k+1:n}$ and $X_{2,n-s+1:n}$ are independent if and only if $F(x_1,x_2) = F_1(x_1)F_2(x_2)$, where $F_1(x_1)$ and $F_2(x_2)$ are the two marginals of the df $F(x_1,x_2)$ (clearly, the condition $F(x_1,x_2) = F_1(x_1)F_2(x_2)$ implies that $G(x_1,x_2) = G_1(x_1)G_2(x_2)$), see Barakat [2-4]. For some discussions on bivariate binomial distributions, see Johnson *et al.* [11].

In the next section, we use the df (1.2) to derive a new univariate five-parameter generalized negative binomial (5-GNB) distribution. Moreover, based on this df, we derive a new univariate three-parameter generalized geometric (3-GG) distribution. Some basic statistical properties of the new geometric distribution are studied. In addition, estimation of the unknown parameters is illustrated. In the third section, we derive a new univariate three-parameter generalized exponential (3-GE) distribution as a limit of the proposed 3-GG distribution. It is worth mentioning that the 3-GG distribution is not only derived in natural manner to be capable of modeling of count data arising from the bivariate Bernoulli scheme, but also it preserves the mathematical tractability and elegance. Moreover, the new univariate exponential distribution besides the proposed 3-GG distribution, as new lifetime distributions, can be used in survival and reliability studies. In Section 5, we introduce an univariate seven-parameter generalized geometric (7-GG) distribution, based on the trivariate Bernoulli scheme, and the corresponding generalized exponential df. Actually, there are many generalizations of geometric distribution in the literature, which more or less are obtained in an artificial manner. Examples of such generalizations are:

1. A generalization of geometric distribution, which is obtained by Tripathi *et al*. [16], by length-biasing some generalized versions of the log-series distribution.

2. A generalization of geometric distribution, which is obtained by Bhati and Joshi [5], by using a weight function and it can be viewed as a discrete analogue of weighted exponential distribution introduced by Gupta and Kundu [10].

3. A generalized inflated geometric distribution obtained by Joshi [12], which is a geometric df inflated at certain counts and it is used to analyze any inflated data at some particular counts.

4. A new three-parameter extension of the generalized geometric distribution of Géomez-Déniz [9] introduced by Bidram *et al*. [6] and it can be considered as discrete analogs of some continuous distributions belonging to the known Marshall-Olkin family (see, Marshall and Olkin [15]).

5. In the last decade many generalizations of geometric distribution have appeared in the literature using one of the techniques (cf. Bhati and Joshi [5]): Discretizing a continuous distribution, e.g., $(p(x) = S_X(x) - S_X(x+1))$, mixed Poisson technique $\left(\int_0^\infty \frac{e^{-\lambda}\lambda^x}{x!} f(\lambda) d\lambda\right)$, where $f(.)$ is density of continuous rv and the technique of constructing discrete analogue of a continuous distribution over integer values on real line.

2. A New Univariate 5-GNB and 3-GG DF's Based on the Bivariate Bernoulli Scheme

Consider the bivariate Bernoulli scheme defined in the preceding section. Furthermore, let the rv Z be the minimum number of required number of trials to get at least k and s of successes of A_{11} and A_{21}, respectively, i.e., the events A_{11} and A_{21} occur at least k and s times, respectively. Clearly, the last definition of Z is equivalent that the rv Z is the required number of trials in the bivariate Bernoulli scheme to get

[A] at least k occurrences of the event A_{11} and exactly s occurrences of the event A_{21}, or

[B] exactly k occurrences of the event A_{11} and at least s occurrences of the event A_{21}, or

[C] exactly k and s occurrences of the events A_{11} and A_{21}, respectively.

