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PERIODIC PERTURBATIONS OF LINEAR SYSTEMS AT RESONANCE

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Second-order linear Hamiltonian systems at resonance with periodic nonlinearity is investigated. An existence result of solutions for such systems is obtained by means of variational methods, saddle point theorem, and an index theory for second-order linear Hamiltonian systems. Meanwhile, two examples and two extensions are presented.

Key words : Nonlinear boundary value problems; critical points; saddle point theorem; index theory; resonance.

1. Introduction

Let us investigate the existence of solutions for Dirichlet boundary value problem

$$\begin{cases} \ddot{x} + A(t)x + g(x) = h, \ \ t \in [0,1], \\ x(0) = 0 = x(1), \end{cases} \tag{1.1}$$

where $A \in L^\infty([0,1]; \mathcal{L}_s(\mathbf{R}^N))$, $\mathcal{L}_s(\mathbf{R}^N)$ is the space of real symmetric matrices of order N, N is a given positive integer, g and h satisfy the following one condition respectively:

(G) $g(x) = (g_1(x_1), \cdots, g_N(x_N))$, each g_i is continuous periodic function (possibly with different minimal periods) with zero mean value.

(H) $h \in C([0,1]; \mathbf{R}^N)$, $h \neq 0$, and $\int_0^1 (h, e) dt = 0$, where e is one nontrivial solution of the linear homogeneous case of (1.1)

$$\begin{cases} \ddot{x} + A(t)x = 0, \ \ t \in [0,1], \\ x(0) = 0 = x(1), \end{cases} \tag{1.2}$$

and $(\cdot, \cdot)$ is the usual inner product in $\mathbf{R}^N$.

One of the first papers to treat ordinary differential equations with the linear part at resonance and periodic nonlinear terms is paper [27]. More precisely, paper [27] solved the principal eigenvalue problem in one-dimensional case. Later on, it was extended to the case of elliptic partial differential equations and the case of higher eigenvalues, see [18, 25]. In 2005, Ca$\tilde{n}$ada and Ruiz [3] considered the following boundary value problem

$$\begin{cases} \ddot{x} + Ax + f(x) = h(t), \quad t \in [0, \pi], \\ x(0) = 0 = x(\pi), \end{cases}$$

where A is a real symmetric matrix of order N such that its greatest eigenvalue is one, with an eigenspace of dimension one, f satisfies the same assumptions as g in (1.1), $h \in C([0, \pi]; \mathbf{R}^N)$.

In addition, the they stated the physical significance of the concerned problem when $N = 2$. Namely, it appears quite naturally in the study of some mechanical models (strongly coupled oscillators under the effect of elastic and external forces), and also in electric circuits under reciprocal induction, see [19, 20]. In addition to the mentioned papers above, the similar systems at resonance with periodic nonlinear terms have been focused on extensively in the past twenty years, see [1, 4-12, 15, 17, 22-24, 26, 28]. To the best of our knowledge, few authors consider (1.1) when A is a real symmetric matrix and its greatest eigenvalue is greater than π^2 (Since t-interval in (1.1) is not $[0, \pi]$ but $[0, 1]$, the eigenvalue of A is need to be multiplied by π^2 correspondingly), let alone the case that A is a L^∞ function.

In all the papers mentioned above, there are mainly three methods in dealing with the resonance problems with periodic nonlinearity. The first one is variational methods and critical point theory, see [3, 11, 12, 18, 25, 26, 28]. The second one is the Lyapunov$-$Schmidt reduction method, see [4-10]. The last one is bifurcation and continuation techniques, see [22-24]. In view of $A \in L^\infty([0, 1]; \mathcal{L}_s(\mathbf{R}^N))$, one can see that Proposition 2.4 in [3] doesn't hold any more for (1.1), and Proposition 2.3, the unique critical point theorem in the paper, can't be applied to here. Meanwhile, the analysis of the oscillatory behavior of the bifurcation equation in [9] can't be also applied here because g is multi-component. In addition, the variational reduction method in [11, 12] can't be also applied to (1.1) because the addition conditions $g \in C^2(\mathbf{R})$ and $\|h\|_{L^2} < \epsilon_0$ are not assumed now. Similarly, bifurcation and continuation techniques in [22] can't be applied to (1.1) because the assumption $A \in C^1([0, 1]; \mathbf{R}^+)$ doesn't exist.

In this paper, we follow the ideas of [25, 27] and investigate the existence of solutions for (1.1). More precisely, we obtain the existence result of solution for (1.1) by applying variational methods and the suitable version of the saddle point theorem. Our result extends Theorem 3.1 in [27] directly.

This paper is organized as follows. In section 2, we mainly recall some useful conclusions in our argument. The first one is a version of the saddle point theorem and its corollary. The classical saddle point theorem is due to Rabinowitz [21] and possesses various versions. Here we quote a version in [25]. Its more general version has been also presented in [16]. Meanwhile, the two weakenings of the Palais-Smale condition and a corollary of the saddle point theorem are stated in [16]. The second one is some results of an index theory for (1.2). The last one is one version of Riemann-Lebesgue lemma.