The df of the rv Z can be interpreted as a natural generalization of the usual negative binomial df based on the univariate Bernoulli scheme. Moreover, we shall show that this new proposed df yields a new 3-GG distribution. Let $P(Z = z) = NB(z|p_{10}, p_{01}, p_{11}; k, s),\ z = k \vee s, (k \vee s) + 1, \ldots$. In order to obtain $NB(z|p_{10},p_{01},p_{11};k,s),\ z=k\vee s,(k\vee s)+1,\dots$, we notice that at the $(z-1)$th trial we should have at least k occurrences of A_{11} and $s-1$ occurrences of A_{21}, or at least s occurrences of A_{21} and $k-1$ occurrences of A_{11}, or $k-1$ and $s-1$ occurrences of A_{11} and A_{21}, respectively. Clearly, the three events defined in $[A],[B]$ and $[C]$ are mutually exclusive. Therefore, adopting the convention that $\sum\limits_{i=a}^{b} C_i=0, b<a$, and relying on the df's (1.1) and (1.2), it is not difficult to get

$$NB(z|p_{10},p_{01},p_{11};k,s)=\begin{cases} p_{11}^k, & z=k=s,\\ p_2\sum\limits_{x=k}^{s-1} b(x|s-1;p_{11},p_{01})+p_{11}b(k-1|s-1;p_{11},p_{01}), & z=s;s>k,\\ p_1\sum\limits_{y=s}^{k-1} b(y|k-1;p_{11},p_{10})+p_{11}b(s-1|k-1;p_{11},p_{10}), & z=k;k>s,\\ p_2\sum\limits_{x=k}^{z-1} B(x,s-1|z-1;p_{10},p_{01},p_{11}) & \\ +p_1\sum\limits_{y=s}^{z-1} B(k-1,y|z-1;p_{10},p_{01},p_{11}) & \\ +p_{11}B(k-1,s-1,r|z-1;p_{10},p_{01},p_{11}), & z=(k\vee s)+1,\dots, \end{cases} \tag{2.1}$$

where $b(x|n;\alpha,\beta)=\frac{n}{x!(n-x)!}\alpha^x\beta^{n-x}, \forall 0\le\alpha,\beta\le 1$.

Remark 2.1 : The df (2.1) enables us to obtain an different expression (although it is complicated to some extent) for jdf of the bivariate (k,s)th order statistics $(X_{1,k:n},X_{2,s:n})$ based on the bivariate df $F(x_1,x_2)$ by putting $p_1=F_1(x_1), q_1=G_1(x_1)$, $p_2=F_2(x_2), q_2=G_2(x_2)$, $p_{11}=F(x_1,x_2)$ and $p_{00}=G(x_1,x_2)$, as

$$F_{k,s:n}(x_1,x_2)=\sum_{z=k\vee s}^{n} NB(z|F_1(x_1)-F(x_1,x_2),F_2(x_2)-F(x_1,x_2),F(x_1,x_2);k,s).$$

As some special cases:

1. $F_{n,n:n}(x_1,x_2)=F^n(x_1,x_2)$,

2. $F_{n,s:n}(x_1,x_2)=NB(n|F_1(x_1)-F(x_1,x_2),F_2(x_2)-F(x_1,x_2),F(x_1,x_2);n,s)$
$=F_1(x_1)\sum\limits_{y=s}^{n-1}\frac{n-1}{y!(n-1-y)!}F^y(x_1,x_2)(F_1(x_1)-F(x_1,x_2))^{n-1-y}$
$+\frac{n-1}{(s-1)!(n-s)!}F^s(x_1,x_2)(F_1(x_1)-F(x_1,x_2))^{n-s}, 1\le s\le n$,

3. $F_{k,n:n}(x_1,x_2) = NB(n|F_1(x_1) - F(x_1,x_2), F_2(x_2) - F(x_1,x_2), F(x_1,x_2); k, n)$
$= F_2(x_2)\sum_{x=k}^{n-1} \frac{n-1}{x!(n-1-x)!} F^x(x_1,x_2)(F_2(x_2) - F(x_1,x_2))^{n-1-x}$
$+\frac{n-1}{(k-1)!(n-k)!} F^k(x_1,x_2)(F_1(x_1) - F(x_1,x_2))^{n-k}, 1 \le k \le n.$

Clearly, in the case that $k = s$, the df (2.1) yields

$$NB(z|p_{10}, p_{01}, p_{11}; k) = NB(z|p_{10}, p_{01}, p_{11}; k)$$

$$= \begin{cases} p_{11}^k, & z = k, \\ p_2 \sum\limits_{x=k}^{z-1} B(x, k-1|z-1; p_{10}, p_{01}, p_{11}) & \\ +p_1 \sum\limits_{y=k}^{z-1} B(k-1, y, r|z-1; p_{10}, p_{01}, p_{11}) & \\ +p_{11} B(k-1, k-1|z-1; p_{10}, p_{01}, p_{11}), & z = k+1, k+2, \dots . \end{cases} \tag{2.2}$$

It is worth mentioning that the df (2.2) is a natural generalization of the usual univariate two-parameter negative binomial df $NB(z|p_{11}; k \parallel p_{01})$, when $p_2 = 1 - q_2 = 1$ (which implies $p_{10} = p_{00} = 0$ and $p_{11} = 1 - p_{01}$). Moreover, the df (2.2) is a natural generalization of the usual univariate two-parameter negative binomial df $NB(z|p_{11}; k \parallel p_{10})$, when $p_1 = 1 - q_1 = 1$ (which implies $p_{01} = p_{00} = 0$ and $p_{11} = 1 - p_{10}$), where $NB(z|p; k \parallel q) = \frac{(z-1)!}{(k-1)!(z-k)!} q^k p^{z-k}, p + q = 1, z = k, k+1, \dots$.

Clearly, if we put $k = 1$ in (2.2), we get a new 3-GG df, which is the df of the rv W, where W is the minimum required trials in the bivariate Bernoulli scheme to get at least one occurrence of each of the events A_{11} and A_{21}. Therefore, the rv W is the required number of trials in the bivariate Bernoulli scheme to get at least one occurrence of the event A_{11} and exactly an occurrence of the event A_{21}, or exactly an occurrence of the event A_{11} and at least one occurrence of the event A_{21}, or exactly one occurrence of each of the events A_{11} and A_{21}.

$$g_W(w|p_{10}, p_{01}, p_{11}) = P(W = w) = \begin{cases} p_{11}, & w = 1, \\ q_1^{w-1} p_1 + q_2^{w-1} p_2 - p_{00}^{w-1} \overline{p}_{00}, & w = 2, 3, \dots, \end{cases} \tag{2.3}$$

where $\overline{p}_{00} = 1 - p_{00}$, $p_{10} \wedge p_{11} \neq 0$ and $p_{01} \wedge p_{11} \neq 0$ (clearly, if $p_{10} = p_{11} = 0$, or $p_{01} = p_{11} = 0$, then $g_W(w|p_{10}, p_{01}, p_{11}) = 0, \forall w$, e.g., if $p_{10} = p_{11} = 0$, then $p_{01} + p_{00} = 1, p_2 = p_{01} = 1 - q_2 = \overline{p}_{00}$ and consequently $g_W(w|p_{10}, p_{01}, p_{11}) = q_2^{w-1} p_2 - p_{00}^{w-1} \overline{p}_{00} = 0$).

The cumulative df $G_W(w|p_{10}, p_{01}, p_{11})$ and the hazard rate function $h_W(w)$ of the distribution

(2.3) have the following elegant forms:

$$G_W(w|p_{10},p_{01},p_{11}) = P(W \le w) = \begin{cases} 0, & w < 1, \\ p_{11}, & 1 \le w < 2, \\ 1 - q_1^{[w]} - q_2^{[w]} + p_{00}^{[w]}, & w \ge 2 \end{cases}$$

and

$$h_W(w) = \frac{g_W(w|p_{10},p_{01},p_{11})}{1 - G_W(w|p_{10},p_{01},p_{11})} = \begin{cases} 0, & w < 1, \\ \frac{p_{11}}{1-p_{11}}, & w = 1, \\ \frac{p_1^{w-1}q_1 + p_2^{w-1}q_2 - p_{00}^{w-1}\overline{p}_{00}}{q_1^w + q_2^w - p_{00}^w}, & w = 2, 3, ..., \end{cases}$$

where $[w]$ denotes the integer part of w.

Special cases of the proposed 3-GG df

1. If $p_2 = 1, q_2 = 0$ (this implies that $p_{10} = p_{00} = 0$), then

$$g_1(w;p_{01}) = \begin{cases} p_{11}, & w = 1, \\ q_1^{w-1}p_1, & w = 2, 3, ... \end{cases} = p_{01}^{w-1}p_{11}, w = 1, 2, ...,$$

   which is the usual geometric df with success probability $p_{11} = 1 - p_{01}$.