In section 3, we state our main result and give its proof by applying the saddle point theorem and its corollary, the index theory for (1.2), and the Riemann-Lebesgue lemma.

In Section 4, we present two examples in order to illustrate our results. In these two examples, A is a piecewise function and a real symmetric matrix whose greatest eigenvalue is $9\pi^2$ respectively. Moreover, we consider two extensions of (1.1).

2. PRELIMINARIES

Let X be a Hilbert space and $\{e_n\}$ be a given orthonormal sequence of X. Moreover, let $k \in \mathbf{N}^*$. Denote

$$\begin{aligned}\mathcal{A}_k^1 &= \{A \subset X | \exists\, \eta \in H_k \ \text{ such that } \ A = \eta(\bar{B}_k)\},\\ \mathcal{A}_k^2 &= \{A \subset X | A \ \text{ is compact and } \ \forall \eta \in H_k : \eta(A) \cap (\bar{x} + X_k^{\perp}) \neq \emptyset\},\end{aligned}$$

where $X_k = span\{e_1, e_2, \ldots, e_k\}$, $B_k(\bar{x}, r)$ is the open ball of $\bar{x} + X_k$ centered at $\bar{x}$ of radius r for given $\bar{x} \in x, r \in \mathbf{R}, r > 0$, H_k is the set of all the deformations η of X which leave invariant ∂B_k, i.e. $\eta \in H_n$ if and only if η is a continuous function from X to X such that $\eta(x) = x$ for all $x \in \partial B_k$. It is noted that the ball B_k can be repeated with trivial modifications for the sets of any other shape like hypercubes or cylinders, therefore we can be flexible in our application and take as $B_k's$ sets of different shape but with the same topological relationships. Moreover, the min-max values

$$c_k^i = \inf_{A \in \mathcal{A}_k^i} \sup_A \varphi, \quad i = 1, 2.$$

***Theorem* 2.1** — *[25, Theorem 3.1]. Let $\varphi : X \to \mathbf{R}$ be a C^1-functional. Suppose that $c_k^i > \sup\limits_{\partial B_k} \varphi$ for any given $k \in \mathbf{N}^*$ and $i = 1, 2$. Then, for any given sequence $\{A_n\} \subset \mathcal{A}_k^i$ such that $\lim\limits_{n\to\infty} \sup\limits_{A_n} \varphi = c_k^i$, there exists a sequence $\{x_n\} \subset X$ such that*

$(i) \lim\limits_{n\to\infty} d(x_n, A_n) = 0,$

$(ii) \lim\limits_{n\to\infty} \|\varphi'(x_n)\| = 0,$

(iii) $\lim_{n\to\infty} \varphi(x_n) = c_k^i$.

Also the condition $c_k^i > \sup_{\partial B_k} \varphi$ *is verified provided* $\inf_{\bar{x}+X_k^\perp} \varphi > \sup_{\partial B_k} \varphi$.

Definition 2.2 — [16, Definition 1.4]. Say that φ verifies the Palais-Smale condition at the level c in shorts, $((PS)_c)$, if any $\{x_n\}$ in X satisfying $\lim_{n\to\infty} \varphi(x_n) = c$, $\lim_{n\to\infty} \|\varphi'(x_n)\| = 0$ has a convergent subsequence.

Definition 2.3 — [16, Definition 1.8]. Say that φ verifies the Palais-Smale condition at the level c and around the set F in shorts, $((PS)_{F,c})$, if any $\{x_n\}$ in X satisfying $\lim_{n\to\infty} \varphi(x_n) = c$, $\lim_{n\to\infty} \|\varphi'(x_n)\| = 0$ and $\lim_{n\to\infty} d(x_n, F) = 0$ has a convergent subsequence.

Definition 2.4 — [16, Definition 3.3]. Say that φ verifies the Palais-Smale condition along a sequence $\{A_n\}$ in $\mathcal{A}_k^1$ or $\mathcal{A}_k^2$, if any $\{x_n\}$ in X satisfying (i), (ii), and (iii) above has a convergent subsequence.

Remark 2.5 : Note that $(PS)_c$ corresponds to $(PS)_{F,c}$ for $F = X$. Meanwhile, $(PS)_{F,c}$ implies φ verifies the Palais-Smale condition along a sequence $\{F \cap A_n\}$ in $\mathcal{A}_k^1$ or $\mathcal{A}_k^2$ if $\lim_{n\to\infty} d(x_n, A_n) = 0$ hold.

Definition 2.6 — [16, Definition 3.4]. Say that a sequence $\{A_n\}$ in $\mathcal{A}_k^i (i = 1, 2)$ is min-maxing for φ if $\lim_{n\to\infty} \sup_{A_n} \varphi = c_k^i$.