2. If $p_1 = 1, q_1 = 0$ (this implies that $p_{01} = p_{00} = 0$), then

$$g_2(w;p_{10}) = \begin{cases} p_{11}, & w = 1, \\ q_2^{w-1}p_2, & w = 2, 3, ..., \end{cases} = p_{10}^{w-1}p_{11}, w = 1, 2, ...,$$

   which is the usual geometric df with success probability $p_{11} = 1 - p_{10}$.

3. If $p_{01} = p_{10} = 0$ (this implies that $p_{11} + p_{00} = 1$), then

$$g_3(w;p_{11}) = \begin{cases} p_{11}, & w = 1, \\ p_{00}^{w-1}p_{11}, & w = 2, 3, ..., \end{cases} = p_{00}^{w-1}p_{11}, w = 1, 2, ...,$$

   which is the usual geometric df with success probability $p_{11} = 1 - p_{00}$.

4. If $p_{00} = p_{11} = 0$ (this implies that $p_{10} + p_{01} = 1$), then

$$g_4(w;p_{10}) = (1 - p_{10})^{w-1}p_{10} + p_{10}^{w-1}(1 - p_{10}) = q^{w-1}p + p^{w-1}q, w = 2, 3, ...$$

The df $g_4(w;p_{10})$ has an interesting statistical meaning in the univariate Bernoulli scheme, with the success probability p (the probability of occurrence of 1) and the failure probability q (the probability of occurrence of 0) that it is the df of the required number of trials by which the sequence of ones or zeros will be broken for the first time. We note in passing that Feller [8] (Chapters VIII and XIII) obtains distributions of waiting times for several patterns of successes and failures arising in infinite sequences of Bernoulli trials.

Remark 2.2 : The df (2.3) enables us to obtain the jdf of the bivariate $(1,1)$th order statistics $(X_{1,1:n}, X_{2,1:n})$ based on the bivariate df $F(x_1,x_2)$, by putting $p_1 = F_1, q_1 = G_1, p_2 = F_2, q_2 = G_2$, $p_{11} = F$ and $p_{00} = G$, as

$$F_{1,1:n}(x_1,x_2) = F(x_1,x_2) + F_1(x_1)\sum_{w=2}^{n} G_1^{w-1}(x_1)$$

$$+F_2(x_2)\sum_{w=2}^{n} G_2^{w-1}(x_2) - \overline{G}(x_1,x_2)\sum_{w=2}^{n} G^{w-1}(x_1,x_2) = 1 - G_1^n(x_1) - G_2^n(x_2) + G^n(x_1,x_2),$$

where $\overline{G}(x_1,x_2) = 1 - G(x_1,x_2)$.

3. The mgf, Moments and the Estimation of the Parameters of the 3-GG df

The mgf of the df (2.3) is given by

$$\phi_W(t) = \mathrm{E}(e^{tW}) = p_{11}e^t + \sum_{w=2}^{\infty}(q_1^{w-2}p_1 + q_2^{w-2}p_2 - p_{00}^{w-1}\overline{p}_{00})$$

$$= p_{11}e^t + \frac{p_1q_1e^{2t}}{1-q_1e^t} + \frac{p_2q_2e^{2t}}{1-q_2e^t} - \frac{p_{00}\overline{p}_{00}e^{2t}}{1-p_{00}e^t},$$

provided that $t < (-\log q_1)\wedge(-\log q_2)\wedge(-\log p_{00})$. Therefore, the ℓth moment of the df (2.3) can be obtained by the formula

$$\mu_W^{(\ell)} = \mathrm{E}(W^\ell) = p_{11} + \Lambda_{q_1}^{(\ell)}(0) + \Lambda_{q_2}^{(\ell)}(0) - \Lambda_{p_{00}}^{(\ell)}(0), \tag{3.1}$$

where $\Lambda_q(t) = \frac{pqe^{2t}}{1-qe^t}, p = 1-q$, and $\Lambda_q^{(\ell)}(0) = \left.\frac{d^\ell\Lambda_q(t)}{dt^\ell}\right|_{t=0}$. For example the mean is given by following elegant formula

$$\mu_W = \mu_W^{(1)} = \frac{q_1}{p_1} + \frac{q_2}{p_2} - \frac{\overline{p}_{00}}{p_{00}}.$$