***Theorem* 2.7** — *[16, Corollary 3.6]. Under the hypothesis of Theorem 2.1 and assuming that φ verifies the $Palais$-$Smale$ condition along a min-maxing sequence $\{A_n\}$, then there exists a sequence $\{x_n\}$ in X with $x_n \in A_n$ that converges to a critical point of φ.*

Proposition 2.8 — [25, Proposition 3.2]. Let $A \in \mathcal{A}_k^1$ for some $k > 1$. Then $A \cap \{e_k\}^\perp \in \mathcal{A}_{k-1}^2$.

From now on, let

$$E \equiv H_0^1([0,1]; \mathbf{R}^N) = \left\{x \in H^1([0,1]; \mathbf{R}^N) | x(0) = 0 = x(1)\right\}$$

with the inner product

$$(x, y) = \int_0^1 \dot{x}(t) \cdot \dot{y}(t)\, dt, \quad \forall x, y \in E,$$

the corresponding norm inducted by the inner product is

$$\|x\| = \left(\int_0^1 |\dot{x}(t)|^2\, dt\right)^{\frac{1}{2}}, \quad \forall x \in E.$$

For any $A \in L^\infty([0,1]; \mathcal{L}_s(\mathbf{R}^N))$, we define

$$q_A(x,y) = \int_0^1 [(\dot{x}(t), \dot{y}(t)) - (A(t)x(t), y(t))]\, dt, \quad \forall x, y \in E.$$

For any $x, y \in E$ if $q_A(x,y) = 0$, we say that x and y q_A-orthogonal. Let E_1 and E_2 be the two subspaces of E. If for any $x \in E_1$ and $y \in E_2$, $q_A(x,y) = 0$, we say that E_1 and E_2 q_A-orthogonal.

Proposition 2.9 — [13, Proposition 2.1]. For any $A \in L^\infty([0,1]; \mathcal{L}_s(\mathbf{R}^N))$, E has a q_A-orthogonal decomposition

$$E = E^+(A) \oplus E^0(A) \oplus E^-(A)$$

such that q_A is positive definite, zero and negative definite on $E^+(A)$, $E^0(A)$ and $E^-(A)$ respectively. Moreover, $E^0(A)$ and $E^-(A)$ are finitely dimensional.

Definition 2.10 — [13, Definition 2.2]. For any $A \in L^\infty([0,1]; \mathcal{L}_s(\mathbf{R}^N))$, we define $i(A) = \dim E^-(A)$, $\nu(A) = \dim E^0(A)$. We call $i(A)$ and $\nu(A)$ the index and nullity of A respectively.

Proposition 2.11 — [13, Proposition 2.3]. For $A \in L^\infty([0,1]; \mathcal{L}_s(\mathbf{R}^N))$, $\nu(A)$ is the dimension of the solution subspace of $x''(t) + A(t)x(t) = 0, x(0) = 0 = x(1)$, $\nu(A) \in \{0, 1, 2, \cdots, N\}$, and $i(A) = \sum\limits_{\lambda<0} \nu(A + \lambda I_N)$.

Proposition 2.12 — [25, Proposition 2.1]. Let f be continuous periodic real function with zero mean value and F be the primitive of f. Let $U \subset H^1[0,1]$ be a bounded set. Assume that $y \in C^1[0,1]$ and $\dot{y}(t) \neq 0$ a.e. in $(0,1)$. Then

$$\lim_{|\alpha|\to\infty} f(x(t) + \alpha y(t)) = 0 \ \text{ weakly } \ \text{in } L^2[0,1],$$

and

$$\lim_{|\alpha|\to\infty} \int_0^1 F(x(t) + \alpha y(t))\, dt = 0,$$

both convergence uniformly for $x \in U$.

In the following we further make an assumption of A and obtain some related lemmas.

(A) $\quad i(A) = k \in \mathbf{N}^*$, $\nu(A) = 1$. Moreover, the derivative of all components of e of (1.2) are not equal to zero a.e. in $(0,1)$.

Consider the functional $\varphi : E \to \mathbf{R}$ defined by

$$\varphi(x) = \psi(x) - \int_0^1 \sum_{i=1}^N G_i(x_i)\, dt, \qquad \forall x \in E, \tag{2.1}$$

where

$$\psi(x) = \frac{1}{2}\int_0^1 \left[|\dot{x}(t)|^2 - (A(t)x(t), x(t))\right]\, dt + \int_0^1 (h(t), x(t))\, dt, \quad \forall x \in E, \tag{2.2}$$

G_i is the primitive of g_i, $i = 1, 2, \cdots, N$. In addition,

$$\langle \varphi'(x), y\rangle = \int_0^1 [(\dot{x}, \dot{y}) - (Ax, y)]\, dt + \int_0^1 (h, y) dt - \int_0^1 (g(x), y)\, dt, \quad \forall y \in E.$$

(G) implies $\varphi \in C^1(E, \mathbf{R})$. Hence, the weak solution for (1.1) corresponds to the critical point of φ.

Now we assume that $\{e_1, \cdots, e_k\}$ is the union of the bases of the solution space of $\ddot{x}(t) + (A(t) + \lambda I_N)x(t) = 0, x(0) = 0 = x(1)$ for all $\lambda < 0$ by Proposition 2.9, Proposition 2.11 and (A). Meanwhile, we can let $e_{k+1} = e$, $E_k = span\{e_1, e_2, \cdots, e_k\}$ and $F = span\{e_{k+1}\}$. Then, F is the solution space of (1.2) and $E = F \oplus F^{\perp}$. In addition, we can assume that $\bar{x}$ is one solution of

$$\begin{cases} \ddot{x}(t) + A(t)x(t) = h(t), & t \in [0, 1], \\ x(0) = 0 = x(1), \end{cases} \tag{2.3}$$

because of (H). Moreover, in view of $h \neq 0$, $\bar{x} \in F^{\perp}$, and so $\bar{x}$ is unique. Let $B_k(\bar{x}, r)$ be the ball of E_k, $r \in \mathbf{R}$, $r > 0$, then $\mathcal{A}_k^1, \mathcal{A}_k^2$ and c_k^1, c_k^2 are determined.

Lemma 2.13 — If (A) and (H) hold, then for some large enough $\bar{r} > 0$,

$$\sup_{\partial B_k(\bar{x}, \bar{r})} \varphi < \inf_{\bar{x} + E_k^{\perp}} \varphi, \tag{2.4}$$

and

$$\sup_{\partial B_k(\bar{x}, \bar{r}) + F} \varphi < \inf_{\bar{x} + E_k^{\perp}} \varphi. \tag{2.5}$$

PROOF : By (2.2) and Proposition 2.9, we know that ψ is concave on $\bar{x} + E_k$. Then,

$$\psi(\bar{x}) = \max_{\bar{x} + E_k} \psi \tag{2.6}$$

because of $\psi'(\bar{x}) = 0$. In addition, φ is bounded below on $\bar{x} + E_k^{\perp}$, ψ and φ go quadratically to $-\infty$ on $\bar{x} + E_k$. Therefore, we can choose some large enough $\bar{r} > 0$ which satisfies (2.4). Since the component belonging to F gives only a bounded contribution for determining φ, we can strengthen (2.4) to (2.5). The proof of Lemma 2.13 is complete.

Lemma 2.14 — If (A), (G) and (H) hold, then

(i) φ satisfies $(PS)_{F^{\perp}, c}$ for all $c \in \mathbf{R}$;

(ii) φ satisfies $(PS)_c$ for all $c \neq \psi(\bar{x})$.

PROOF : (i) Suppose $\{x_n\} \subset E$ such that $\varphi(x_n) \to c$, $\|\varphi'(x_n)\| \to 0$ and $d(x_n, F^{\perp}) \to 0$. Then,

$$\int_0^1 (\dot{x}_n, \dot{y})\, dt = \langle \varphi'(x_n), y\rangle + \int_0^1 (Ax_n, y)dt - \int_0^1 (h, y)dt + \int_0^1 (g(x_n), y)dt, \quad \forall y \in E.$$

Since x_n can be written in

$$x_n = y_n + z_n = c_n^{k+1} e_{k+1} + z_n, \tag{2.7}$$

we have

$$\int_0^1 (\dot{z}_n, \dot{y})dt = \langle \varphi'(x_n), y\rangle + \int_0^1 (Az_n, y)dt - \int_0^1 (h, y)dt + \int_0^1 (g(x_n), y)dt, \quad \forall y \in E. \tag{2.8}$$

Observing that that $\{y_n\}$ is bounded in E since $d(x_n, F^{\perp}) \to 0$, we will verify that $\{z_n\}$ is bounded in E which implies that $\{x_n\}$ is bounded. Let $\|\cdot\|_\infty$ be the norm of $C([0,1]; \mathbf{R}^N)$, then it is enough to verify that $\{z_n\}$ is bounded in $C([0,1]; \mathbf{R}^N)$ by (2.8). If not, $\|z_n\|_\infty \to +\infty$ and we set $w_n = z_n/\|z_n\|_\infty$. Furthermore, we can choose $y = z_n$, and (2.8) will become

$$\int_0^1 |\dot{w}_n|^2\, dt = \frac{1}{\|z_n\|_\infty}\langle \varphi'(x_n), w_n\rangle + \int_0^1 (Aw_n, w_n)dt - \frac{1}{\|z_n\|_\infty}\int_0^1 (h, w_n)dt + \frac{1}{\|z_n\|_\infty}\int_0^1 (g(x_n), w_n)dt. \tag{2.9}$$