Estimators based on proportion of zeros and ones

Let $\hat{p}_{10}, \hat{p}_{01}$ and $\hat{p}_{11}$ be the known observed proportion of (1,0)'s (i.e., the proportion of occurrences of $A_{11}A_{20}$), (0,1)'s (i.e., the proportion of occurrences of $A_{10}A_{21}$) and (1,1)'s (i.e., the proportion of occurrences of $A_{11}A_{21}$), respectively, in the sample of size N from the bivariate Bernoulli scheme (A_1, A_2). Then, the estimators $\hat{p}_1 = \hat{p}_{10} + \hat{p}_{11}$, $\hat{p}_2 = \hat{p}_{01} + \hat{p}_{11}$ and $\hat{p}_{11}$ of p_1, p_2 and p_{11}, respectively, are unbiased and consistent. Moreover, the jdf of these estimators is described by (1.1), as $P(N\hat{p}_1 =$

$k, N\hat{p}_2 = s, N\hat{p}_{11} = r) = B(k, s, r|N; p_{10}, p_{01}, p_{11}), k, s = 0, ..., N, r = 0 \vee (k+s-N), ..., k \wedge s$. Furthermore, the jdf of the estimators $\hat{p}_1$ and $\hat{p}_2$ is described by (1.2), as $P(N\hat{p}_1 = k, N\hat{p}_2 = s) = B(k, s|N; p_{10}, p_{01}), k, s = 0, 1, ..., N$. Finally, the estimators $N\hat{p}_1, N\hat{p}_2$ and $N\hat{p}_{11}$ are distributed as the binomial df with parameters $(N, p_1), (N, p_2)$ and (N, p_{11}), respectively.

4. A NEW UNIVARIATE 3-GE DF, BASED ON THE 3-GG DF

As we have seen before the 3-GG distribution has the interpretation that it is the df of the minimum number of required trials in the bivariate Bernoulli scheme to get at least one occurrence (success) of each of the events A_{11} and A_{21}. Consider a regime when the probabilities of successes are very small, such that $p_{10} = \frac{\alpha_{10}}{n}, p_{01} = \frac{\alpha_{01}}{n}, p_{11} = \frac{\alpha_{11}}{n}$ and consider $w = xn$. Then, for all $x < 0$, we get $G_W(nx|p_{10}, p_{01}, p_{11}) = 0, \forall n$, and for all $x \geq 0$, we get

$$G_W(nx|p_{10}, p_{01}, p_{11}) = \frac{\alpha_{11}}{n} + (1 - \frac{\alpha_{10}+\alpha_{11}}{n}) \left[1 - (1 - \frac{\alpha_{10}+\alpha_{11}}{n})^{[nx]-1}\right]$$
$$+(1 - \frac{\alpha_{01}+\alpha_{11}}{n}) \left[1 - (1 - \frac{\alpha_{01}+\alpha_{11}}{n})^{[nx]-1}\right]$$
$$-(1 - \frac{\alpha_{10}+\alpha_{01}+\alpha_{11}}{n}) \left[1 - (1 - \frac{\alpha_{10}+\alpha_{01}+\alpha_{11}}{n})^{[nx]-1}\right].$$

Thus, we get

$$\Psi(x|\lambda_1, \lambda_2, \lambda_{11}) = \lim_{n\to\infty} G_W(nx|p_{10}, p_{01}, p_{11}) = 1 - e^{-\lambda_1 x} - e^{-\lambda_2 x} + e^{-\lambda_{11} x}, \qquad (4.1)$$