(2.9) deduces that $\{w_n\}$ is bounded in E. Without loss of generality, we still assume $w_n \rightharpoonup w_0$ in E, then $w_n \to w_0$ in $C([0,1]; \mathbf{R}^N)$ and $\|w_0\|_\infty = 1$. In addition, (2.8) follows that

$$\int_0^1 [(\dot{w}_n, \dot{y}) - (Aw_n, y)]\, dt = \frac{1}{\|z_n\|_\infty}\langle \varphi'(x_n), y\rangle + \frac{1}{\|z_n\|_\infty}\int_0^1 [(g(x_n), y) - (h, y)]\, dt, \quad \forall y \in E. \tag{2.10}$$

Now we can pass to the limit of both sides of (2.10) and obtain

$$\int_0^1 [(\dot{w}_0, \dot{y}) - (Aw_0, y)]\, dt = 0, \quad \forall y \in E. \tag{2.11}$$

(2.11) shows that w_0 is a solution of (1.2). But (1.2) has no nontrivial solution in $F^{\perp}$. This is a contradiction, and it follows that $\{z_n\}$ is bounded. Hence, there exists a subsequence $\{x_{n_j}\}$ such that $x_{n_j} \rightharpoonup x_0$ in E and $x_{n_j} \to x_0$ in $C([0,1]; \mathbf{R}^N)$. In view of (2.8), we have

$$\int_0^1 (\dot{x}_{n_j} - \dot{x}_0, \dot{y})dt = \langle \varphi'(x_{n_j}), y\rangle + \int_0^1 \Big(A(x_{n_j} - x_0), y)dt + \int_0^1 (g(x_{n_j}) - g(x_0), y\Big)dt, \quad \forall y \in E. \tag{2.12}$$

Set $y = x_{n_j} - x_0$, we get $\|x_{n_j} - x_0\| \to 0$ from (2.12), i.e. $x_{n_j} \to x_0$ in E. Hence, $\{x_n\}$ has one convergent subsequence.

(ii) It's easy to see that we only need to prove that $\{y_n\}$ is still bounded when $c \neq \psi(\bar{x})$. In other words, if $\{y_n\}$ is not bounded then $c = \psi(\bar{x})$. Without loss of generality, we can assume that $|c_n^{k+1}| \to \infty$ as $n \to \infty$ and $z_{n_j} \rightharpoonup z$ as $j \to \infty$. Substituting z_n by z_{n_j} in (2.8), we have

$$\int_0^1 (\dot{z}_{n_j}, \dot{y})\, dt = \langle \varphi'(x_{n_j}), y \rangle + \int_0^1 (Az_{n_j}, y)\, dt - \int_0^1 (h, y)\, dt + \int_0^1 (g(x_{n_j}), y)\, dt, \quad \forall y \in E. \tag{2.13}$$

In addition, $z_{n_j} \rightharpoonup z$ implies

$$\int_0^1 (\dot{z}_{n_j}, \dot{y})\, dt \to \int_0^1 (\dot{z}, \dot{y})\, dt \quad \text{and} \quad \int_0^1 (Az_{n_j}, y)\, dt \to \int_0^1 (Az, y)\, dt. \tag{2.14}$$

Moreover, we deduce that $|c_{n_j}^{k+1}| \to \infty$ as $j \to \infty$, and $\dot{e}_{k+1}^i \neq 0$ a.e. on $(0, 1)$ from Hypothesis (A). Hence, we can assure that

$$\begin{aligned} \lim_{n_j \to \infty} \int_0^1 (g(x_{n_j}), y)\, dt &= \lim_{n_j \to \infty} \int_0^1 \left(g(c_{n_j}^{k+1} e_{k+1} + z_{n_j}), y \right)\, dt \\ &= \lim_{n_j \to \infty} \int_0^1 \sum_{i=1}^N \left(g_i(c_{n_j}^{k+1} e_{k+1}^i + z_{n_j}^i), y_i \right)\, dt \\ &= \sum_{i=1}^N \lim_{n_j \to \infty} \int_0^1 \left(g_i(c_{n_j}^{k+1} e_{k+1}^i + z_{n_j}^i), y_i \right)\, dt \\ &= 0 \end{aligned} \tag{2.15}$$

thanks to (G) and Proposition 2.12. Now taking the limit of both sides of (2.13), we can deduce

$$\int_0^1 (\dot{z}, \dot{y})\, dt = \int_0^1 [(Az, y) - (h, y)]\, dt, \quad \forall y \in E \tag{2.16}$$

from (2.14) and (2.15). (2.16) means that z is also a solution of (2.3), i.e. $z = \bar{x}$. Furthermore, (2.1), (2.2) and (2.7) imply

$$\begin{aligned} \psi(x_n) &= \psi(z_n) + c_n^{k+1} \int_0^1 [(\dot{e}_{k+1}, \dot{z}_n) - (Ae_{k+1}, z_n)]\, dt + \\ &\quad \frac{1}{2} \left(c_n^{k+1} \right)^2 \int_0^1 [|\dot{e}_{k+1}|^2 - (Ae_{k+1}, e_{k+1})]\, dt - c_n^{k+1} \int_0^1 (h, e_{k+1}) dt. \end{aligned} \tag{2.17}$$

But by (A) and (H), we know that

$$\begin{aligned} &\int_0^1 (h, e_{k+1})\, dt = 0, \ \int_0^1 [(\dot{e}_{k+1}, \dot{z}_n) - (Ae_{k+1}, z_n)] dt = 0, \\ &\text{and } \int_0^1 [|\dot{e}_{k+1}|^2 - (Ae_{k+1}, e_{k+1})] dt = 0. \end{aligned} \tag{2.18}$$

(2.17) combining with (2.18) implies

$$\psi(x_n) = \psi(z_n). \tag{2.19}$$

Hence, we obtain

$$\varphi(x_n) \;=\; \psi(z_n) - \int_0^1 \sum_{i=1}^{N} G_i(c_n^{k+1} e_{k+1}^i + z_n^i)\, dt \tag{2.20}$$

from (2.7) and (2.19). Then, employing (G) and Proposition 2.12 again to (2.20) we have

$$\lim_{n\to\infty} \int_0^1 \sum_{i=1}^{N} G_i(c_n^{k+1} e_{k+1}^i + z_n^i)\, dt = 0. \tag{2.21}$$

(2.20) and (2.21) imply $\lim\limits_{n\to\infty} \varphi(x_n) = \lim\limits_{n\to\infty} \psi(z_n)$. But $\varphi(x_n) \to c$ and $\psi(z_n) \to \psi(z) = \psi(\bar{x})$ since $z_n \rightharpoonup z$. Hence, $c = \psi(\bar{x})$. The proof of Lemma 2.14 is complete.

3. Main Result and its Proof

***Theorem* 3.1** — *If (A), (G) and (H) hold, then (1.1) has at least one solution.*

PROOF : We follow Solimini's idea in [25] to complete the proof. We distinguish the two cases and find one critical point of φ.

Case 1 : $c_k^1 \neq \psi(\bar{x})$ or $c_k^2 \neq \psi(\bar{x})$.

We apply Theorem 2.1, Lemma 2.13, and Lemma 2.14 to φ and find that φ satisfies $(PS)_{c_k^1}$ or $(PS)_{c_k^2}$. It implies that φ has one critical point at the level c_k^1 or c_k^2 by using Theorem 2.7 and Remark 2.5.

Case 2 : $c_k^1 = c_k^2 = \psi(\bar{x})$.

Clearly, $(PS)_c$ can't be applied here at the level c_k^1 or c_k^2 and we need to find the other level. For each $\alpha \in \mathbf{N}$, let $C(\alpha) = B_k(\bar{x}, \bar{r}) + (B(0,\alpha) \cap F)$. By taking $C(\alpha)$ as B_k, we consider $\mathcal{A}_{k+1}^1$ and c_{k+1}^1. Let

$$b(\alpha) = \sup_{B_k + (\partial B(0,\alpha) \cap F)} \varphi \qquad \text{and} \qquad c(\alpha) = \sup_{\partial C(\alpha)} \varphi. \tag{3.1}$$

We can deduce $\varphi(x + \alpha \sum\limits_{i=k+1}^{m} e_i) = \psi(x)$ from (A) and (H). Then, (2.6) implies

$$\lim_{\alpha\to\infty} b(\alpha) = \psi(\bar{x}). \tag{3.2}$$

But from (2.5) and the fact that $\inf\limits_{\bar{x}+E_k^\perp} \varphi \le c_k^2 = \psi(\bar{x})$, we deduce

$$\sup_{\partial B_k(\bar{x},\bar{r})+F} \varphi < \psi(\bar{x}). \tag{3.3}$$

Hence, (3.1)-(3.3) imply that for large enough α,

$$c(\alpha) = b(\alpha). \tag{3.4}$$

If there exists an $\alpha \in \mathbf{N}$ such that $c_{k+1}^1 > c(\alpha)$, then for such α, $\psi(\bar{x}) = c_k^2 = \inf\limits_{A\in\mathcal{A}_k^2} \sup\limits_A \varphi \le \sup\limits_{\partial C(\alpha)} \varphi = c(\alpha) < c_{k+1}^1$. Hence, we can apply Theorem 2.1, Lemma 2.13, and Lemma 2.14 to φ and find that φ satisfies $(PS)_{c_{k+1}^1}$. Therefore, Theorem 2.7 and Remark 2.5 can be applied again and there exists the other critical point for φ at the level c_{k+1}^1.