where $\lambda_1 = \alpha_{10} + \alpha_{11}$, $\lambda_2 = \alpha_{01} + \alpha_{11}$ and $\lambda_{11} = \alpha_{10} + \alpha_{01} + \alpha_{11}$. Clearly, $\Psi(x|\alpha_{10}, \alpha_{01}, \alpha_{11})$ is monotone non-decreasing function for being it is a limit of the df. Moreover, $\Psi(0|\alpha_{10}, \alpha_{01}, \alpha_{11}) = 0$ and $\lim_{x\to\infty} \Psi(x|\lambda_1, \lambda_2, \lambda_{11}) = 1$. Therefore, $\Psi(x|\alpha_{10}, \alpha_{01}, \alpha_{11})$ is a df, which may be interpreted as the df of the minimum waiting time (continuous time) to get at least one success of each of the events A_1 and A_2. When $\alpha_{11} = 0$, we get

$$\Psi(x|\alpha_{10}, \alpha_{01}, 0) = 1 - e^{-\alpha_{10}x} - e^{-\alpha_{01}x} + e^{-(\alpha_{10}+\alpha_{01})x} = (1 - e^{-\alpha_{10}x})(1 - e^{-\alpha_{01}x}),$$

which is the df of $\xi_1 \vee \xi_2$, where ξ_1 and ξ_2 are two independent rv's distributed as exponential df's with parameters α_{10} and α_{01}, respectively.

The mgf of the df (4.1) is given by

$$\Phi(t) = \frac{\lambda_1}{\lambda_1 - t} + \frac{\lambda_2}{\lambda_2 - t} - \frac{\lambda_{11}}{\lambda_{11} - t}, t < \lambda_1 \wedge \lambda_2 \wedge \lambda_{11},$$

which yields the mean and variance as $\frac{1}{\lambda_1} + \frac{1}{\lambda_2} - \frac{1}{\lambda_1+\lambda_2-\lambda_{11}}$ and $\frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} - \frac{1}{(\lambda_1+\lambda_2-\lambda_{11})^2}$, respectively.

5. A Generalization of the Geometric and Exponential df's Based on the Trivariate Bernoulli Scheme

In this section we introduce an univariate 7-GG df based on the trivariate Bernoulli scheme $(A_{11}, A_{10}; A_{21}, A_{20}; A_{31}, A_{30})$ with probabilities $p_{000} = P(A_{10}A_{20}A_{30})$, $p_{100} = P(A_{11}A_{20}A_{30})$, $p_{010} = P(A_{10}A_{21}A_{30})$, $p_{001} = P(A_{10}A_{20}A_{31})$, $p_{110} = P(A_{11}A_{21}A_{30})$, $p_{101} = P(A_{11}A_{20}A_{31})$, $p_{011} = P(A_{10}A_{21}A_{31})$ and $p_{111} = P(A_{11}A_{21}A_{31})$, where $p_{000} + p_{100} + p_{010} + p_{001} + p_{110} + p_{101} + p_{011} + p_{111} = 1$. Let $p_1 = p_{111} + p_{110} + p_{101} + p_{100} = 1 - q_1$, $p_2 = p_{111} + p_{011} + p_{110} + p_{010} = 1 - q_2$ and $p_3 = p_{111} + p_{101} + p_{011} + p_{001} = 1 - q_3$, then it can be shown that the 7-GG df based on the trivariate Bernoulli scheme corresponding to the df (2.3) is given by

$$g(w : p_{100}, p_{010}, p_{001}, p_{110}, p_{101}, p_{011}, p_{111}) = \begin{cases} p_{111}, & w = 1, \\ \sum_{i=1}^{3} q_i^{w-1} p_i - p_{100}^{w-1}\overline{p}_{100} - p_{010}^{w-1}\overline{p}_{010} & \\ -p_{001}^{w-1}\overline{p}_{001} - 2p_{000}^{w-1}\overline{p}_{000}, & w = 2, 3, ... \end{cases} \tag{5.1}$$