If not, then for all $\alpha \in \mathbf{N}$, $c_{k+1}^1 = c(\alpha)$. That is, for each $\alpha \in \mathbf{N}$ there exists $A_\alpha \in \mathcal{A}_{k+1}^1$ such that

$$\sup_{A_\alpha} \varphi < c(\alpha) + \frac{1}{\alpha}. \tag{3.5}$$

By Proposition 2.8, it follows that $A_\alpha \cap F^\perp \in \mathcal{A}_k^2$. (3.4) and (3.5) imply

$$c_k^2 \le \lim_{\alpha\to\infty} \sup_{A_\alpha \cap F^\perp} \varphi \le \lim_{\alpha\to\infty} \sup_{A_\alpha} \varphi \le \lim_{\alpha\to\infty} c(\alpha) = \lim_{\alpha\to\infty} b(\alpha) = \psi(\bar{x}) = c_k^2,$$

i.e. $A_\alpha \cap F^\perp \in \mathcal{A}_k^2$. Now we can employ Theorem 2.1, Lemma 2.13, and Lemma 2.14 to φ again and find that φ verifies the Palais-Smale condition along a sequence $A_\alpha \cap F^\perp$ in $\mathcal{A}_k^2$. Therefore, we apply Theorem 2.7 and Remark 2.5 to φ again and obtain another critical point of φ at the level c_k^2. The proof of Theorem 3.1 is complete.

Remark : (1) Theorem 3.1 extends Theorem 3.1 in [27], and generalizes Theorem 3.1 in [9] and Theorem 2.1 in [10] partially.

(2) Theorem 3.1 is a necessary supplement of Theorem 3.1 and Theorem 3.3 in [13]. We can illuminate it in one dimensional case. Consider the problem

$$\ddot{x} + 4\pi^2 x + \sin x = \cos^2 \pi t, \quad x(0) = 0 = x(1). \tag{3.6}$$

On one hand, Theorem 3.1 ensures that (3.6) has one solution. On the other hand, we can let $V'(t,x) = 4\pi^2 x + \sin x - \cos^2 \pi t$ and $B(t,x) = 4\pi^2$ so that $V'(t,x) - B(t,x)x$ is bounded for any $x \in \mathbf{R}^N$. But we can't find B_1, B_2 such that $B_1(t) \le 4\pi^2 \le B_2(t), i(B_1) = i(B_2), \nu(B_2) = 0$

by Definition 2.10 and Proposition 2.11. In other words, we don't know whether (3.6) has one solution because one condition of Theorem 3.3 in [13] can't be satisfied. Similarly, in view of $V''(t,x) = 4\pi^2 + \cos x$, we can't also find B_1, B_2 such that $B_1(t) \le V''(t,x) \le B_2(t)$ with $i(B_1) = i(B_2), \nu(B_2) = 0$ because $\cos x$ is bound to change its sign for any $r > 0$ and all x with $|x| \ge r$. Hence, we don't know whether (3.6) has one solution by Theorem 3.3 in [13].

4. EXAMPLES AND EXTENSIONS

Example 1 : Let $k \in \mathbf{N}^*$ be even, $t_1 \in (0,1)$, $T > 0$. Moreover, let $E = H_0^1([0,1];\mathbf{R})$ and

$$A(t) = \begin{cases} \frac{(k+1)^2\pi^2}{4t_1^2}, & t \in (0,t_1). \\ \frac{(k+1)^2\pi^2}{4(1-t_1)^2}, & t \in (t_1,1). \end{cases}, \quad e(t) = \begin{cases} \sin\frac{(k+1)\pi t}{2t_1}, & t \in [0,t_1]. \\ \sin\frac{(k+1)\pi(t+1-2t_1)}{2(1-t_1)}, & t \in [t_1,1]. \end{cases},$$

$$g(x) = \sin\tfrac{2\pi x}{T},\ x \in E, \quad h(t) = \begin{cases} \sin\frac{\pi t}{2t_1}, & t \in [0,t_1]. \\ -\sin\frac{\pi(t+1-2t_1)}{2(1-t_1)}, & t \in [t_1,1]. \end{cases}.$$

By simple calculations, we get that $i(A) = k$ and $\nu(A) = 1$ from Definition 2.10 and Proposition 2.10. Hence, (A) is satisfied since $\dot{e}$ has only $k+1$ zeros in $(0,1)$. In addition, one can verify that (G) and (H) are satisfied. Therefore, (1.1) has at least one solution by Theorem 3.1.

Example 2 : Let $T_i > 0,\ i = 1,2,3$. Moreover, let $E = H_0^1([0,1];\mathbf{R}^3)$ and

$$A(t) = \begin{pmatrix} 4\pi^2 & 6\pi^2 & 3\pi^2 \\ 6\pi^2 & \pi^2 & -2\pi^2 \\ 3\pi^2 & -2\pi^2 & 4\pi^2 \end{pmatrix}, \quad e(t) = (3\sin 3\pi t, 2\sin 3\pi t, \sin 3\pi t),$$

$$g(x) = (\sin\tfrac{2\pi x_1}{T_1}, \sin\tfrac{2\pi x_2}{T_2}, \sin\tfrac{2\pi x_3}{T_3}), (x_1,x_2,x_3) \in E, \quad h(t) = (\sin\pi t, \cos\pi t, \sin 2\pi t).$$

We can work out that $i(A) = 2$, $\nu(A) = 1$. Meanwhile, we find that the derivative of each component of e has three zeros in $(0,1)$. Hence, (A) is satisfied naturally. In addition, one can verify that (G) and (H) are satisfied. Therefore, (1.1) has one solution by Theorem 3.1.