(it can be easily show that $\sum_{i=1}^{\infty} g(w : p_{100}, p_{010}, p_{001}, p_{110}, p_{101}, p_{011}, p_{111}) = 1$). Moreover, the corresponding mgf is given by

$$\Phi(t) = p_{111}e^t + \sum_{i=1}^{3} \frac{p_i q_i e^{2t}}{1 - q_i e^t} - \frac{p_{100}\overline{p}_{100}e^{2t}}{1 - p_{100}e^t} - \frac{p_{010}\overline{p}_{010}e^{2t}}{1 - p_{010}e^t} - \frac{p_{001}\overline{p}_{001}e^{2t}}{1 - p_{001}e^t} - \frac{2p_{000}\overline{p}_{000}e^{2t}}{1 - p_{000}e^t},$$

where $t < -\log(\max\{q_1, q_2, q_3, p_{100}, p_{010}, p_{001}, p_{000}\})$. Finally, some especial cases of the df (5.1) are given by:

1. If $p_2 = p_3 = 1$ (this implies $q_2 = q_3 = 0, p_{100} = 1 - p_{111}$); or $p_1 = p_3 = 1$ (this implies $q_1 = q_3 = 0, p_{010} = 1 - p_{111}$); or $p_1 = p_2 = 1$ (this implies $q_1 = q_2 = 0, p_{001} = 1 - p_{111}$); or $p_{100} = p_{010} = p_{001} = p_{110} = p_{101} = p_{011} = 0$, then we get $g_{100}(w) = p_{100}^{w-1}p_{111}, w = 1, 2, ...$; or $g_{010}(w) = p_{010}^{w-1}p_{111}, w = 1, 2, ...$; or $g_{001}(w) = p_{001}^{w-1}p_{111}, w = 1, 2, ...$; or $g_{000}(w) = p_{000}^{w-1}p_{111}, w = 1, 2, ...$, respectively.

2. If $p_1 = 1$, then we get

$$g(w : p_{100}, 0, 0, p_{110}, p_{101}, 0, p_{111}) = \begin{cases} p_{111}, & w = 1, \\ p_2^{w-1}q_2 + p_3^{w-1}q_3 - p_{100}^{w-1}\overline{p}_{100}, & w = 2, 3, ... \end{cases}$$

   If $p_2 = 1$, then we get

$$g(w : 0, p_{010}, 0, p_{110}, 0, p_{011}, p_{111}) = \begin{cases} p_{111}, & w = 1, \\ p_1^{w-1}q_1 + p_3^{w-1}q_3 - p_{010}^{w-1}\overline{p}_{010}, & w = 2, 3, ... \end{cases}$$

If $p_3 = 1$, then we get

$$g(w:0,0,p_{001},0,p_{101},p_{011},p_{111}) = \begin{cases} p_{111}, & w=1, \\ p_1^{w-1}q_1 + p_2^{w-1}q_2 - p_{001}^{w-1}\overline{p}_{001}, & w=2,3,\dots . \end{cases}$$

3. If $p_{100} = p_{010} = p_{001} = 0$ (this implies that $q_1 = p_{000} + p_{011}$, $q_2 = p_{000} + p_{101}$, $q_3 = p_{000} + p_{110}$, $p_1 = p_{101} + p_{110} + p_{111}$, $p_2 = p_{011} + p_{110} + p_{111}$ and $p_3 = p_{011} + p_{101} + p_{111}$), then

$$g(w:0,0,0,p_{110},p_{101},p_{011},p_{111}) = \begin{cases} p_{111}, & w=1, \\ \sum\limits_{i=1}^{3} q_i^{w-1}p_i - 2p_{000}^{w-1}\overline{p}_{000}, & w=2,3,\dots . \end{cases} \tag{5.2}$$

Consider a regime when the probabilities of successes in the df (5.2) are very small, such that $p_{110} = \frac{\alpha_{110}}{n}$, $p_{101} = \frac{\alpha_{101}}{n}$, $p_{011} = \frac{\alpha_{011}}{n}$, $p_{111} = \frac{\alpha_{111}}{n}$ and consider $w = xn$. Then,

$$\Psi(x|\lambda_1,\lambda_2,\lambda_3,\lambda_{111}) = \lim_{n\to\infty} \sum_{w=1}^{[nx]} g(w|0,0,0,p_{110},p_{101},p_{011},p_{111})$$

$$= 1 - e^{-\lambda_1 x} - e^{-\lambda_2 x} - e^{-\lambda_3 x} + 2e^{-\lambda_{111} x}, \tag{5.3}$$

where $\lambda_1 = \alpha_{110} + \alpha_{101} + \alpha_{111}$, $\lambda_2 = \alpha_{011} + \alpha_{110} + \alpha_{111}$, $\lambda_3 = \alpha_{011} + \alpha_{101} + \alpha_{111}$ and $\lambda_{111} = \alpha_{011} + \alpha_{101} + \alpha_{110} + \alpha_{111}$.

The new four-parameter generalized exponential df (5.3) (provided that $\lambda_{111} > \lambda_1 \vee \lambda_2 \vee \lambda_3$ and $\lambda_1 \wedge \lambda_2 \wedge \lambda_3 > 0$ – note that if $\lambda_i = 0, i \in \{1,2,3\}$, then $\lambda_{j_1} = \lambda_{j_2} = \lambda_{111}, j_1, j_2 \in \{1,2,3\}, j_1, j_2 \neq i$, and consequently this implies that $\Psi(x|\lambda_1,\lambda_2,\lambda_3,\lambda_{111}) = 1, \forall x > 0$), as a lifetime df, has an interesting application that it represents the df of the minimum time (continuous time) of the failure of some machine that consists of three parallel connected components, where the success means here the occurrence of the failure. On the other hand, these components are dependent on each other in such a way that if any one of them failed, then at least one of the others should fail.

ACKNOWLEDGEMENT

The author wishes to thank the Editor-in-Chief, Professor Gadadhar Misra, and the referee for some valuable comments and suggestions which led improvement over an earlier version of the paper.

REFERENCES

1. I. Bairamov and O. E. Gultekin, Discrete distributions connected with the bivariate binomial, *Hacettepe J. Math. Statist.*, **39**(1) (2010), 109-120.

2. H. M. Barakat, Limit theorems for lower-upper extreme values from two-dimensional distribution function, *J. Statist. Plan. Inf.*, **24** (1990), 69-79.

3. H. M. Barakat, Asymptotic properties of bivariate random extremes, *J. Statist. Plan. Inf.*, **61** (1997), 203-217.

4. H. M. Barakat, Asymptotic distribution theory of bivariate order statistics, *Ann. Inst. Statist. Math.*, **53**(3) (2001), 487-497.

5. D. Bhati and S. Joshi, Weighted geometric distribution with a new characterization of geometric distribution, *arXiv:1512.07139v2 [stat.ME]*, (2016), 1-16.

6. H. Bidram, R. Roozegar, and V. Nekoukhou, Exponentiated generalized geometric distribution: A new discrete distribution, *Hacettepe*, **45**(6) (2016), 1767-1779.

7. B. Chandrasekar and N. Balakrishnan, Some properties and a characterization of trivariate and multivariate binomial distributions, *Statistics*, **36** (2002), 211-218.

8. W. Feller, *An introduction to probability theory and its applications*, volume 1, 3rd edition, John Wiley & Sons, New York, 1968.

9. E. Géomez-Déniz, Another generalization of the geometric distribution, *Test*, **19**(2) (2010), 399-415.

10. R. D. Gupta and D. Kundu, A new class of weighted exponential distributions, *Statistics*, **43** (2009), 621-634.

11. N. L. Johnson, S. Kotz, and N. Balakrishnan, *Discrete multivariate distributions*, John Wiley & Sons, New York, 1997.

12. R. D. Joshi, *A generalized inflated geometric distribution*, Ph.D. thesis. Marshall University Marshall Digital Scholar-Theses, Dissertations and Capstones, Paper 916, 2015.

13. S. Kocherlakota and K. Kocherlakota, *Bivariate discrete distributions*, Marcel Dekker Inc. New York, 1992.

14. A. W. Marshall and I. Olkin, A Family of bivariate distributions generated by the bivariate Bernoulli distribution, *J. Amer. Statist. Assoc.*, **80**(390) (1985), 332-338.

15. A. W. Marshall and I. Olkin, A new method for adding a parameter to a family of distributions with application to the exponential and Weibull families, *Biometrika*, **84**(3) (1997), 641-652.

16. R. C. Tripathi, R. C. Gupta, and T. J. White, Some generalizations of the geometric distribution, *Sankhya: Indian J. Statist., Series B*, **49**(3) (1987), 218-223.