In [2], a classification theory for the following problem

$$\begin{cases} \ddot{x}(t) + C\dot{x}(t) + A(t)x(t) = 0, \\ x(0) = 0 = x(1), \end{cases} \tag{4.1}$$

where C is an $N-$order antisymmetric matrix was investigated. The Hilbert space $E \equiv H_0^1([0,1];\mathbf{R}^N)$ with the norm $\|x\| = \left(\int_0^1 \left(|\dot{x}(t)|^2 + |x(t)|^2\right)\, dt\right)^{\frac{1}{2}}$, $\forall x \in E$. In addition, the index and nullity of A, denoted by $i_C(A)$ and $\nu_C(A)$ respectively, were defined. Hence, we claim that the following Dirichlet problem

$$\begin{cases} \ddot{x}(t) + C\dot{x}(t) + A(t)x + g(x) = h(t), \\ x(0) = 0 = x(1) \end{cases} \tag{4.2}$$

can be investigated. Specially, the following result hold.

***Theorem* 4.1** — *Assume that* $i_C(A) = k \in \mathbf{N}^*$, $\nu_C(A) = 1$, *and the derivative of all components of the nontrivial solution of (4.1) are not equal to zero a.e. in* $(0,1)$. *Moreover, if* (G) *and* (H) *hold, then (4.2) has at least one solution.*

In [14], Dong developed a classification theory for the following Lagrangian system satisfying Sturm-Liouville boundary conditions

$$\begin{cases} (\Lambda(t)\dot{x}(t))' + A(t)x = 0, \\ x(0)\cos\alpha - \Lambda(0)\dot{x}(0)\sin\alpha = 0, \\ x(1)\cos\beta - \Lambda(1)\dot{x}(1)\sin\beta = 0, \end{cases} \tag{4.3}$$

where $\Lambda \in C([0,1]; \mathcal{L}_s(\mathbf{R}^N))$, $\Lambda(t)$ is positive definite for $t \in [0,1]$, $E \equiv E_{\alpha,\beta}$, $E_{\alpha,\beta} = H^1([0,1];\mathbf{R}^N)$ as $\alpha, \beta \in (0,\pi)$; $E_{0,\beta} = \{x \in H^1([0,1];\mathbf{R}^N) | x(1) = 0\}$ as $\beta \in (0,\pi)$; $E_{\alpha,\pi} = \{x \in H^1([0,1];\mathbf{R}^N) | x(0) = 0\}$ as $\alpha \in (0,\pi)$; $E_{0,\pi} = \{x \in H^1([0,1];\mathbf{R}^N) | x(0) = 0 = x(1)\}$. E is a Hilbert space with the inner product $(x,y) := \int_0^1 [(\dot{x}(t),\dot{y}(t)) + (x(t),y(t))]\,dt$, $0 \le \alpha < \pi, 0 < \beta \le \pi$. Similarly, he defined the following bilinear form

$$q_{\Lambda,A,\alpha,\beta}(x,y) \;=\; \int_0^1 [(\Lambda(t)\dot{x}(t),\dot{y}(t)) - (A(t)x(t),y(t)) - (x(1),y(1))k(\beta) + (x(0),y(0))k(\alpha)]\,dt, \quad \forall x, y \in E$$

where $k(t) = cot\, t$ as $t \in (0,\pi)$, $k(t) = 0$ as $t = 0$ or π.

In addition, the index and nullity of A, denoted by $i_{\Lambda,\alpha,\beta}(A)$ and $\nu_{\Lambda,\alpha,\beta}(A)$ respectively, were defined. Hence, we claim that the following Sturm-Liouville problem

$$\begin{cases} (\Lambda(t)\dot{x}(t))' + A(t)x + g(x) = h(t), \\ x(0)\cos\alpha - \Lambda(0)\dot{x}(0)\sin\alpha = 0, \\ x(1)\cos\beta - \Lambda(1)\dot{x}(1)\sin\beta = 0 \end{cases} \tag{4.4}$$

can be investigated. Specially, the following result hold.

***Theorem* 4.2** — *Assume that* $i_{\Lambda,\alpha,\beta}(A) = k \in \mathbf{N}^*$, $\nu_{\Lambda,\alpha,\beta}(A) = 1$*, and the derivative of all components of the nontrivial solution of (4.3) are not equal to zero a.e. in* $(0,1)$*. Moreover, if* (G) *and* (H) *hold, then (4.4) has at least one solution.*

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